Degree Sequences of Optimally Edge-Connected Multigraphs

Peter Dankelmann*
University of Natal
Durban 4041
South Africa

Ortrud Oellermann †
University of Winnipeg
515 Portage Avenue
MB R3B 2E9, Canada

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Abstract

Let u, v be distinct vertices of a multigraph G with degrees d_u and d_v, respectively. The number of edge-disjoint u, v-paths in G is bounded above by $\min\{d_u, d_v\}$. A multigraph G is optimally edge-connected if for all pairs of distinct vertices u and v this upper bound is achieved. If G is a multigraph with degree sequence D, then we say G is a realisation of D. We characterize degree sequences of multigraphs that have an optimally edge-connected realisation as well as those for which every realisation is optimally edge-connected.

Let $G = (V, E)$ be a finite multigraph, i.e., a graph with multiple edges but without loops. Let u and v be distinct vertices of G and let $\lambda_G(u, v)$ ($\kappa_G(u, v)$) be the maximum number of edge disjoint (internally vertex disjoint) u, v-paths in G. The minimum of $\lambda_G(u, v)$ over all $u, v \in V$, called the *edge-connectivity* of G, and also the minimum of $\kappa_G(u, v)$ over all $u, v \in V$, called the *connectivity* of G, have been studied widely. Recently also the average of $\lambda_G(u, v)$ ($\kappa_G(u, v)$) over all $u, v \in V$, the *average edge-connectivity* $\overline{\lambda}(G)$ (*average connectivity* $\overline{\kappa}(G)$), received attention [1, 3, 5, 7].

If u and v are vertices of G denote their respective degrees by $\deg_G u$ and $\deg_G v$. Clearly, $\min\{\deg_G u, \deg_G v\}$ is an upper bound for $\lambda_G(u, v)$ and $\kappa_G(u, v)$. This implies an upper bound on $\overline{\lambda}(G)$, in terms of the degrees of the vertices of G,

$$\overline{\lambda}(G) \leq \binom{|V|}{2}^{-1} \sum_{\{u,v\} \subset V} \min\{\deg_G u, \deg_G v\}. \qquad (1)$$

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We consider graphs that achieve this upper bound.

We call a finite sequence D of non-negative integers *multigraphical* (or *graphical*) if D is the degree sequence of some multigraph, (respectively, of some graph). If G is a multigraph having D as its degree sequence, then we say G is a *realisation* of D. If $D: d_1 \geq d_2 \geq \ldots \geq d_n$ has a connected realisation, then necessarily $\sum_{i=1}^{n} d_i \geq 2(n-1)$ and if $n \geq 2$, $d_n \geq 1$. Moreover, if $\sum_{i=1}^{n} d_i \geq 2(n-1)$ and $d_n \geq 1$ for $n \geq 2$, then there is a connected realisation of D as we now see. For $n = 1$ or 2 this is certainly the case. Suppose now that $n \geq 3$. Let G be a realisation of D with the fewest number of components. Since $d_n \geq 1$, each component of G is nontrivial. Moreover, if G has at least two components, then some component must have a cycle. Let uv be an edge of G that belongs to a cycle and let xy be an edge in a component of G that does not contain uv. Then the graph obtained from G by deleting the edges uv, xy and adding the edges ux, vy is a realisation of D that has fewer components than G. This is contrary to our choice of G. Hence D has a connected realisation.

A (multi)graph G is *optimally edge-connected* if it achieves the bound (1), i.e., if for all pairs u, v of distinct vertices of G,

$$\lambda_G(u, v) = \min\{\deg_G u, \deg_G v\}. \tag{2}$$

The definition of an *optimally connected* (multi)graph is analogous.

We call a (multi)graphical sequence D *edge-optimal* if and only if there exists an optimally edge-connected (multi)graph G with degree sequence D. In this case we say G is an *edge-optimal realisation* of D.

We make use of the following well-known characterisation of multigraphical degree sequences. For further results or notions not defined here we refer the reader to [2].

Lemma 1 Hakimi [6] *A sequence $d_1 \geq d_2 \geq \ldots \geq d_n$ is multigraphical if and only if $\sum_{i=1}^{n} d_i$ is even and $d_1 \leq \sum_{i=2}^{n} d_i$.*

The problem of characterising those graphical sequences which are optimal or edge-optimal was posed in [5]. The aim of this paper is to give a characterization of multigraphical degree sequences that are edge-optimal. Since zero entries result in isolated vertices which together with any other vertex trivially satisfy condition (2), a degree sequence D containing zeros is edge-optimal if and only if the sequence obtained from D by discarding all zero entries is edge-optimal. If D contains exactly two terms, then D is necessarily edge-optimal since in this case $d_1 = d_2$. Hence, it suffices to consider multigraphical sequences of length at least 3 with only positive entries. For a positive integer i we denote the number of terms in the sequence D which equal i by n_i.

Theorem 1 *Let $D: d_1 \geq d_2 \geq \ldots \geq d_n$, $n \geq 3$, be a multigraphical sequence with $d_n > 0$. Then D is edge-optimal if and only if*
(i) $n_1 \leq d_1 - d_2$ or
(ii) $D: n-1, 1, 1, \ldots, 1$ where D contains $n-1$ terms equal to 1.

Proof. If D is edge-optimal, then all vertices of degree 1 are necessarily adjacent to the same vertex, and this vertex has degree d_1. Hence $n_1 \leq d_1 - d_2$ or $D: n-1, 1, 1, \ldots, 1$. For the converse suppose D is a multigraphical sequence for which (i) and (ii) hold. We show D is edge-optimal. Suppose that D is a counter example for which $\sum_{i=1}^{n} d_i$ is as small as possible.

CLAIM 1: $d_n \geq 3$.
Suppose first that $d_n = 1$. Let D have n_1 terms equal to 1. Let D' be the degree sequence obtained from D by discarding all terms that equal 1 and reducing d_1 by n_1. If $D': 0, 0, \ldots, 0$ and thus $D: n-1, 1, 1, \ldots, 1$ where the latter sequence has $n-1$ terms equal to 1, then the star $K_{1,n-1}$ is an edge-optimal realisation of D, a contradiction to the choice of D. If, on the other hand, D' is not $0, 0, \ldots, 0$, then D' is a multigraphical sequence and satisfies condition (i). By the choice of D, there exists an edge-optimal realisation G' of the sequence D'. Let G be the graph obtained from G' by appending n_1 end vertices to a vertex of degree $d_1 - n_1$ in G'. Then G is an edge-optimal realisation of D, a contradiction.
Now suppose that $d_n = 2$. Then let D' be obtained from D by discarding d_n. By the choice of D, the sequence D' has an edge-optimal realisation G'. Let G be obtained from G' by subdividing an arbitrary edge. Then G has degree sequence D and it is easy to see that G is an edge-optimal, realisation of D, a contradiction.

CLAIM 2: $d_1 \leq (\sum_{i=2}^{n} d_i) - 2$.
Since D is multigraphical, we have $d_1 \leq \sum_{i=2}^{n} d_i$. If $d_1 = \sum_{i=2}^{n} d_i$, then let G be the multigraph with vertex set $\{v_1, v_2, \ldots, v_n\}$ in which vertex v_i is joined to v_1 by d_i edges and no other edges are present. Then G is an edge-optimal realisation of D, a contradiction. Hence, $d_1 \leq (\sum_{i=2}^{n} d_i) - 1$. By the handshake lemma, the inequality is strict. Hence Claim 2 follows.

CLAIM 3: $d_1 = d_2$.
Suppose not. Then let D' be the sequence $d'_1, d'_2, \ldots, d'_n$ where $d'_1 = d_1 - 1$ and $d'_n = d_n - 1$ and $d'_i = d_i$ for $2 \leq i \leq n-1$. Then D' has an edge-optimal realisation G' with vertices $v_1, v_2, \ldots, v_n$ such that $\deg_{G'} v_j = d'_j$. Let $G = G' + v_1 v_n$. Then G has degree sequence D. To see that G is optimally edge-connected consider two vertices v_a and v_b. If $a, b \notin \{1, n\}$, then $\lambda_G(v_a, v_b) \geq \lambda_{G'}(v_a, v_b) = \min\{d_a, d_b\}$. Also if $b \neq 1$, $a \neq 1, n$, then $\lambda_G(v_1, v_a) \geq \lambda_{G'}(v_1, v_a) = \min\{d'_1, d'_a\} = \min\{d_1, d_a\}$. Now consider $\lambda_G(v_a, v_n)$ for $a \in \{1, 2, \ldots, n-1\}$. Since v_n has degree $d_n - 1$ in G', there exist $d_n - 1$ edge-disjoint $v_a - v_n$ paths and $d_a \geq 3$ edge-disjoint $v_a - v_1$

paths in G'. To separate v_a and v_n in G, one has to remove either v_1v_n and at least one edge from each path in any collection of $d_n - 1$ edge-disjoint $v_a - v_n$ paths or one has to delete at least one edge from each path in any collection of $d_a(\geq d_n)$ edge-disjoint $v_1 - v_a$ paths. In either case, at least d_n edges have to be removed and so $\lambda_G(v_a, v_n) \geq \min\{d_a, d_n\}$. Therefore, G is an edge-optimal realisation of D. This contradiction proves $d_1 = d_2$.

CLAIM 4: $d_i - d_{i+1} \leq 1$ for $i = 2, 3, \ldots, n-1$.
Suppose, to the contrary, that $d_i - d_{i+1} \geq 2$ for some $i \geq 2$. Let $D' = d'_1, \ldots, d'_n$ be obtained from D by subtracting 2 from $d_1, d_2, \ldots, d_i$. The new sequence is multigraphical since $d'_1 = d'_2$ and $\sum_{i=1}^{n} d'_i$ is even. As D' contains no 1's, it satisfies condition (i). Hence D' has an edge-optimal realisation G' with vertex set $\{v_1, v_2, \ldots, v_n\}$ such that $deg_{G'}v_j = d'_j$ for $j = 1, 2, \ldots, n$. Let G be the multigraph obtained from G' by adding a cycle through vertices of degree $d_1 - 2, d_2 - 2, \ldots, d_i - 2$. Then G has degree sequence D. To prove that G is optimally edge-connected consider two vertices v_a and v_b. If $a, b > i$ or $a > i$ and $b \leq i$, then $\lambda_G(v_a, v_b) \geq \lambda_{G'}(v_a, v_b) = \min\{d_a, d_b\}$. If $a, b \leq i$, then the added cycle provides two additional edge-disjoint paths between v_a and v_b that do not contain edges in G'. Hence $\lambda_G(v_a, v_b) \geq \lambda_{G'}(v_a, v_b) + 2 = \min\{d_a, d_b\}$, a contradiction.

CLAIM 5: $d_n = 3$.
This follows as in the proof of Claim 4 by considering the sequence $D' = d_1 - 2, d_2 - 2, \ldots, d_n - 2$ and adding a cycle through all n vertices of an edge-optimal realisation of D'.

CLAIM 6: $n_i \leq i - 1$ for $i = 1, 2, \ldots, d_1$.
Suppose, to the contrary, that $n_i \geq i$ for some i with $i \leq d_1$. Let D' be the sequence obtained from D by discarding $i - 1$ terms that equal i. Then the sum of the terms in D' is even. By Claims 4 and 5, D and thus D' contain the subsequence $d_1, d_1 - 1, d_1 - 2, \ldots, 3$. Therefore, the second condition of Lemma 1 holds if $d_1 \geq 5$. If d_1 is 4 or 3, then D and D' both contain a positive even number of 3's and again, the second condition of Lemma 1 holds. Thus D' is multigraphical. By Claim 1, $n_1 = 0$ and thus D' satisfies condition (i). Hence D' has an edge-optimal realisation G'. Since $n_i \geq i$, there exists a vertex v of degree i in G'. Let G be the multigraph obtained from G' by replacing v by a complete graph K_i and making each of the i edges of G incident with v now incident with a different vertex of K_i. Then the graph G has degree sequence D and it is easy to prove that G is optimally edge-connected.

CLAIM 7: $n_{d_1} \geq 3$.
We already know that $d_1 = d_2$. We next show that $d_1 = d_3$. Suppose, to the contrary, that $d_1 = d_2 > d_3$. Then let $D' = d_1 - 1, d_2 - 1, d_3, d_4, \ldots, d_n$. Then D' is multigraphical and satisfies condition (i). Hence D' has an

edge-optimal realisation G'. Adding an edge between two vertices of degree $d_1 = d_2$ yields a multigraph G which is easily seen to be optimally edge-connected, a contradiction. Hence $d_1 = d_3$ and thus $n_{d_1} \geq 3$.

CLAIM 8: There exists some i with $d_i = n_{d_1}$.
By Claims 7 and 5 we have $n_{d_1} \geq 3 = d_n$. By Claim 6 we have $n_{d_1} \leq d_1 - 1$. Since, by Claim 3, the difference $d_j - d_{j+1}$ is at most 1, Claim 8 follows.

To complete the proof let $k = n_{d_1}$ and consider the sequence D' obtained from D by subtracting 1 from each of the terms $d_1, d_2, \ldots, d_k$ and discarding a term d_i with $d_i = k$. Then D' is multigraphical and satisfies condition (i). Hence D' has an edge-optimal realisation G'. Let G be the graph obtained from G' by adding a new vertex v to G' and making it adjacent to k vertices of maximum degree. Clearly, G has degree sequence D. To see that G is optimally edge-connected consider two vertices v_a and v_b of degree d_a and d_b, respectively, in G and d'_a and d'_b, respectively in G'. Suppose $v_a, v_b \neq v$. If v_a and v_b are not adjacent to v in G, then $\lambda_G(v_a, v_b) = \min\{d'_a, d'_b\} = \min\{d_a, d_b\}$. If exactly one of v_a and v_b, say, v_a is adjacent to v, then $d_a > d_b$ and, as above, we obtain $\lambda_G(v_a, v_b) = \min\{d'_a, d'_b\} = d_b$. If both, v_a and v_b, are adjacent to v in G, then vertex v provides an additional path of length 2 from v_a to v_b. Hence $\lambda_G(v_a, v_b) = 1 + \min\{d'_a, d'_b\} = \min\{d_a, d_b\}$. Finally, consider $\lambda_G(v, v_a)$. Let $S \subset E(G)$ be a minimum edge-cut separating v_a and v in G. If S contains all edges incident with v, then $\lambda_G(v, v_a) = |S| \geq \deg_G(v) = k$. If some edge incident with v, say vv_i, is not in S then, by $\lambda_G(v_i, v_a) = \min\{d_i, d_a\}$, the edge-cut S contains at least $\min\{d_i, d_a\} = d_a$ edges. In both cases, we have $\lambda_G(v, v_a) \geq \min\{d_a, k\}$, as desired.

Hence G is an edge-optimal realisation of D. This contradiction completes the proof. □

We remark that the proof yields a polynomial time algorithm to construct an edge-optimal realisation of a given edge-optimal multigraphical degree sequence.

Not every multigraphical sequence that satisfies the conditions of Theorem 1 has the property that each one of its realisations is edge-optimal. For example, if G is obtained from two copies of K_n, $n \geq 3$, by adding exactly one edge between the two copies of K_n yields a connected (multi)graph that is not edge-optimal. Moreover, its degree sequence satisfies the conditions of Theorem 1. The next result characterises those degree sequences for which every realisation is edge-optimal.

Theorem 2 *Let $D : d_1 \geq d_2 \geq \ldots \geq d_n$ be a multigraphical sequence with $d_n > 0$. Then every realisation of D is optimally edge-connected if and only* *if*

$$d_1 \geq \sum_{i=2}^{n-1} d_i. \tag{3}$$

Proof. Let D be a multigraphical sequence satisfying $d_n > 0$ and (3). Let $b := \frac{1}{2}(\sum_{i=2}^{n} d_i - d_1)$. Since D is multigraphical, $\sum_{i=2}^{n} d_i - d_1$ is even and nonnegative. Hence, by our hypothesis,

$$0 \leq 2b \leq d_n.$$

We show that each realisation G of D is optimally edge-connected. Let $v_1, v_2, \ldots, v_n$ be the vertices of G with degrees $d_1, d_2, \ldots, d_n$, respectively. It suffices to show that

$$\lambda_G(v_1, v_i) = d_i \quad \text{for } i = 2, 3, \ldots, n, \tag{4}$$

since we then have, for $i > j > 1$,

$$\lambda_G(v_i, v_j) \geq \min\{\lambda_G(v_i, v_1), \lambda_G(v_j, v_1)\} = d_j = \min\{d_i, d_j\},$$

and G is optimally edge-connected. Denote the graph $G - v_1$ by H. Then H has exactly b edges and for each vertex v_j, $1 < j < n$, we have

$$\deg_H v_j \leq b \leq \frac{1}{2} d_n \leq \frac{1}{2} d_j. \tag{5}$$

Consider a vertex v_i $(1 < i \leq n)$. If there are $a_{i,j}$ edges from v_i to v_j, then by (5) there are at least $a_{i,j}$ edges from v_j to v_1. Hence there exist $a_{i,j}$ edge-disjoint paths v_i, v_j, v_1 for each v_j, $j \neq 1, i$. Together with the $a_{i,1}$ edges from v_i to v_1, we obtain $\sum_{j \neq i} a_{i,j} = d_i$ edge-disjoint paths from v_i to v_1. This proves (4). Hence G is optimally edge-connected.

For the converse consider a multigraphical sequence $D : d_1, \ldots, d_n$ with $d_1 < \sum_{i=2}^{n-1} d_i$. We show that D has a realisation G that is not optimally edge-connected. Let $c = \lceil \frac{1}{2}(d_n + 1) \rceil$. Consider the sequence $D' : d'_1, d'_2, \ldots, d'_{n-1}$ where $d'_i = d_i$ for $i = 1, 2, \ldots, n-2$ and $d'_{n-1} = d_{n-1} + d_n - 2c$. We first show that D' is multigraphical. Clearly $\sum_{i=1}^{n-1} d'_i$ is even. Since $d'_{n-1} < d_{n-1} \leq d_1 = d'_1$, the entry d'_1 is the largest entry of the sequence D'. The second condition of Lemma 1 holds since $d'_1 = d_1 < \sum_{i=2}^{n-1} d_i = \sum_{i=2}^{n-1} d'_i + 2c - d_n \leq \sum_{i=2}^{n-1} d'_i + 1$, hence D' is multigraphical. Let H be a realisation of D' with vertices $v_1, \ldots, v_{n-1}$ of degree $d'_1, d'_2, \ldots, d'_{n-1}$, respectively. Let G be the graph obtained from H as follows. Add a vertex v_n, replace $d_n - c$ edges that, in H, join v_{n-1} to other vertices by $d_n - c$ edges now joining v_n to those vertices, and add c edges between v_{n-1} and v_n. Then G is a realisation of D.

It remains to show that G is not optimally edge-connected. Since only d'_{n-1} edges join a vertex in $\{v_{n-1}, v_n\}$ to a vertex in $\{v_1, \ldots, v_{n-2}\}$, we have

$$\lambda_G(v_{n-1}, v_1) \leq d'_{n-1} = d_{n-1} + d_n - 2c < d_{n-1}.$$

implying that G is not optimally edge-connected, as desired. □

The problems of characterising the graphical degree sequences which are optimal or edge-optimal remains open. If $D : d_1 \geq d_2 \geq \ldots \geq d_n$ is graphical, then as for multigraphs D is edge-optimal if and only if the sequence obtained from D by deleting all terms equal to 0 is graphical. Suppose now that $D : d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ is graphical and contains n_1 terms equal to 1. Then D is edge-optimal if and only if $d_2 = 1$ and $d_1 = n_1$ or if $d_1 - n_1 \geq d_2$ and $D' : d_1 - n_1, d_2, \ldots, d_{n-n_1+1}$ is edge-optimal. So it suffices to characterize edge-optimal graphical sequences D where $D : d_1 \geq d_2 \geq \ldots \geq d_n \geq 2$. We conjecture that all graphical sequences with smallest term at least 2 are edge-optimal. Edmonds showed in [4], that if $D : d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ is graphical, then there is a graph G having D as its degree sequence and edge-connectivity at least d_n if and only if $\sum_{i=1}^{n} d_i \geq 2(n-1)$.

Examples show that not all such graphical sequences are optimal graphical sequences. For example, $D : 5, 5, 3, 3, 3, 3$ is graphical and $K_2 + (2K_2)$ is the unique graph having D as its degree sequence. Since the two vertices of degree 5 form a vertex cut, not every pair of vertices of degree 3 are connected by three internally disjoint paths.

References

[1] L.W. Beineke, O.R. Oellermann, and R.E. Pippert, The average connectivity of a graph. *Discr. Math.* **252** (2002) 31-45.

[2] G. Chartrand, O.R. Oellermann, *Applied and Algorithmic Graph Theory.* Mc-Graw Hill, New York (1993).

[3] P. Dankelmann and O.R. Oellermann, Bounds on the average connectivity of a graph. *Discr. Appl. Math.* **129** (2003) 305-318.

[4] J. Edmonds, Existence of k-edge connected ordinary graphs with prescribed degrees. *J. Res. Nat. Bur. Std.-B, Mathematics and Mathematical Physics* **68B** (1964).

[5] G. Fricke, O. R. Oellermann and H.C. Swart, The edge-connectivity, average edge-connectivity and degree conditions. Technical Report, 1999.

[6] S.L. Hakimi, On the realizability of a set of integers as degrees of the vertices of a graph. *SIAM J. Appl. Math.* 10 (1962), 496-506.

[7] M.A. Henning and O.R. Oellermann, The average connectivity of a digraph. *Discr. Appl. Math.* (To appear.)