Bipartite Dominating Sets in Hypercubes

Mark Ramras*

Abstract

If G is a bipartite graph with bipartition (X, Y), a subset S of X is called a one-sided dominating set if every vertex $y \in Y$ is adjacent to some $x \in S$. If S is minimal as a one-sided dominating set (i.e. if it has no proper subset which is also a one-sided dominating set,) it is called a **bipartite dominating set** (see [4],[5], and [6]). We study bipartite dominating sets in hypercubes.

Definition 1 *Let G be a bipartite graph with bipartition (X, Y). A subset S of X is called a* **one-sided dominating set** *if every vertex $y \in Y$ is adjacent to some $x \in S$, i.e. if $N(S) = Y$. S is a* **minimal** *one-sided dominating set if no proper subset of S is a one-sided dominating set. It is a* minimum *one-sided dominating set if no one-sided dominating set contained in X has smaller cardinality. In that case, S is called a* **bipartite dominating set**.

Bipartite dominating sets have been studied by Haynes, Hedetniemi, and Slater [4] and by Hedetniemi and Laskar [5], [6].

Remark 1 *A subset S of X is a one-sided dominating set $\Leftrightarrow$ the only maximal independent set containing S is X.*

Notation. For any graph G, $\gamma(G)$ denotes the minimum size of a dominating set in G.

We denote by Q_n the n-dimensional hypercube. Its bipartition (X, Y) is given by $X = \{x \in Q_n \mid wt(x) \text{ is even}\}$, $Y = \{y \in Q_n \mid wt(y) \text{ is odd}\}$ where $wt(z)$, the weight of z, is the number of 1's in the n-tuple z. Alternatively, if we think of the vertices of Q_n as the subsets of $\{1, 2, \ldots, n\}$, X consists of the subsets of even cardinality, and Y consists of the subsets of odd cardinality. We will also at times consider Q_n to be a group under component-wise addition of n-tuples (or, if the vertices are thought of as subsets of $\{1, 2, \ldots, n\}$, then under the operation of symmetric difference).

The next proposition basically restates the Hamming Bound (see [9], p. 413), for Q_n for single-error-correcting codes.

*Department of Mathematics, Northeastern University, Boston, MA 02115 (ramras@neu.edu), Tel: 617-373-5651, Fax: 617-373-5658.

Proposition 1 (i) *A one-sided dominating set S on Q_n must have at least $\frac{2^{n-1}}{n}$ elements.*
(ii) *A dominating set D on Q_n must have at least $\frac{2^n}{n+1}$ elements, i.e. $\gamma(Q_n) \leq \frac{2^n}{n+1}$.*

Proof. (i) Each even node is adjacent to n odd nodes. Thus to dominate all 2^{n-1} odd nodes, at least $\frac{2^{n-1}}{n}$ even nodes are needed.

(ii) Each closed neighborhood of a node in Q_n has cardinality $n+1$. Thus for γ closed neighborhoods to cover the 2^n nodes of Q_n, we must have $\gamma(n+1) \geq 2^n$, from which the desired result follows. □

Remark 2 *Q_4 has a bipartite dominating set of size 2. Furthermore,* any *2 even nodes of Q_3 form a bipartite dominating set. Q_5 has a bipartite dominating set of size 4.*

Proof. For Q_4: let $S = \{\emptyset, \{1,2,3,4\}\}$. Each node of weight one is adjacent to $\emptyset$ and each node of weight 3 is adjacent to $\{1,2,3,4\}$.

For Q_3: Since the distance between any 2 even nodes in Q_3 is 2, we may assume with no loss in generality that $S = \{\emptyset, \{1,2\}\}$. Then each node of weight 1 is adjacent to $\emptyset$ and $\{1,2,3\}$ is adjacent to $\{1,2\}$. So S is a one-sided dominating set, and by Proposition 1, no single node can form a one-sided dominating set in Q_3. Thus S is a bipartite dominating set.

By Proposition 1, a one-sided dominating set for Q_5 must have at least $\lceil 16/5 \rceil = 4$ elements. Now $\{\emptyset, \{1,2,3,4\}, \{1,5\}, \{2,3,4,5\}\}$ is a one-sided dominating set, and therefore is a bipartite dominating set. □

Proposition 2 *Let $n \geq 2$. In Q_n any set S of $2^{n-1} - n + 1$ even nodes is a one-sided dominating set.*

Proof. Let y be an odd node. $|N(y)| = n$. If $S \cap N(y) = \emptyset$, then $|S \cup N(y)| = |S| + |N(y)| = (2^{n-1} - n + 1) + n = 2^{n-1} + 1$. But Q_n has only 2^{n-1} even nodes. This contradiction means that $S \cap N(y) \neq \emptyset$ and therefore $y \in N(S)$. Thus $N(S) = \{\text{odd nodes}\}$ and so S is a one-sided dominating set. □

Next we show that $2^{n-1} - n + 1$ can not be replaced by a smaller number.

Proposition 3 *Q_n has sets of even nodes of size $2^{n-1} - n$ which are not one-sided dominating sets.*

Proof. Let y be any odd node of Q_n, and let $S = X \setminus N(y)$. Since Q_n is n-regular, $|S| = |X| - n = 2^{n-1} - n$. Since $S \cap N(y) = \emptyset$, $y \notin N(S)$ and so S is not a one-sided dominating set. □

Next we give a construction which generalizes the construction of Q_{n+1} from Q_n. Let $G = (X, Y)$ be any bipartite graph. Let $\overline{G} = G \square K_2$. So $\overline{G}$ consists of two copies of G, say G and G' with a an isomorphism ϕ from G to G'. Note that $\{\langle v, \phi(v)\rangle \mid v \in V(G)\}$ is a perfect matching joining the two copies of X and also joining the two copies of Y. Define $\phi : G \longrightarrow \overline{G}$ by $\phi(x) = x$ for $x \in X$, and for $y \in Y, \phi(y) = y'$, where y' is the vertex in the second copy of Y which is matched with y. Note that in G', the two partite sets are $Y' = \phi(Y)$ and $X' = \phi(X)$. Thus $\overline{G}$ is bipartite, with partite sets $X \cup Y'$ and $X' \cup Y$.

Proposition 4 *For a bipartite graph G and any subset S of $V(G)$, S dominates G if and only if $S_X \cup \phi(S_Y)$ dominates $Y \cup X'$.*

Proof. ($\Longrightarrow$) Suppose that S dominates G. Let $v \in Y \cup X'$. If $v \in Y$ then since S dominates Y, v is adjacent to some $s \in S$. Now let $v \in X'$. Then $v = \phi(x)$ for some $x \in X$. If $x \in S_X$ then v is adjacent to an element of S_X. If $x \notin S_X$, then since S dominates G, x is adjacent to some $y \in S_Y$. Hence $\phi(x)$ is adjacent to $\phi(y) \in \phi(S_Y)$. Thus $S_X \cup \phi(S_Y)$ dominates $Y \cup X'$.

($\Longleftarrow$) Assume that $S_X \cup \phi(S_Y)$ dominates $Y \cup X'$. We must show that S dominates G. First let $y \in Y$. Since y can not be adjacent to anything in X', and by assumption it is adjacent to some vertex of $S \cup \phi(S)$, it must be adjacent to some $s \in S_X$. Next, let $x \in X$. If $x \in S_X$ there is nothing to prove. If $x \in X \setminus S_X$ then $\phi(x) \in Y \setminus \phi(S_X)$. Hence, by the hypothesis, $\phi(x)$ is adjacent to some $y' = \phi(y) \in \phi(S_Y)$, where $y \in S_Y$. Therefore, x must be adjacent to y. Hence $S_X \cup \phi(S_Y)$ dominates $Y \cup X'$. □

Proposition 5 *For all $n \geq 3$, Q_n has a bipartite dominating set of size 2^{n-2}.*

Proof. Choose S to be the set of all even weight vertices in $Q_n^{(0)} \simeq Q_{n-1}$. Since S is a maximal independent set in $Q_n^{(0)}$ it dominates all odd nodes of $Q_n^{(0)}$. Each odd node of $Q_n^{(1)}$ is adjacent to a unique member of S. So S dominates the set of all odd vertices of Q_n. Clearly $|S| = 2^{n-2}$. S is minimal, for if any $x \in S$ is deleted, its neighbor in $Q_n^{(1)}$ will not be dominated. □

Next we show that for $n \geq 4$ there are one-sided dominating sets half the size of those of the preceding proposition. It follows that there are bipartite dominating sets at least that small.

Proposition 6 *For $n \geq 4$, Q_n has a one-sided dominating set of cardinality 2^{n-3}.*

Proof. In Q_4, $S = \{\emptyset, \{1,2,3,4\}\}$ is a one-sided dominating set by Remark 2. Now for $n > 4$ we proceed by induction. Let S_{n-1} be an one-sided dominating set of size 2^{n-4} in $Q_n^{(0)} \simeq Q_{n-1}$. Let $T = S_{n-1}\Delta\{1,n\}$. Since $\{1,n\}$ has weight 2 and every vertex in S_{n-1} has even weight, so does every vertex of T. Since $Q_n^{(0)}\Delta\{1,n\} \simeq Q_n^{(1)}$, T dominates all odd weight vertices of $Q_n^{(1)}$. Thus $S_n = S_{n-1} \cup T$ dominates all odd weight vertices of $Q_n^{(0)} \cup Q_n^{(1)} = Q_n$, i.e. it is a one-sided dominating set. Clearly, $|S_n| = 2 \cdot |S_{n-1}| = 2 \cdot 2^{n-4} = 2^{n-3}$. □

These results can be improved.

Lemma 1 *For* $n = 2^k - 1$, $\gamma(Q_n) = 2^{n-k}$.

Proof. The hypercubes of dimension $2^k - 1$ are precisely the ones that have perfect single-error-correcting codes (Hamming codes). The cardinality of such a code is 2^{n-k}, and it certainly is a dominating set. It therefore achieves the sphere-packing bound [8], Theorem 10, p. 23, of $\frac{2^n}{n+1}$, and so must be a minimum dominating set. □

The following general result about the domination number of a product graph is given as Proposition 8.14 in [7].

Lemma 2 *For any graphs* G *and* H, $\gamma(G\Box H) \leq \min\{\gamma(G)|V(H)|, \gamma(H)|V(G)|\}$.

Corollary 1 *Let* G *be any bipartite graph, with bipartition* (X,Y). *Then* $G\Box I$ *is bipartite and* $\gamma(G\Box I) \leq 2\gamma(G)$. *In particular, for any* n, $\gamma(Q_{n+1}) \leq 2\gamma(Q_n)$.

Corollary 2 *For* $m \geq n = 2^k - 1$, $\gamma(Q_m) \leq 2^{m-n}\gamma(Q_n)$.

Example 1 *The following set of* 10 *vertices of* Q_6 *is a one-sided dominating set. (It is not a minimum bipartite dominating set since, by the preceding proposition,* Q_6 *has a one-sided dominating set of size* $2^3 = 8$.)
$S =$

$$\{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{3,4,5,6\}, \{2,4,5,6\}, \{2,3,5,6\}, \{2,3,4,6\}, [6]\}$$

where $[n]$ *denotes the set* $\{1, 2, \ldots, n\}$.

Proof. To see that S is a one-sided dominating set, note that every vertex of weight 1 is adjacent to $\emptyset$, and every vertex of weight 5 is adjacent to $[6]$. For $2 \leq i < j \leq 6$, $\{1,i,j\}$ is adjacent to $\{1,i\} \in S$. Next, there are 4 vertices in S of weight 4, all of which contain the element 6. If 6 is deleted from these vertices, we get the 4 subsets of $\{2,3,4,5\}$ of weight 3. Thus any vertex $\{i,j,k\}$ of weight 3 which is contained in $\{2,3,4,5\}$ is adjacent to $\{i,j,k,6\} \in S$. Finally, the only remaining vertices of weight 3 are those of the form $\{i,j,6\}$, with $2 \leq i < j < 6$. Let k be one of the 2 members of $\{2,3,4,5\} \setminus \{i,j\}$. Then $\{i,j,6\}$ is adjacent to $\{i,j,k,6\} \in S$.

For minimality it suffices to show that for each $s \in S$ there is a vertex of odd weight which is adjacent to s and no other element of S. Now S consists of five elements and their antipodes (set complements). Thus we need only check the condition for $\emptyset$ and the four 2-element subsets containing 1. For $\emptyset$, that element is $\{6\}$. For $\{1,j\}$, with $2 \leq j \leq 5$, that element is $\{1,j,6\}$. □

Lemma 3 *If A and B are adjacent in Q_n, then so are $\bar{A}$ and $\bar{B}$, where $\bar{C}$ denotes the complement of C in $\{1,2,\ldots,n\}$.*

Proof. This follows from the fact that taking complements is an automorphism of Q_n. (In fact, it is the antipodal map.) □

Proposition 7 *Let $n \equiv 0 \pmod 4$, say $n = 4k$. Let $\mathcal{A} = \{1,2,\ldots,2k\}$. Let*

$$\mathcal{S}_j = \{C \in Q_n \mid |C| = j, |C \cap \mathcal{A}| \equiv 0 \pmod 2\}.$$

$$\mathcal{S} = \bigcup_{\substack{j \equiv 0 \pmod 4 \\ 0 \leq j \leq n}} \mathcal{S}_j$$

Then $\mathcal{S}$ is a one-sided dominating set in Q_n.

Proof. Let $D \in Y = Q_n \setminus X$, and let $\mathcal{B} = [n] \setminus \mathcal{A}$. Since $|D|$ is odd, exactly one of $|D \cap \mathcal{A}|$ and $|D \cap \mathcal{B}|$ is odd. Without loss of generality we may assume that $|D \cap \mathcal{A}|$ is odd. First suppose that $|D| \equiv 1 \pmod 4$. Let $i \in D \cap \mathcal{A}$. Then $D - \{i\} \in \mathcal{S}$, and D is adjacent to $D - \{i\}$. Next, suppose that $|D| \equiv 3 \pmod 4$. Then $|\bar{D}| \equiv 1 \pmod 4$. Thus by the first case, there is a $C \in \mathcal{S}$ such that $\bar{D}$ and C are adjacent. Hence, by Lemma 1, $\bar{C}$ and D are adjacent. It is easy to see that $\mathcal{S}$ is closed under complementation, and so $\bar{C} \in \mathcal{S}$. Thus $\mathcal{S}$ is a one-sided dominating set. □

It should be noted that the one-sided dominating set produced by Proposition 4 is *not* minimal. For example, when $n = 8$ it produces a set of cardinality 40. The following subset of that one-sided dominating set, of cardinality 16, is a minimal one-sided dominating set, i.e. a bipartite dominating set.

Example 2 *A bipartite dominating set on Q_8.*

$$\begin{array}{cccc} \{1,2,3,4\} & \{1,2,5,6\} & \{1,2,7,8\} & \{1,4,6,8\} \\ \{5,6,7,8\} & \{3,4,7,8\} & \{3,4,5,6\} & \{2,3,5,7\} \\ \{1,3,5,8\} & \{1,3,6,7\} & \{1,4,5,7\} & \emptyset \\ \{2,4,6,7\} & \{2,4,5,8\} & \{2,3,6,8\} & \{1,2,3,4,5,6,7,8\} \end{array}$$

The next two propositions establish a 1–1 correspondence between dominating sets of Q_n and one-sided dominating sets in Q_{n+1}, by means of a parity check bit. Define $\mathcal{F} : Q_n \longrightarrow Q_{n+1}$ by

$$\mathcal{F}(x_1, x_2, \ldots, x_n) = \begin{cases} x0 & \text{if } wt(x) \text{ is even} \\ x1 & \text{otherwise} \end{cases}$$

This mapping $\mathcal{F}$ is known in the literature as *extension* (see Pless [8], p. 35.)

Proposition 8 *A subset S of Q_n is a dominating set for $Q_n \Leftrightarrow \mathcal{F}(S)$ is a one-sided dominating set in Q_{n+1}.*

Proof. ($\Rightarrow$) Let $y = y_1, y_2, \ldots, y_{n+1}$ be an odd vertex of Q_{n+1}. If $y_{n+1} = 0$ then $y_1, y_2, \ldots, y_n$ is an odd vertex of Q_n. Suppose first that $y_1, y_2, \ldots, y_n \in S$. Then $y_1, y_2, \ldots, y_n 1 \in \mathcal{F}(S)$. Since this vertex is adjacent to y, we have $y \in N(\mathcal{F}(S))$. On the other hand, if $y_1, y_2, \ldots, y_n \notin S$, then since S is a dominating set for Q_n $y_1, y_2, \ldots, y_n$ is adjacent to some $s_1, s_2, \ldots, s_n \in S$. Hence $s_1, s_2, \ldots, s_n$ is even, and so $\mathcal{F}(s_1, s_2, \ldots, s_n) = s_1, s_2, \ldots, s_n, 0$. Thus $y = y_1, y_2, \ldots, y_n, 0$ is adjacent to an element of $\mathcal{F}(S)$.

Now suppose that $y_{n+1} = 1$. Then $y_1, y_2, \ldots, y_n$ is even. If $y_1, y_2, \ldots, y_n \in S$, then $y_1, y_2, \ldots, y_n, 0 \in \mathcal{F}(S)$ and is adjacent to $y_1, y_2, \ldots, y_n, 1 = y$. If $y_1, y_2, \ldots, y_n \notin S$, then since S dominates Q_n, there is some $s \in S$ which is adjacent to $y_1, y_2, \ldots, y_n$. In particular, s is therefore odd. Hence $\mathcal{F}(s) = s1$. Since s is adjacent to $y_1, y_2, \ldots, y_n$, it follows that $s1$ is adjacent to $y_1, y_2, \ldots, y_n, 1$. So $y_1, y_2, \ldots, y_n \in \mathcal{F}(S)$.

($\Leftarrow$) Assume that $\mathcal{F}(S)$ is a one-sided dominating set in Q_{n+1}. Let $x \in Q_n$, $x \notin S$.

Case 1: x is even. Then $x1 \in Q_{n+1}$ is odd. Therefore $x1$ is adjacent to $\mathcal{F}(s)$ for some $s \in S$. If s is even, then $\mathcal{F}(s) = s0$. Since $x1$ is adjacent to $s0$, we must have $x = s$. But then $x \in S$, contrary to our assumption. So s must be odd and therefore $\mathcal{F}(s) = s1$. Hence $s1$ is adjacent to $x1$, which implies that s is adjacent to x.

Case 2: x is odd. Then $\mathcal{F}(x) = x1 \notin \mathcal{F}(S)$. Now $x0$ must also be odd, and so $x0$ is adjacent to $\mathcal{F}(s)$, for some $s \in S$. If s is odd, then $\mathcal{F}(s) = s1$, and $x0$ is adjacent to $s1$. It follows that $x = s$, contradicting the fact that $s \notin S$. So s must be even. Therefore $\mathcal{F}(s) = s0$, and so $x0$ is adjacent to $s0$. Hence x is adjacent to $s \in S$.

Thus S is a dominating set for Q_n. □

Now we define a map $\mathcal{G} : Q_{n+1} \longrightarrow Q_n$ by

$$\mathcal{G}(x_1, x_2, \ldots, x_{n+1}) = x_1, x_2, \ldots, x_n$$

which is a left inverse to $\mathcal{F}$, but more importantly, a right inverse to $\mathcal{F}$ when restricted to the even nodes of Q_{n+1}. This mapping is known in the literature as *puncturing* (see Pless [8], p. 35).

Proposition 9 (i) *$\mathcal{F} \circ \mathcal{G}$ is the identity on the set of even nodes of Q_{n+1}.* (ii) *Let A be any subset of the even weight nodes of Q_{n+1}. Then A is a one-sided dominating set $\Leftrightarrow \mathcal{G}(A)$ is a dominating set in Q_n.*

Proof. (i) Let $x = x_1, x_2, \ldots, x_n, x_{n+1}$ be any even node of Q_{n+1}. $\mathcal{G}(x) = x_1, x_2, \ldots, x_n$, whose weight is congruent (mod 2) to x_{n+1}. Hence $\mathcal{F}(\mathcal{G}(x)) = \mathcal{F}(x_1, x_2, \ldots, x_n) = x_1, x_2, \ldots, x_n, x_{n+1} = x$.

(ii) It suffices to show that if A is any subset of the even nodes of Q_{n+1}, then $A = \mathcal{F}(S)$ for some subset S of the nodes of Q_n. For then $\mathcal{G}(A) = \mathcal{G}(\mathcal{F}(S) = S$, and so the result follows from the preceding proposition. But if $a = a_1, a_2, \ldots, a_{n+1}$ is any even node in Q_{n+1}, $a = \mathcal{F}(a_1, a_2, \ldots, a_n)$. □

Corollary 3 *There is a 1–1 correspondence between dominating sets in Q_n and one-sided dominating sets in Q_{n+1}. This correspondence is inclusion-preserving, and therefore preseves minimality and maximality.*

The one-sided dominating set of Example 2 belongs to a special class of bipartite dominating sets in Q_n which we introduce now.

Definition 2 *A set C of even weight vertices of Q_n is a* **half-perfect single error correcting code** *if every odd weight vertex is adjacent to exactly one member of C.*

Lemma 4 *A set C of even weight vertices is a half-perfect single error correcting code in $Q_n \Leftrightarrow \forall x, y \in C, d(x,y) \geq 4$* and $n \cdot \mid C \mid = 2^{n-1}$.

Proof. ($\Rightarrow$) If $x, y \in C$ and $d(x,y) = 2$, there exists a $z \in Q_n$ such that z is adjacent to both x and y. Since the weight of z is necessarily odd, this contradicts the assumption that C is half-perfect. So $d(x,y) \geq 4$. Since each odd vertex is adjacent to a unique $x \in C$, $\bigcup_{x \in C} N(x) = \{z \in Q_n \mid wt(z) \equiv 1 \pmod 2\}$ (where $N(x)$ denotes the set of vertices adjacent to x, and $wt(z)$ denotes the weight of z) and the neighbor sets $N(x)$, for $x \in C$, are pairwise disjoint. Thus

$$\sum_{x \in C} \mid N(x) \mid = 2^{n-1}$$

The desired conclusion now follows from the n-regularity of Q_n.

($\Leftarrow$) If $x, y \in \mathcal{C}$, then since $d(x,y) > 2, N(x) \cap N(y) = \emptyset$. Thus no odd weight vertex can be adjacent to more than one member of $\mathcal{C}$. Since $n \cdot \mid \mathcal{C} \mid = 2^{n-1}$ and the family $\{N(x) \mid x \in \mathcal{C}\}$ consists of pairwise disjoint sets, we have

$$\mid \bigcup_{x \in \mathcal{C}} N(x) \mid = 2^{n-1}$$

and so $\bigcup_{x \in \mathcal{C}} N(x) = \{z \in Q_n \mid wt(z) \equiv 1 \pmod 2\}$. Thus $\mathcal{C}$ is half-perfect and single error correcting. □

In particular, if Q_n has a half-perfect single error correcting code, then n must be a power of 2. The next proposition says that by means of an additional parity-check bit, we can convert a perfect single error correcting code on Q_{n-1} to a half-perfect single error correcting code on Q_n.

Proposition 10 *Let $\mathcal{F}$ be the extension mapping from codes on Q_{n-1} to codes on Q_n, and let $\mathcal{G}$ be the puncturing mapping which is its right inverse. If $\mathcal{S}$ is a perfect single error correcting code on Q_{n-1}, then $\mathcal{C} = \mathcal{F}(\mathcal{S})$ is a half-perfect single error correcting code on Q_n.*

Conversely, if $\mathcal{C}$ is a half-perfect single error correcting code on Q_n, then $\mathcal{G}(\mathcal{C})$ is a perfect single error correcting code on Q_{n-1}. Moreover, $\mathcal{C} = \mathcal{F}(\mathcal{G}(\mathcal{C}))$.

Proof. Let $\mathcal{S}$ be a perfect single error correcting code on Q_{n-1}. Then for all $x, y \in \mathcal{S}, d(x,y) \geq 3$. If $wt(x)$ and $wt(y)$ have the same parity then $d(x,y)$ must be even. Therefore $d(x,y) \geq 4$ and $d(\mathcal{F}(x), \mathcal{F}(y)) = d(x,y)$. Now suppose that $wt(x)$ and $wt(y)$ have opposite parity, say $wt(x)$ is even and $wt(y)$ is odd. Then $\mathcal{F}(x) = x0$, $\mathcal{F}(y) = y1$. Hence $d(\mathcal{F}(x), \mathcal{F}(y)) = 1 + d(x,y) \geq 4$. Finally, since $\mathcal{S}$ is perfect, and each closed neighborhood $\{x\} \cup N(x)$ has cardinality n, $n \cdot \mid \mathcal{S} \mid = 2^{n-1} >$, and so $n \cdot \mid \mathcal{C} \mid = 2^{n-1}$. The desired result now follows from Lemma 5.

Now suppose that $\mathcal{C}$ is a half-perfect single-error correcting code on Q_n. Then for all $x, y \in \mathcal{C}$, $d(x,y) \geq 4$. We must show that $d(\mathcal{G}(x), \mathcal{G}(y)) \geq 3$. Now $d(\mathcal{G}(x), \mathcal{G}(y)) = wt(\mathcal{G}(x) + \mathcal{G}(y))$. Since $\mathcal{G}$ is a group homomorphism, the latter is $wt(\mathcal{G}(x+y))$. Now for any $z \in Q_n$, $wt(z) \leq 1 + wt(\mathcal{G}(z))$. So if $wt(\mathcal{G}(x+y)) \leq 2$, then $wt(x+y) \leq 3$, contrary to our assumption. Thus $\mathcal{G}(\mathcal{C})$ is a single-error correcting code on Q_{n-1}.

Next, we must show that the single-error correcting code $\mathcal{G}(\mathcal{C})$ on Q_{n-1} is perfect. Thus we must show that for $x, y \in \mathcal{G}(\mathcal{C})$, $n|\mathcal{G}(\mathcal{C})| = 2^{n-1}$. By Lemma 5, $n|\mathcal{C}| = 2^{n-1}$. Hence it suffices to show that $|\mathcal{G}(\mathcal{C})| = |\mathcal{C}|$. For this, it suffices to show that on $\mathcal{C}$, the map $\mathcal{G}$ is one-to-one. But if $\mathcal{G}(x) = \mathcal{G}(y)$, then $x_1 x_2 \cdots x_{n-1} = y_1 y_2 \cdots y_{n-1}$. Call this $(n-1)$-tuple $z_1 z_2 \cdots z_{n-1}$. Then $x = z_1 z_2 \cdots z_{n-1}0$ and $y = z_1 z_2 \cdots z_{n-1}1$, or vice versa.

Either way, this implies that $d(x, y) = 2$, contradicting the fact that for $x, y \in \mathcal{C}, d(x, y) \geq 4$. Finally, it is easy to see that $\mathcal{F} \circ \mathcal{G}$ is the identity map on the even weight vertices of Q_n, and so $\mathcal{C} = \mathcal{F}(\mathcal{G}(\mathcal{C}))$. □

Definition 3 *For a graph G, subsets S and S′ of V(G) will be said to be isomorphic if there is an automorphism of G which carries S to S′.*

Lemma 5 *If $\mathcal{T}$ is an independent subset in Q_n, then $Q_n - \mathcal{T}$ contains an independent set isomorphic to $\mathcal{T}$.*

Proof. Let $u \in Q_n$ be any vertex of weight one. We claim that the set $u + \mathcal{T}$ has the desired properties. It is clearly isomorphic to $\mathcal{T}$ since the map $x \mapsto u + x$ is an automorphism of Q_n. We must show that it is disjoint from $\mathcal{T}$. If not, then for some $t_1, t_2 \in \mathcal{T}, t_2 = u + t_1$. But then since $wt(u) = 1$, t_2 is adjacent to t_1, contradicting the independence of $\mathcal{T}$. □

For a perfect single error correcting group code, and for the half-perfect single error correcting code obtained from it, we can say a good deal more.

Proposition 11 *Let $\mathcal{C}$ be a perfect single error correcting group code on Q_n. Then $V(Q_n)$ has a partition into isomorphic copies of $\mathcal{C}$.*

Proof. $\mathcal{C}$ is a subgroup of Q_n. Thus the cosets of $\mathcal{C}$ in Q_n partition $\mathcal{C}$. Each coset is of the form $x+\mathcal{C}$ and is thus the image of $\mathcal{C}$ under the automorphism $z \mapsto x + z$. □

Proposition 12 *Let $\mathcal{C}$ be a perfect single error correcting group code on Q_n and let $\mathcal{S}$ be the half-perfect single error correcting code on Q_{n+1} derived from $\mathcal{C}$. Then $V(Q_{n+1})$ has a partition into isomorphic copies of $\mathcal{S}$.*

Proof. The map extension mapping $\mathcal{F}$ is a group homomorphism. So $\mathcal{S} = \mathcal{F}(\mathcal{C})$ is a subgroup of Q_{n+1}, and just as in the lemma, the cosets of $\mathcal{S}$ in Q_{n+1} yield the desired partition. □

Proposition 13 *Suppose $\mathcal{S}$ is a subset of the even weight vertices of Q_{n-1}. Define $\mathcal{S}^* = \mathcal{S}_0 \cup \mathcal{S}_1$, where*

$$\mathcal{S}_0 = \{x0 \mid x \in \mathcal{S}\} \text{ and } \mathcal{S}_1 = \{x1 + \vec{e_1} \mid x \in \mathcal{S}\}.$$

Then $\mathcal{S}$ is a minimal one-sided dominating set on $Q_{n-1} \Leftrightarrow \mathcal{S}^*$ is a minimal one-sided dominating set on Q_n.

Proof. ($\Rightarrow$) It is easy to check that φ preserves adjacency, and also the parity of the weight of a vertex. Note that by construction, every vertex in $\mathcal{S}^*$ has even weight. First we shall show that $\mathcal{S}^*$ is an one-sided dominating set in Q_n. So let $u \in Q_n$ be any vertex of odd weight. Then $\varphi(u)$ has odd weight, and so it is adjacent to some $x_1, x_2, \ldots, x_{n-1} \in \mathcal{S}$. If $u_n = 0$ then $u = u_1, u_2, \ldots, u_{n-1}, 0$ is adjacent to $x = x_1, x_2, \ldots, x_{n-1}, 0 \in \mathcal{S}_0 \subset \mathcal{S}^*$. On the other hand, if $u_n = 1$, then $\bar{u_1}, u_2, \ldots, u_{n-1} = \varphi(u)$ is adjacent to x and so $\bar{u_1}, u_2, \ldots, u_{n-1}, 1$ is adjacent to $x1$. Hence $u_1, u_2, \ldots, u_{n-1}, 1$ is adjacent to $\bar{x_1}, x_2, \ldots, x_{n-1}, 1 \in \mathcal{S}_1 \subset \mathcal{S}^*$. So in either case, u is adjacent to some $z \in \mathcal{S}^*$, and so $\mathcal{S}^*$ is an one-sided dominating set.

To prove the minimality of $\mathcal{S}^*$, suppose that $\mathcal{T} \subset \mathcal{S}^*$, where $\mathcal{T}$ is an one-sided dominating set. Then $\varphi(\mathcal{T}) \subset \varphi(\mathcal{S}^*) = \mathcal{S}$ and $\varphi(\mathcal{T})$ is an one-sided dominating set. So, by the assumed minimality of $\mathcal{S}$, $\varphi(\mathcal{T}) = \mathcal{S}$. Now suppose $y \in \mathcal{S}^* - \mathcal{T}$. If $y_n = 0$, then $\varphi(y) = y_1, y_2, \ldots, y_{n-1}$, so $y_1, y_2, \ldots, y_{n-1} \in \mathcal{S}$, and therefore $y = y_1, y_2, \ldots, y_{n-1}, 0 \in \mathcal{S}^*$. On the other hand, if $y_n = 1$, then $\varphi(y) = \bar{y_1}, y_2, \ldots, y_{n-1} \in \mathcal{S}$. Therefore $y_1, y_2, \ldots, y_{n-1}, 1 \in \mathcal{S}^*$, *i.e.* $y \in \mathcal{S}^*$, which is a contradiction. Thus $\mathcal{T} = \mathcal{S}^*$, and so $\mathcal{S}^*$ is minimal.

($\Leftarrow$) Suppose $\mathcal{S}^*$ is a minimal one-sided dominating set in Q_n. Let $x \in Q_{n-1}$ be any odd weight vertex. Then $x0 \in Q_n$ has odd weight and thus is adjacent to some $s^* \in \mathcal{S}^*$. If $s^* = y0$ then $y \in \mathcal{S}$ and x is adjacent to y. If $s^* = y1$ then $y + \vec{e_1} \in \mathcal{S}$ and $x = y$. But x is adjacent to $x + \vec{e_1} = y + \vec{e_1}$. Thus $\mathcal{S}$ is a one-sided dominating set.

Now let $\mathcal{T} \subset \mathcal{S}$ be a bipartite dominating set. Then by the first half of this proof, $\mathcal{T}^*$ is a bipartite dominating set in Q_n. Since $\mathcal{T}^* \subset \mathcal{S}^*$, and by hypothesis $\mathcal{S}^*$ is minimal, $\mathcal{T}^* = \mathcal{S}^*$. Now let $z \in \mathcal{S}$. Then $z0 \in \mathcal{S}^* = \mathcal{T}^*$. Hence $z \in \mathcal{T}$. Thus $\mathcal{S} = \mathcal{T}$, and so $\mathcal{S}$ is minimal. □

Definition 4 *A subset $\mathcal{S}$ of a graph G is* **irredundant** *if for all proper subsets $\mathcal{S}'$ of $\mathcal{S}$, $N(\mathcal{S}') \neq N(\mathcal{S})$.*

Lemma 6 *Let G be a bipartite graph with bipartition (X, Y), and assume that G has a perfect matching. Let S be a subset of X. Then S is a minimal one-sided dominating set $\Leftrightarrow$ S is irredundant and maximal among the irredundant subsets of X.*

Proof. ($\Rightarrow$) Assume that S is a bipartite dominating set. Suppose that $T \subset S$ and $N(T) = N(S)$. Then T is also a one-sided dominating set and so, by the minimality of S, $T = S$. So S is irredundant. Now suppose that $S \subset W \subset X$. Then $Y = N(S) \subset N(W) \subset N(X) = Y$, and so $N(S) = N(W)$. So if $S \neq W$ then W is not irredundant. Hence S is maximal among the irredundant subsets of X.

($\Leftarrow$) Assume that S is maximal among the irredundant subsets of X. First we show that S is a one-sided dominating set, i.e. that $N(S) = Y$. Suppose not. Then $V = Y - N(S) \neq \emptyset$. By hypothesis, G has some perfect matching, say $\mathcal{M}$. Let U be the subset of X matched with V under $\mathcal{M}$. Then $N(U) = V$ and U is minimal among subsets with this property. Let $W = S \cup U$. Then $S \subset W \subset X$ and $S \neq W$. Note that $N(W) = N(S) \cup N(U) = N(S) \cup V = Y$. We claim that W is irredundant. For suppose $W' \subset W$ and $N(W') = N(W) = Y$. Since $W' = (W' \cap S) \cup (W' \cap U)$, we have

$$Y = N(W') = N(W' \cap S) \cup N(W' \cap U). \tag{1}$$

Now $N(W' \cap S) \subset N(S)$, and $V \cap N(S) = \emptyset$, so $V \subset N(W' \cap U)$. But $N(W' \cap U) \subset N(W) = V$, and so we have $N(W' \cap U) = V$. By the minimality of U, we get $W' \cap U = U$, and so $U \subset W'$. Next, since $N(W' \cap S) \subset N(S)$ and $N(S) \cap V = \emptyset$, it follows from Equation 1 that $N(W' \cap S) = N(S)$. Since S is irredundant it follows that $W' \cap S = S$. Hence $S \subset W'$. Since we also have $U \subset W'$, it follows that $W = S \cup U \subset W'$. Hence $W' = W$. Thus W is irredundant. But by hypothesis, S is maximal among irredundant subsets of X, so we have a contradiction. Therefore $N(S) = Y$, i.e. S is a one-sided dominating set.

Finally, if $S' \subset S$ is a one-sided dominating set, then $N(S') = N(S) = Y$. But since S is irredundant, we must have $S' = S$. Thus S is a bipartite dominating set. □

Lemma 7 *Let G be an irredundant subset of the even weight vertices of Q_n and suppose that G is a subgroup of Q_n. If G is a one-sided dominating set, then G is a bipartite dominating set. Otherwise, there exists a subgroup H consisting of even weight vertices, $H \supset G$, which is also irredundant.*

Proof. First suppose that G is an one-sided dominating set. If $S \subset G$ and S is also a one-sided dominating set, then $N(S) = \{\text{odd vertices of } Q_n\} = N(G)$. Since G is irredundant, it follows that $S = G$. Thus G is a bipartite dominating set.

Now suppose that G is *not* a one-sided dominating set and that no H of the desired form exists. Let x be a vertex of odd weight such that $x \notin N(G)$. For $1 \leq i \leq n$ let $y_i = x + \vec{e_i}$. So y_i is adjacent to x. Then $wt(y_i)$ is even and $y_i \notin G$. Let $H_i = G + y_i = \{g + y_i \mid g \in G\}$. Then H_i is a subgroup of the group of vertices of even weight, and $H_i \supset G$. So by our assumption, H_i is not irredundant and therefore for some $g \in G$, $N(H_i - (g + y_i)) = N(H_i)$. Thus $N(g + y_i) \subset N(H_i - (g + y_i))$. Now since x is adjacent to y_i, $g + x$ is adjacent to $g + y_i$. So $g + x \in N(H_i - (g + y_i))$. Hence $g + x$ is adjacent to $g' + y_i, g' \neq g, g' \in G$. Therefore $x + y_i$ is adjacent to $g + g' \neq 0$. In other words, for each $1 \leq i \leq n$, $\vec{e_i} \in N(G - \{0\})$. This contradicts the assumed irredundance of G, so an H of the desired form must exist. □

Proposition 14 *Let G be an irredundant subset of the even weight vertices of Q_n and suppose that G is a subgroup of Q_n. Then there exists a subgroup $G' \supset G$ which is a bipartite dominating set.*

Proof. Let G' be maximal among those subgroups of the group of even weight vertices which contain G and are irredundant. It follows from the preceding lemma that G' is a one-sided dominating set and hence a bipartite dominating set. □

References

[1] K. Arumugam, Domination parameters of hypercubes, *J. Indian Math. Soc.* (N.S.) **65** (1998) 31–38.

[2] R.W. Hamming, Error detecting and error correcting codes, *Bell System Tech. J.* **29** (1950), 147–160.

[3] J. Harant and A. Pruchnewski, A Note on the Domination Number of a Bipartite Graph, *Ann. Comb.* **5** (2001), 175–178.

[4] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, *Monographs and Textbooks in Pure and Applied Mathematics,* 208. Marcel Dekker, Inc., New York, 1998.

[5] S.T. Hedetniemi and R.C. Laskar, *A bipartite theory of graphs*: I. *Congr. Numer.*, **55** (1986) 5–14.

[6] S.T. Hedetniemi and R.C. Laskar, *A bipartite theory of graphs*: II. *Congr. Numer.*, **64** (1988) 137–146.

[7] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition*, Wiley, New York, 2000.

[8] V. Pless, *Introduction to the Theory of Error-Correcting Codes*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc. 1998.

[9] F.S. Roberts, *Applied Combinatorics*, Prentice-Hall (1984).