Associated Graphs of p-dimensional (0,1)-matrices

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Abstract: The *associated graph of a (0,1)-matrix* has as its vertex set the lines of the matrix with vertices adjacent whenever their lines intersect at a 1. This association relates the (0,1)-matrix and bipartite graph versions of the König-Egerváry Theorem. We extend this graph association to higher dimensional matrices. We characterize these graphs, modulo isolated vertices, using a coloring in which every path between each pair of vertices contains the same two colors. We rely on previous results about p-dimensional gridline graphs, where vertices are 1's in a higher dimensional matrix and vertices are adjacent whenever they are on a common line. Also important is the dual property that the doubly iterated clique graph of a diamond- and simplicial vertex-free graph is isomorphic to the original.

Keywords: (0,1)-matrix; Clique graph; Gridline graph; Vertex coloring

1. Introduction

Every (0,1)-matrix A has an associated bipartite graph, namely, the one where the rows are the vertices in the first class, the columns are the vertices in the second class, and vertices are adjacent whenever, in the matrix, they intersect at a 1. This association relates two versions of the König-Egerváry Theorem, one for bipartite graphs and one for (0,1)-matrices. In this paper we extend this association to higher-dimensional matrices. Such matrices occur, for example, in the Taylor expansion for multivariable functions and in computer data structures. We associate with a p-dimensional (0,1)-matrix a graph having the lines of the matrix as vertices and vertices adjacent whenever the lines intersect at a 1. We characterize these graphs, except possibly for some isolated vertices.

Our notation for graph theory is standard and follows Bondy and Murty [1]; see it for any undefined terms or notation. A graph $G = (V, E)$ is undirected and has no multiple edges or loops. With a common abuse of language, we often refer to a vertex or edge as being in a graph G, and write, for example, $v \in G$ or $uw \in G$ instead of $v \in V(G)$ or $uw \in E(G)$. The cardinality of V (and, consequently, of E) is finite or denumerable. A complete graph with n vertices is denoted K_n. We take a *clique* to be a *maximal* complete subgraph. The *clique graph* $K(G)$ of a graph G has as its vertex set the cliques of G, with two vertices adjacent whenever they have some vertex of G in common. Cliques are typically denoted using capital letters. Hence, vertices of clique graphs are sometimes denoted by capital rather than small letters. A *hole*, or *n-hole* of a graph, $n \geq 4$, is an induced subgraph that is an n-cycle – that is, an n-cycle with no *chord*. If G and H are graphs then G is *H-free* means that no induced subgraph of G is isomorphic to H. A *diamond* is a K_4 minus one edge. A coloring in this paper is always a vertex-coloring and is proper.

Formally, a *p-dimensional* (or, for brevity, p-d) *(0,1)-matrix* A, where $p \in \mathbf{N}$ (we take N to be the positive integers), is a function $\mathbf{a} : N_1 \times \ldots \times N_p \rightarrow \{0, 1\}$ where, for $k = 1, \ldots, p$, $N_k = \{1, \ldots, n_k\}$ for some $n_k \in \mathbf{N} \cup \aleph_0$. The *dimensions* of A are the numbers n_k, $k = 1, 2, \ldots, p$. A *line* of A is a restricted to a maximal subset of $N_1 \times \ldots \times N_p$ in which the elements agree in every component but the k^{th} component, some $k \in \{1, \ldots, p\}$. We use the same type of terminology as with normal (2-dimensional) matrices. For example, a line refers to the subset of $N_1 \times \ldots \times N_p$ to which it is restricted, lines contain the (function) values, lines intersect (if the subsets of $N_1 \times \ldots \times N_p$ to which they are restricted intersect), and intersecting lines intersect at a 0 or a 1. We will sometimes use geometric terms when speaking of a p-d (0,1)-matrix $\mathbf{A}$, such as parallel, perpendicular, line, and hyperplane. In this case we are thinking of the value $\mathrm{a}(i_1, \ldots, i_p)$ as being at $(i_1, \ldots, i_p)$ in $\mathbf{N}^p$. Lines [resp., hyperplanes] are always parallel [resp., perpendicular] to an axis. This usage of line is equivalent to the definition of line given above. The *associated graph* $G_{\mathbf{A}}$ of a p-d (0,1)-matrix $\mathbf{A}$ is the graph in which each vertex is a line of $\mathbf{A}$ and vertices are adjacent whenever, as lines of $\mathbf{A}$, they intersect at a 1.

2. Previous Results

All of the results from this section are from Peterson [3]. The first lemma is well known and had been used elsewhere.

Lemma 2.1: *A graph is diamond-free if and only if any two vertices are in at most one clique. Moreover, in this case, if A and B are intersecting cliques then, except for edges incident to the vertex in $A \cap B$, no edge is incident to a vertex in A and to a vertex in B.* ∎

Lemma 2.2: *Suppose G is a diamond-free graph and that $\{A_j : j \in J \subseteq \mathbf{N}\}$ is a set of cliques in G that pairwise intersect. Then there exists a unique vertex $v \in G$ such that $A_i \cap A_j = \{v\}$ whenever $i \neq j$.* ∎

Lemma 2.3: *If G is a diamond-free graph then so is $K(G)$.* ∎

Lemma 2.4: *Suppose G is a diamond-free graph. Then $K(G)$ has a 5-hole if (and, in fact, only if) G has a 5-hole.* ∎

Lemma 2.5: *Suppose G is a diamond-free graph and Z is a cycle in G containing the vertex v. If the two vertices consecutive to v in Z are not adjacent, then v is in a hole whose vertices are in Z.* ∎

Given a colored graph, the color c *separates* vertices u and v whenever c appears on the *interior* – that is, on some internal vertex – of every (u, v)-path. If u and v are in different components then they are separated by every color in the coloring.

Lemma 2.6: *Suppose G is a diamond-free graph and γ is a p-coloring of G in which every pair of vertices at distance greater than two is separated by at least two colors. Then, using the coloring γ, no hole contains any color exactly once if and only if G is 5-hole-free and every 4-hole is colored with only two colors.* ∎

A *p-dimensional* (or, for brevity, p-d) *gridline graph*, where $p \in \mathbf{N}$, is a graph G that is isomorphic to some graph $\overline{G}$ whose vertices are a subset of $\mathbf{N}^p$ and vertices $\mathbf{x} = (x_1, \ldots, x_p)$ and $\mathbf{x}' = (x'_1, \ldots, x'_p)$ are adjacent whenever they differ in exactly one entry. That is, G can be realized in $\mathbf{N}^p$ such that no two vertices are colocated and two vertices are adjacent whenever they are on a common line that is parallel to some axis of $\mathbf{N}^p$. A *realization* is a graph $\overline{G}$ as given in the definition, and the term *line*, in the context of a realization, always refers to a line parallel to some axis.

The following proposition and theorem each characterize p-d gridline graphs. There is a natural geometric interpretation for each of these; see Peterson [3].

Proposition 2.7: *A graph G is a p-d gridline graph if and only if it is is diamond-free and $K(G)$ is p-colorable such that (a) no hole contains some color exactly once, and (b) every pair of vertices at distance greater than two is separated by at least two colors.* ∎

A *p-gridline coloring* of a graph is a p-coloring in which (a) every 4-hole is colored with only two colors and (b) every pair of vertices at distance greater than two is separated by at least two colors. A graph that admits a p-gridline coloring is said to be *p-gridline colorable.*

Theorem 2.8: *A graph G is a p-d gridline graph if and only if it is diamond- and 5-hole-free and $K(G)$ is p-gridline colorable.* ∎

3. Results on Associated Graphs of p-d (0,1)-Matrices

We give here all of the additional definitions and notation needed for the rest of the paper. The *union of two graphs* G and H, where $V(G)$ and $V(H)$ are disjoint, denoted $G \cup H$, is the graph $(V(G) \cup V(H), E(G) \cup E(H))$. The *clique number* of G is the number of vertices in a maximum clique, and is denoted $\omega(G)$. A vertex is isolated whenever it has degree zero. The set I_G consists of the set of isolated vertices in graph G. The graph I_n consists of n isolated vertices. A vertex is *simplicial* whenever it is in only one clique. Given a graph G, the set S_G consists of the simplicial vertices and the set S_G^* consists of those simplicial vertices each of whose cliques contain exactly one nonsimplical vertex. The set $\overline{K}_G$ consists of exactly one vertex from each component of the graph G that is a complete subgraph with at least two vertices (such a set exists by the Axiom of Choice, which is typically assumed as part of the axiomatic system of set theory – see e.g. Hewitt and Stromberg [2]). If H and G are isomorphic we write $H \cong G$, if H is an induced subgraph of G we write $H \subseteq G$, and if H is isomorphic to some induced subgraph of G we write $H \lesssim G$. Given a p-d (0,1)-matrix A, the *induced* p-d *gridline graph* H_{A} is the p-d gridline graph (realization) obtained by taking $(i_1, \ldots, i_p)$ as a vertex whenever $\mathrm{a}(i_1, \ldots, i_p) = 1$. Note the difference between H_{A} and the *associated graph* G_{A} defined in Section 1. The *induced* p-d *(0,1)-matrix* A_H of a p-d gridline graph realization H in N^p

is the p-d matrix $\mathbf{A}$ such that $H = H_{\mathbf{A}}$. That is, $\mathrm{a}(i_1, \dots, i_p) = 1$ whenever $(i_1, \dots, i_p)$ is a vertex of H. (We do not worry about the dimensions of $\mathbf{A}_H$; for our purposes we may let each dimension be infinite.)

The first lemma of this section shows that, under certain conditions, there exists a dual relationship between a graph and its clique graph.

Lemma 3.1: *If the graph G is diamond-free and has no nonisolated simplicial vertices then $K(K(G)) \cong G$. Moreover, $K(G)$ also satisfies the above hypotheses.*

Proof: We may assume that G has no isolated vertices since these remain (isomorphically) unchanged in the clique graph. Define $\mathcal{C} : V(G) \to V(K(K(G)))$ as follows: $\mathcal{C}(v)$ consists of those cliques of G containing v. We first show $\mathcal{C}(v)$ is in fact a clique of $K(G)$, and thus a vertex in $K(K(G))$. By the definition, $\mathcal{C}(v)$ is a clique of $K(G)$. Since G has no simplicial vertex, $\mathcal{C}(v)$ contains at least two cliques of G, and by Lemma 2.2 no other clique of G (not containing v) can be added to $\mathcal{C}(v)$ so that they still pairwise intersect. Thus the set of cliques of G in $\mathcal{C}(v)$ is maximal, as desired.

We now show that $\mathcal{C}$ is an isomorphism. Suppose $\mathcal{C}'$ is a clique in $K(G)$. We claim $\mathcal{C}'$ contains at least two cliques of G. Suppose in contradiction that $\mathcal{C}'$ contains only one clique A of G. Then A is an isolated vertex in $K(G)$. Now any vertex $v \in A$ is nonsimplicial in G and thus contained in another clique B of G. Thus A and B are adjacent in $K(G)$, a contradiction. We can now invoke Lemma 2.2 to see that $\mathcal{C}$ is an isomorphism: The two or more cliques of G in $\mathcal{C}'$ intersect at a unique vertex $v \in G$, so $\mathcal{C}' = \mathcal{C}(v) \neq \mathcal{C}(u)$ where $u \neq v$.

To see the edge correspondence, observe that $uv \in E(G)$ iff u and v are in a common clique of G iff $\mathcal{C}(u)$ and $\mathcal{C}(v)$ intersect iff $\mathcal{C}(u)\mathcal{C}(v) \in E(K(K(G)))$.

Finally we prove the last statement of the lemma. By Lemma 2.3,

$K(G)$ is diamond-free. Since G has no isolated vertex, any clique A of G contains (at least) two vertices u and v. Thus A is in (at least) two cliques $C(u)$ and $C(v)$ of $K(G)$, so $K(G)$ has no simplicial vertex. ∎

We make two observations related to this lemma. First, if G is diamond-free but is not simplicial vertex-free, there still exists a diamond-free graph H such that $K(H) \cong G$. The construction of H is given by Roberts and Spencer [5]: Begin with $K(G) \cup I_{|V(G)|}$ where each vertex v' of $I_{|V(G)|}$ corresponds to a vertex v of G, then join v' to the vertices of $K(G)$ that, as cliques of G, contain v. Thus a graph is diamond-free if and only if it is the clique graph of a diamond-free graph. Though $K(H) \ncong G$, we do have that $K(H) \subsetneqq G$. Second, this lemma is an example of Theorem 3.1 in Prisner [4] regarding convergence of iterated clique graphs. This result requires the *clique-Helly* property that if a set of cliques pairwise intersect then their intersection is nonempty. By Lemma 2.2, this property is satisfied in diamond-free graphs.

Lemma 3.2: *If* $\mathbf{A}$ *is a p-d (0,1)-matrix then* $G_{\mathbf{A}}$ *is diamond-free and each of its cliques is an isolated vertex or has exactly* p *vertices.*

Proof: To see that $G_{\mathbf{A}}$ is diamond-free, suppose that $\mathbf{A}^{(1)}\mathbf{A}^{(2)}\mathbf{A}^{(3)}$ and $\mathbf{A}^{(2)}\mathbf{A}^{(3)}\mathbf{A}^{(4)}$ are triangles in $G_{\mathbf{A}}$. Since $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$, $\mathbf{A}^{(3)}$ are, as lines of $\mathbf{A}$, perpendicular to one another, it follows that they all intersect at a unique element of $\mathbf{A}$ (which is a 1). Since the same is true of $\mathbf{A}^{(2)}, \mathbf{A}^{(3)}, \mathbf{A}^{(4)}$, all four lines intersect at the same element. Thus they induce a K_4 in $G_{\mathbf{A}}$.

An isolated vertex in $G_{\mathbf{A}}$ results from a line of all 0's in $\mathbf{A}$. Any edge in $G_{\mathbf{A}}$ results from a 1 in $\mathbf{A}$, which is at the intersection of p lines and thus yields edges between p vertices in $G_{\mathbf{A}}$, that is, yields a clique of p vertices. ∎

One way of viewing $G_{\mathbf{A}}$ in the lemma is as a hypergraph, where each hyperedge consists of p vertices and any two hyperedges intersect in at most one vertex.

The following operation will be used in Lemmas 3.3, 3.4, and Theorem 3.6. The "NC" stands for "nearly constructs" and the operation will be used to nearly construct an associated graph of a p-d (0,1)-matrix from another graph.

Step NC: Extend every clique to p vertices, that is, for each clique C' with q vertices, where $q < p$, add a K_{p-q} and join every vertex in the K_{p-q} to every vertex in C'.

The following technical lemma will be needed in the proofs of Lemma 3.4 and Theorem 3.6.

Lemma 3.3: *Suppose G is a diamond-free graph, each clique of which contains exactly one or exactly p vertices, where $p \in \mathrm{N} \setminus \{1\}$. Suppose further that $G' = (G \setminus S_G) \cup \overline{K}_G$. Then Step* NC *applied to G' yields a graph $\hat{G}$ such that:*

- $G' \subsetneqq \hat{G} \subsetneqq G$
- *Each clique of $\hat{G}$ contains exactly p vertices*
- $V(G \setminus \hat{G}) = I_G \cup S_G^{*\prime}$ *where $S_G^{*\prime}$ is a subset of S_G^**

Proof: Suppose C is a clique of G with p vertices. Suppose first that C contains at least two nonsimplicial vertices. Then those two vertices are in G'. By Lemma 2.1, these vertices are contained in only one clique of G (namely C) and in only one clique of G' (say C'). Using that each clique in G having at least two vertices contains exactly p vertices, Step NC reconstructs C from C'. (With an abuse of language we say we are reconstructing vertices of G when we add vertices isomorphic to those in G.) Conversely, by Lemma 2.1, every clique C' in G' with at least two vertices corresponds to a clique C in G (with exactly p vertices).

Suppose every vertex in C is simplicial, that is, C is a component that is a K_p. Since $\overline{K}_G$ is in G', we have that G' contains exactly one vertex of C, and it is isolated in G'. Then Step **NC** reconstructs C from that vertex.

Suppose an isolated vertex $c \in G'$ is not in $\overline{K}_G$. Then c is the only nonsimplicial vertex in a component of G consisting of two or more cliques that pairwise intersect at c. Then Step **NC** reconstructs one of the cliques containing c.

We have that Step **NC** reconstructs no vertices not originally in G. Any nonisolated vertex v of G (i.e. v is in some clique having at least two vertices) that has not been reconstructed belongs to a clique of G that contains exactly one nonsimplicial vertex. That is, $v \in S_G^*$. Finally, Step **NC** reconstructs no isolated vertices of G. ∎

Suppose $\mathbf{A}$ is a p-d (0,1)-matrix where $p \geq 2$. Table 3.1 gives several relationships between $\mathbf{A}$, $H_\mathbf{A}$, $G_\mathbf{A}$, and $K(H_\mathbf{A})$. This table will be useful in proving Lemma 3.4 and Theorem 3.6. Row one is immediate. Rows 2-3 follow since each 1 in $\mathbf{A}$ has a corresponding clique of p vertices in $G_\mathbf{A}$. Row four follows from rows 2-3.

The point of the following lemma is that, given a p-d (0,1)-matrix $\mathbf{A}$, $K(H_\mathbf{A}) \cong G_\mathbf{A}$ except possibly for some simplicial vertices of $G_\mathbf{A}$. Furthermore, if $\mathbf{A}$ is not given, $K(H_\mathbf{A})$ can still be constructed from $G_\mathbf{A}$ and $G_\mathbf{A}$ can be nearly constructed from $K(H_\mathbf{A})$.

Lemma 3.4: *Suppose* $\mathbf{A}$ *is a p-d (0,1)-matrix where* $p \in \mathbb{N} \setminus \{1\}$. *Then statements (1)-(3) below hold.*

(1) $K(H_\mathbf{A}) \lesseqqgtr G_\mathbf{A}$, *with isomorphism if and only if every line of* $\mathbf{A}$ *has at least two 1's.*

(2) $K(H_\mathbf{A}) \cong (G_\mathbf{A} \setminus S_{G_\mathbf{A}}) \cup \overline{K}_{G_\mathbf{A}}$.

In A (or $H_{\mathbf{A}}$)		In $G_{\mathbf{A}}$		In $K(H_{\mathbf{A}})$
A line with no 1's [no vertices]	$\Leftrightarrow$	An isolated vertex	$\Rightarrow$	Nothing (no vertex)
A line with $\geq$ two 1's [vertices]	$\Leftrightarrow$	A nonsimplicial vertex	$\Rightarrow$	A vertex
A line with exactly one 1 [vertex]	$\Leftrightarrow$	A simplicial vertex in a clique of p vertices	$\Rightarrow$	An isolated vertex or nothing (no vertex)
A line with exactly one 1 [vertex] where the 1 [vertex] is contained in exactly q lines that each contain $\geq$ two 1's [vertices]	$\Leftrightarrow$	One of exactly $p-q$ simplicial vertices in a common clique (if $q=0$ this is a component that is a K_p)	$\Rightarrow$	An isolated vertex (iff $q=0$ and this line is chosen to represent the clique of $H_{\mathbf{A}}$ consisting of this vertex) or nothing (no vertex)

Table 3.1: Relationships between A, $H_{\mathbf{A}}$, $G_{\mathbf{A}}$, and $K(H_{\mathbf{A}})$

(3) Step NC *applied to $K(H_{\mathbf{A}})$ yields a graph $\hat{G}$ such that:*

- $K(H_{\mathbf{A}}) \subseteqq \hat{G} \subseteqq G_{\mathbf{A}}$
- *Each clique of $\hat{G}$ contains exactly p vertices*
- $V(G_{\mathbf{A}} \setminus \hat{G}) = I_{G_{\mathbf{A}}} \cup S^{*\prime}_{G_{\mathbf{A}}}$ *where $S^{*\prime}_{G_{\mathbf{A}}}$ is a subset of $S^{*}_{G_{\mathbf{A}}}$*

Proof: (1) Each line of **A** corresponds to a vertex of $G_{\mathbf{A}}$. It also corresponds to a line in $H_{\mathbf{A}}$, which may yield a clique of $H_{\mathbf{A}}$, that is, a vertex of $K(H_{\mathbf{A}})$ (see rows 1-3 of Table 3.1). To show $K(H_{\mathbf{A}})$ is induced, if two vertices $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ of $G_{\mathbf{A}}$ have corresponding vertices a_1 and a_2 of $K(H_{\mathbf{A}})$, then $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are adjacent in $G_{\mathbf{A}}$ iff $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ intersect (as lines of **A**) at a 1 iff a_1 and a_2 intersect (as lines in the realization $H_{\mathbf{A}}$) at a vertex iff a_1 and a_2 are adjacent in $K(H_{\mathbf{A}})$. The isomorphism follows from statement (2) (or by a simple direct argument).

(2) We will use the second and third columns of Table 3.1 to identify which vertices of $G_{\mathbf{A}}$ to delete to obtain (a graph isomorphic to) $K(H_{\mathbf{A}})$. First suppose $\mathbf{A}^{(1)}$ is a vertex of $G_{\mathbf{A}}$ that is simplicial but not in $\overline{K}_{G_{\mathbf{A}}}$; we show it must be deleted. We have that $\mathbf{A}^{(1)}$ is either (i) isolated, (ii) nonisolated and not in a component that is a K_p, or (iii) in a component that is a K_p, exactly one other vertex of which is in $\overline{K}_{G_{\mathbf{A}}}$. If (i) holds, then by row one of Table 3.1, $\mathbf{A}^{(1)}$ must be deleted. If (ii) holds, then by row four with $q \neq 0$, $\mathbf{A}^{(1)}$ must be deleted. If (iii) holds, then by row four with $q = 0$, $\mathbf{A}^{(1)}$ must be deleted (by the first column of row four, this corresponds to an isolated vertex in the realization $H_{\mathbf{A}}$, which yields an isolated vertex in $K(H_{\mathbf{A}})$).

Now suppose a vertex $\mathbf{A}^{(1)}$ of $G_{\mathbf{A}}$ was not deleted; we show it has a corresponding vertex in $K(H_{\mathbf{A}})$. Since $\mathbf{A}^{(1)}$ was not deleted it was either (i) nonsimplicial or (ii) it was in a component that is a K_p, the other $p-1$ vertices of which were deleted. If (i) holds then, by row two of Table 3.1, $\mathbf{A}^{(1)}$ has a corresponding vertex in $K(H_{\mathbf{A}})$. If (ii) holds then, by row four with $q = 0$, $\mathbf{A}^{(1)}$ corresponds to an isolated vertex in $H_{\mathbf{A}}$, which yields an isolated vertex in $K(H_{\mathbf{A}})$. In either case $\mathbf{A}^{(1)}$ has a corresponding vertex in $K(H_{\mathbf{A}})$, as claimed.

(3) By Lemma 3.2 and statement (2), this is exactly Lemma 3.3 with $G = G_{\mathbf{A}}$ and $G' = K(H_{\mathbf{A}})$. ∎

Recall the definition of p-gridline colorable that preceded Theorem 2.8. We will use this in Lemma 3.5 and Theorem 3.6.

Lemma 3.5: *Suppose G is a graph and G' is an induced subgraph such that every vertex of $G \setminus G'$ is simplicial (in G). Then, if G is p-gridline colorable then so is G'. If, in addition, G is diamond- and 5-hole-free and $\omega(G) \leq p$, then the converse also holds.*

Proof: For brevity, we refer to the part of p-gridline colorability for 4-holes simply as (a) and the part for vertices at distance greater than two as (b).

We show the p-gridline coloring used for G works for G'; the argument assumes this coloring. Clearly (a) holds for G'. Suppose (b) does not hold; then G' has vertices a and b such that $d_{G'}(a, b) > 2$ and there are less than two colors that separate a and b. Then, since all paths of G' are also in G, there are less than two colors that separate a and b in G. Thus $d_G(a, b) \leq 2$ in G, and since G' is induced, this implies that $d_G(a, b) = 2$ in G. Then a and b are both adjacent to a vertex c that was removed from G. But then c is simplicial in G, so a, b, and c are in a common clique in G and are therefore pairwise adjacent; hence a and b are adjacent so $d_{G'}(a, b) = 1$, a violation.

Now suppose that the converse does not hold. It will be convenient to take $G' = G \setminus \{a\}$ for some simplicial vertex $a \in V(G)$; we may do this by taking a minimal G violating the converse and a maximal subgraph G' satisfying the lemma hypotheses. (Formally, such graphs exist by Zorn's Lemma, which is logically equivalent to the Axiom of Choice – see Hewitt and Stromberg [2]). By hypothesis G' is p-gridline colorable, and we can easily extend the p-coloring to G: Vertex a has at most $p - 1$ neighbors since it is simplicial and its clique has by hypothesis at most p vertices, so there is at least one color available to color a. To see that (a) holds in G, note that a is in no 4-hole, since it is simplicial implying its neighbors are adjacent. To show that (b) indeed holds in G, suppose that $d_G(x, y) > 2$ in G. If neither of x or y is a, then they do not violate (b). For, if a is in an (x, y)-path of G then a can be removed, since it is simplicial and the vertices on either side of it are adjacent; thus x, y are separated at least by the same colors as in G'. We have then that (b) can only be violated by vertices a, b in G for some vertex b. Suppose a is in clique C (in G). Suppose P is an (a, b)-path in G. Then the vertex adjacent to a in P is a vertex $c \in C$. We show that in fact there are at least two colors, each of which separates a and b in G. We consider two cases.

Case 1: $d_G(a, b) > 3$. Then $d_G(c, b) > 2$. Since $c, b \neq a$ then, as noted above, there are at least two colors, say colors 1 and 2, each of which separates c and b in G. If every (a, b)-path passes through c then a, b are also separated by colors 1 and 2. If there is an (a, b)-path P' not passing through c, then it passes through $c' \in C$ where $c' \neq c, a$. But then the portion of P' from b to c', with c concatenated to the end, is a (b, c)-path in G', and thus contains the colors 1 and 2 in its interior. It follows that P' contains the colors 1 and 2 in its interior.

Case 2: $d_G(a, b) = 3$. We may assume P is a shortest (a, b)-path $acdb$ where, say, c has color 1 and d has color 2. We show that every (a, b)-path Q contains the colors 1 and 2 in its interior. Observe that, since G' is an induced subgraph of G, it is diamond- and 5-hole-free. Thus we will be able to apply Lemmas 2.5 and 2.6. We now consider four subcases.

Case 2a: Path Q contains c and d. Then Q contains colors 1 and 2 in its interior.

Case 2b: Path Q contains c but not d. Then Q contains the color 1 in its interior, at c. The portion of Q from b to c together with d forms a cycle in G'. Vertices b and c are not adjacent, or else $d_G(a, b) < 3$. By Lemmas 2.5 and 2.6, the portion of Q from b to c contains the color 2 in its interior.

Case 2c: Path Q contains d but not c. Then Q contains the color 2 in its interior, at d. Let c' be the first vertex in C that Q hits, going from b to a. Then the portion of Q from d to c', together with c, forms a cycle in G'. Vertices c' and d are not adjacent, or else $\{a, c, c', d\}$ induces a diamond in G. Again by Lemmas 2.5 and 2.6, the portion of Q from d to c' contains the color 1 in its interior.

Case 2d: Path Q contains neither c nor d. Let c' be as in case 2c. If $d_{G'}(b, c') > 2$ in G' then, since $b, c' \neq a$, there are at least two colors each of which separates b and c'; by path $bdcc'$ two of these colors are 1 and 2. Then the portion of Q from b to c' contains the colors 1 and 2 in its interior. Suppose now that $d_{G'}(b, c') = 2$ in G' (note $d_{G'}(b, c') \geq 2$ since $d_G(b, a) = 3$ in G). Then there is a path $bd'c'$. Now $d \neq d'$ since c' and d are not adjacent, as observed in case 2c. Then $bdcc'd'$ is a 5-cycle. There is no chord from b or else $d_G(b, a) < 3$, violating the hypothesis of case 2. There is no chord from c to d' or c' to d, or else a diamond is induced. Thus dd' is the only chord. Then $cc'd'd$ is a 4-hole, so by (a), which was shown to hold, c' has color 2 and d' has color 1. Now rename $P = acdb$ to $P = ac'd'b$, reverse colors 1 and 2, and apply either case 2a or 2b (depending on whether Q contains d'). We conclude that Q contains the colors 1 and 2 in its interior. ∎

If some vertex of $G \setminus G'$ is not simplicial then the lemma does not hold. For example, if G is a 6-cycle with chords between vertices 1 and 4, 2 and 6, and 3 and 5, then it is not gridline colorable (for any p). But removing any vertex yields a graph that is 3-gridline colorable.

The following theorem is the main result of the paper. It is a characterization, except possibly for some isolated vertices, of associated graphs of p-d (0,1)-matrices.

Theorem 3.6: *Suppose G is a graph and $p \in \mathbb{N}$. Then $G \cup I_n \cong G_{\mathbf{A}}$ for some p-d (0,1)-matrix* $\mathbf{A}$ *and some $n \in \mathbb{N} \cup \{0\} \cup \aleph_0$ if and only if (1) G is diamond- and 5-hole-free, (2) each clique of G has exactly one or exactly p vertices, and (3) G is p-gridline colorable. In particular, $G_{\mathbf{A}}$ satisfies conditions (1)-(3).*

Proof: In the case $p = 1$, $G_{\mathbf{A}}$ is an isolated vertex. If G is 1-gridline colorable then it too is an isolated vertex, since if there were at least two (isolated) vertices then there would only be one color to separate them.

Suppose $p \geq 2$.

("only if") It suffices to prove that $G_{\mathbf{A}}$ satisfies (1)-(3). For then it is immediate that $G \cong G_{\mathbf{A}} \setminus I_n$ also satisfies (1)-(3). By Lemma 3.2, $G_{\mathbf{A}}$ is diamond-free and (2) holds. To show $G_{\mathbf{A}}$ has no 5-hole, by Lemma 3.4 (1) $K(H_{\mathbf{A}}) \subsetneqq G_{\mathbf{A}}$, and by Proposition 2.7 $K(H_{\mathbf{A}})$ has no 5-hole. By Lemma 3.4 (2), any vertex a of $G_{\mathbf{A}} \setminus K(H_{\mathbf{A}})$ is simplicial. Thus a's neighbors are adjacent and so it is in no 5-hole. Thus (1) holds. Finally, $K(H_{\mathbf{A}})$ is p-gridline colorable by Theorem 2.8, and by Lemma 3.5 $G_{\mathbf{A}}$ is also p-gridline colorable.

("if") Suppose G satisfies (1)-(3). We will construct a p-d (0,1)-matrix $\mathbf{A}$ satisfying the condition. We may assume:

(i) G has no isolated vertex. To create isolated vertices in $G_{\mathbf{A}}$ we can insert into $\mathbf{A}$ hyperplanes of all 0's to obtain at least as many lines of 0's – which correspond to isolated vertices in $G_{\mathbf{A}}$ – as needed. (Note that this argument is not valid for $p = 1$.)

(ii) Every clique of G has at least two nonsimplicial vertices. To create a clique C in $G_{\mathbf{A}}$ with at most one nonsimplicial vertex a, where a has a corresponding line $\mathbf{A}^{(1)}$ in $\mathbf{A}$, we can insert into $\mathbf{A}$ a hyperplane perpendicular to $\mathbf{A}^{(1)}$ and give all elements in the new hyperplane the value 0, except give the element at the intersection with $\mathbf{A}^{(1)}$ the value 1. If a has no corresponding line in $\mathbf{A}$ (this occurs where there is a component containing at most one nonsimplicial vertex), then we add a hyperplane to $\mathbf{A}$ as described in (i) above, choose one of the lines in this hyperplane to correspond to a, and proceed as in the previous sentence.

Delete S_G from G and call the resulting graph G'. We claim that no vertex $a \in G'$ is simplicial. Since $a \notin S_G$, a is in (at least) two cliques C_1 and C_2 of G. By (ii), there are vertices $a_i \in C_i$ for $i = 1, 2$ such that $a_i \notin S_G$ and $a_i \neq a$. By Lemma 2.1, a_1 and a_2 are distinct (or else the pair $\{a, a_1\}$ is in both C_1 and C_2) and nonadjacent (since C_1 and C_2 are nonadjacent except at a). Thus a has (at least) two neighbors in G' that are not adjacent, so a is not simplicial, as claimed. We can now apply Lemma 3.1: There exists a graph H – namely $K(G')$ – such that $K(H) \cong G'$, H is diamond-free, and H contains no simplicial vertex. By Lemma 2.4 H has no 5-hole and by Lemma 3.5 G' is p-gridline colorable. Then, by Theorem 2.8, H is a p-d gridline graph, which we may assume to be a realization in $\mathbf{N}^p$. We show that $\mathbf{A} = \mathbf{A}_H$ satisfies the condition of the theorem.

Now by (ii) $\overline{K}_G$ is empty, so deleting S_G from G was the same as deleting $S_G \setminus \overline{K}_G$. Thus, by Lemma 3.3, Step NC reconstructs G from G' except for I_G and possibly some vertices of S^*_G. But by (i) and (ii) I_G and S^*_G are empty, so Step NC applied to G' yields G. Now by Lemma 3.4 (3), Step NC applied to $K(H) \cong G'$ reconstructs $G_{\mathbf{A}_H}$ except for $I_{G_{\mathbf{A}_H}}$ and possibly some vertices of $S^*_{G_{\mathbf{A}_H}}$. We show that $S^*_{G_{\mathbf{A}_H}}$ is empty. This will complete the proof, for then Step NC applied to G' yields (graphs isomorphic to) both G and $G_{\mathbf{A}_H} \setminus I_{G_{\mathbf{A}_H}}$. Suppose $G_{\mathbf{A}_H}$ contains a vertex of $S^*_{G_{\mathbf{A}_H}}$. By row four of Table 3.1 with $q = 1$, there is a vertex v of H contained by only one line with at least two vertices. But then v is simplicial in H, a contradiction. ∎

Observe that if $p = 2$ then any bipartite graph satisfies conditions (1)-(3) of Theorem 3.6. Further, in that case, we may take $n = 0$ in I_n, that is, $G \cong G_{\mathbf{A}}$. For, in two dimensions we can delete single lines of $\mathbf{A}$ that are 0 everywhere, which deletes single isolated vertices in $G_{\mathbf{A}}$.

For $p \geq 3$ we may not be able to delete a single line that is 0 everywhere from a p-d (0,1)-matrix $\mathbf{A}$.

4. Conclusion

We have associated with any p-d (0,1)-matrix **A** a graph G having the lines of **A** as vertices and edges being between vertices that as lines of **A** intersect at a 1. This association is often used between a bipartite graph G and a (2-d) (0,1)-matrix. In any dimension, the associated graph $G_{\mathbf{A}}$ is closely associated to the p-d gridline graph $H_{\mathbf{A}}$ induced by **A**, in that $K(H_{\mathbf{A}}) \cong G_{\mathbf{A}}$ modulo some simplical vertices. Using this idea, we have nearly characterized graphs associated with p-d (0,1)-matrices. A graph G, modulo isolated vertices, is associated with a p-d (0,1)-matrix whenever it is diamond- and 5-hole-free, each clique has exactly one or p vertices, and it is p-colorable such that every 4-hole is colored with only two colors and every pair of vertices at distance greater than two is separated by at least two colors.

Among the questions remaining are:

- *What full characterization is there for graphs associated with p-d (0,1)-matrices?*

- *What is the minimum number of isolated vertices that must be added to a graph such that it becomes associated with some p-d (0,1)-matrix?*

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