Graph Operations and Point-Distinguishing Chromatic Indices*

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Abstract

The point-distinguishing chromatic index of a graph $G = (V, E)$ is the smallest number of colors assigned to E so that no two different points are incident with the same color set. In this paper we discuss the bounds of the point-distinguishing chromatic indices of graphs resulting from the graph operations. We emphasize that almost all of these bounds are best possible.

1 Introduction

Let $G = (V, E)$ be a graph, where V is the vertex set of G and E is the edge set of G. A mapping f from E onto the k-set $N_k = \{1, 2, \cdots, k\}$ is called an edge k-coloring of G. Suppose that u is a vertex of G with neighborhood $\{v_1, v_2, \cdots, v_r\}$. The color set of u induced by f is the set $\overline{f(u)} = \{f(uv_1), f(uv_2), \cdots, f(uv_r)\}$. We say that an edge k-coloring f is a point-distinguishing coloring of G if $\overline{f(u)} \neq \overline{f(w)}$ for any two different points u and w. The point-distinguishing chromatic index, denoted by $\chi_0(G)$, is the minimum number of colors used in any point-distinguishing coloring of G.

Let's first indicate which graphs have a point-distinguishing edge coloring.

Lemma 1.1. Let $G = (V, E)$ be a graph. Then G has a point-distinguishing coloring if and only if G has at most one isolated vertex and no K_2 components.
Proof. We know that for any edge coloring of G, the endpoints of the component K_2 have the same color set and the color set of an isolated point must be an empty set. So, any graph $G = (V, E)$ which has a point-distinguishing coloring must have no K_2 components and at most one isolated point.

On the other hand, if G is a graph which has no K_2 components and at most one isolated vertex, coloring each edge with a distinct color, we get an edge coloring of G. Since any two different points of G are not incident to the

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same set of edges, the coloring is point-distinguishing. ◇

From now on, we will call a graph G a point-distinguishable graph if G has no K_2 components and at most one isolated point.

Corollary 1.2. For any point-distinguishable graph G, the point-distinguishing chromatic index exists.

In the study of point-distinguishing colorings and chromatic indices, F. Harary and M. Plantholt[2] obtained bounds for $\chi_0(G)$ and determined $\chi_0(G)$ for certain classes of graphs. Later M. Hornak and R. Sotak[3,4] and N. Z. Salvi[5,6] studied the point-distinguishing chromatic index of $K_{n,n}$. A. C. Burris and R. H. Schelp[1] proved that $\chi_0(G) \leq C \cdot max\{n_i^{1/i} | 1 \leq i \leq \Delta\}$ where C is a constant depending only on the maximum degree Δ and n_i is the number of vertices of degree i in G. They also get some results about $\chi_0(G)$ for special classes of graphs including trees.

Usually, in studying parameters of a graph, it is convenient to consider the effects caused by applying the graph operations on the graph. We categorize the graph operations into two types : unitary operations Ψ_1's(such as edge-deletion, vertex-deletion, edge-addition, edge-splitting, vertex-splitting, taking a subgraph, taking a complement) and binary operations Ψ_2's (such as union, Cartesian product, composition, join). In this paper, we consider the problems : If $\chi_0(G)$ and $\chi_0(H)$ are known, what are the upper bounds and lower bounds of $\chi_0(\Psi_1(G))$ and $\chi_0(\Psi_2(G, H))$? We get the upper bound and lower bound for each operation. We emphasize that almost all these bounds that we get here are best possible, i.e., these bounds can be attained by infinitely many graphs(we will call these graphs the critical graphs for the operations respectively).

2 Unitary Operations On Graphs

Let $G = (V, E)$ be a graph, and $e \in E$. The edge-deletion graph $G - e$ means the graph (V, E') where $E' = E - \{e\}$. About this operation, we find the following property.

Theorem 2.1. Let $G = (V, E)$ be a graph with $\chi_0(G) = n$. Suppose that $e \in E$, and $G - e$ is a point-distinguishable graph. Then $n - 1 \leq \chi_0(G - e) \leq n + 2$ and both bounds are best possible.

Proof. Suppose that $\chi_0(G - e)$=$k \leq n - 2$. Let φ be a point-distinguishing k-coloring of $G - e$. We can extend φ to get a point-distinguishing $(k+1)$-coloring ϕ of G as follows. For each edge $f \in E$,

$$\phi(f) = \begin{cases} \varphi(f) & \text{if} \quad f \neq e; \\ k+1 & \text{if} \quad f = e. \end{cases}$$

Then $\chi_0(G) \leq k + 1 \leq n - 1$ and it contradicts the assumption $\chi_0(G) = n$. Hence we have $\chi_0(G - e) \geq n - 1$.

On the other hand, suppose that ψ is a point-distinguishing n-coloring of G, and $e = uv$. Let $H = G - e$. If $\psi|_H$ is still a point-distinguishing n-coloring of H, then $\chi_0(H) \leq n \leq n+2$. If $\psi|_H$ is not a point-distinguishing n-coloring of H, then let us consider first the case $deg_H(u) = 0$ or $deg_H(v) = 0$ but not both $deg_H(u)$ and $deg_H(v)$ equal to 0(otherwise H is not point-distinguishable). Without loss of generality, we may assume that $deg_H(u) = 0$. Then there is exactly one vertex, say w, such that $\overline{\psi|_H(v)} = \overline{\psi(w)}$. If $vw \notin E(H)$ or $deg_H(v) \neq 1$, then color an edge $f(\neq vw)$ incident to v with the color $n+1$ and we have a point-distinguishing $(n+1)$-coloring of H. If $vw \in E(H)$ and $deg_H(v) = 1$, then $deg_H(w) \neq 1$(otherwise H is not point-distinguishable). And then if we color an edge $g(\neq vw)$ incident to w with the color $n+1$, then we have a point-distinguishing $(n+1)$-coloring of H. Before considering the other cases, let us note that: if $deg_H(u) = 1$(or $deg_H(v) = 1$), and $N_H(u) = \{x\}$(or $N_H(v) = \{y\}$), then $deg_H(x) \geq 2$(or $deg_H(y) \geq 2$)(otherwise H is not a point-distinguishable graph). If $|N_H(u)| = 1$, then we take x to be the only vertex in $N_H(u)$ and $y \in N_H(v)$ with $\overline{\psi(y)} \neq \overline{\psi(v)} - \{\psi(e)\}$; if $|N_H(v)| = 1$ then we take y to be the only vertex in $N_H(v)$ and x to be a vertex in $N_H(u)$ such that $\overline{\psi(x)} \neq \overline{\psi(u)} - \{\psi(e)\}$; otherwise we take x to be a vertex in $N_H(u)$ such that $\overline{\psi(x)} \neq \overline{\psi(u)} - \{\psi(e)\}$ and y to be a vertex in $N_H(v)$ such that $\overline{\psi(y)} \neq \overline{\psi(v)} - \{\psi(e)\}$. Define a $(n+2)$-coloring λ of H as following.

$$\lambda(f) = \begin{cases} \psi(f) & \text{if} \quad f \neq ux \text{ and } f \neq vy; \\ n+1 & \text{if} \quad f = ux; \\ n+2 & \text{if} \quad f = vy. \end{cases}$$

Then it is clear that λ is a point-distinguishing $(n+2)$-coloring of H. Hence no matter what case it is, we have $\chi_0(H) \leq n+2$.

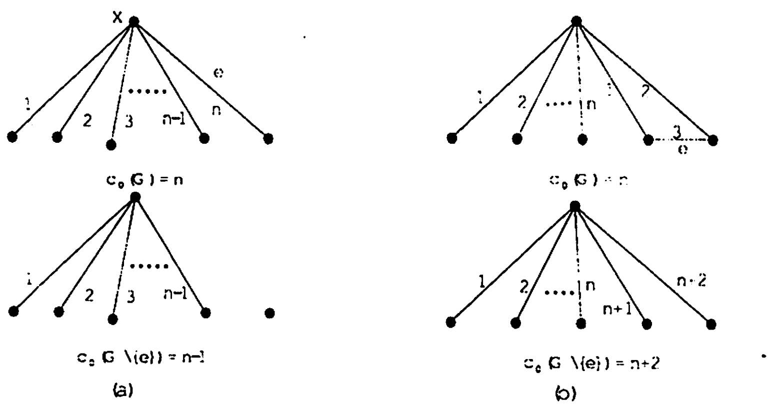


Figure 1: The critical graphs for the edge deletion

For each $n \geq 3$, Figure 1(a) illustrates a graph G with $\chi_0(G) = n$ and $\chi_0(G - e) = n - 1$ and Figure 1(b) illustrates a graph G with $\chi_0(G) = n$ and

$\chi_0(G-e) = n+2$ respectively. Hence all the bounds in Theorem 2.1 are best possible. ◇

Let $G = (V, E)$ be a graph, and v be a vertex of G. The vertex-deletion graph $G - v$ is a graph (V', E'), where $V' = V \setminus \{v\}$ and $E' = E \setminus \{vv_i | v_i \in N(v)\}$.

Theorem 2.2. Let $G = (V, E)$ be a graph with $\chi_0(G) = n$, and $v \in V$ with $deg_G(v) = k$. If $G - v$ is a point-distinguishable graph, then $n-1 \leq \chi_0(G-v) \leq n+k$ and both bounds are best possible.

Proof. First of all, we assume that the neighborhood of v are $v_1, v_2, \ldots, v_k$.

By contrary suppose that $\chi_0(G-v) = l \leq n-2$. Let φ be a point-distinguishing l-coloring of $G-v$. We can extend φ to be a point-distinguishing $(l+1)$-coloring ϕ of G as follows.

$$\phi(f) = \begin{cases} \varphi(f) & \text{if} \quad f \in E(G-v); \\ l+1 & \text{if} \quad f = vv_1, vv_2, \ldots, \text{or}, vv_k. \end{cases}$$

Then $\chi_0(G) \leq l+1 \leq n-1$, and it contradicts the assumption $\chi_0(G) = n$. Hence we have $\chi_0(G-v) \geq n-1$.

On the other hand, let ψ be a point-distinguishing n-coloring of G. We can get a point-distinguishing $(n+k)$-coloring λ of $G-v$ as follows. Since $G-v$ is point-distinguishable, at most one point is isolated(in this case the isolated point has no incident edge to color). Without loss of generality we may assume that v_1v_1', v_2v_2', ..., $v_kv_k' \in E(G)$. Defined λ to be

$$\lambda(f) = \begin{cases} \psi(f) & \text{if} \quad f \in E(G') \setminus \{v_iv_i' : i = 1, 2, \ldots, k\}; \\ n+1 & \text{if} \quad f = v_1v_1'; \\ n+2 & \text{if} \quad f = v_2v_2'; \\ & \vdots \\ n+k & \text{if} \quad f = v_kv_k'. \end{cases}$$

Then λ is a point-distinguishing $(n+k)$-coloring of $G-v$. Hence $\chi_0(G-v) \leq n+k$.

In Figure 2(a) we give an example for $\chi_0(G) = n$ and $\chi_0(G-v) = n-1$ for $n \geq 2$, while in Figure 2(b) we give an example for $\chi_0(G) = n$ and $\chi_0(G-v) = n+k$ where $3 \leq k \leq \binom{n}{2}$ for all $n \geq 3$. These show that the bounds in Theorem 2.2 are best possible. ◇

Let $G = (V, E)$ be a graph, and $e = uv \notin E$, where $u, v \in V$. The graph edge-addition graph $G \bigcup e$ is the graph (V, E'), where $E' = E \bigcup \{e\}$.

Theorem 2.3. Let $G = (V, E)$ be a graph with $\chi_0(G) = n$, and $e = uv \notin E$, where $u, v \in V$. Then $n-2 \leq \chi_0(G \bigcup e) \leq n+1$, and both bounds are best possible.

Proof. Suppose that $\chi_0(G \bigcup e) = k$. Then by Theorem 2.1, we have $k-1 \leq \chi_0(G) = n \leq k+2$. Hence we have $n-2 \leq \chi_0(G \bigcup \{e\}) = k \leq n+1$.

Figure 3(a) and (b) illustrate the graphs G's such that $\chi_0(G) = n$ and $\chi_0(G \bigcup e) = n - 2$ and $n + 1$ respectively.(In fact edge-addition is an inverse operation of edge-deletion, so Figure 3 is just another expression of Figure 2.) This shows that both bounds in Theorem 2.3 are best possible.

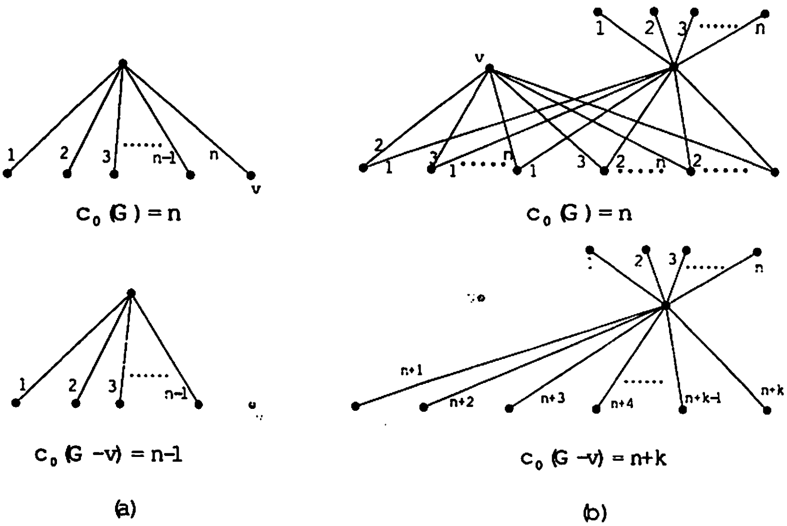


Figure 2: The critical graphs for vertex-deletions

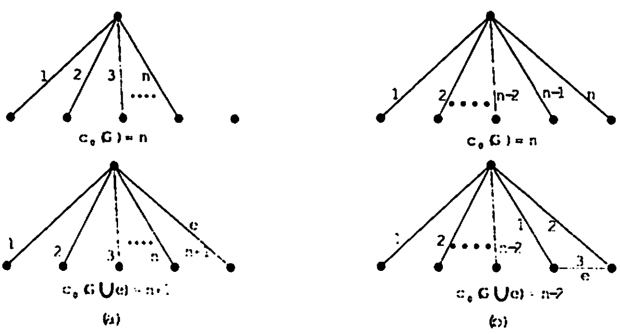


Figure 3: The critical graphs for edge-additions

Let $G = (V, E)$ be a graph, $e = xy$ be an edge of G, and $v \notin V$. The edge splitting graph is the graph $G_e^s = (V', E')$ where $V' = V \bigcup \{v\}$ and $E' = (E - \{e\}) \bigcup \{xv, vy\}$.

Theorem 2.4. Let $G = (V, E)$ be a graph with $\chi_0(G) = n$, and $e = xy$ be an edge of G. Then $n - 1 \leq \chi_0(G_e^s) \leq n + 1$, and the upper bound is best possible.
Proof. Since G is a point-distinguishable graph, we have $deg_{G_e^s}(x) \geq 2$ or $deg_{G_e^s}(y) \geq 2$.

By contrary, suppose that $\chi_0(G_e^s) = k \leq n-2$. Let φ be a point-distinguishing k-coloring of G_e^s.
Case 1. One of $deg_{G_e^s}(x)$ and $deg_{G_e^s}(y)$ is 1.

Without loss of generality, suppose that $deg(x) \geq 2$ and $deg(y) = 1$. We can get a point-distinguishing $(k + 1)$-coloring ϕ of G as follows.

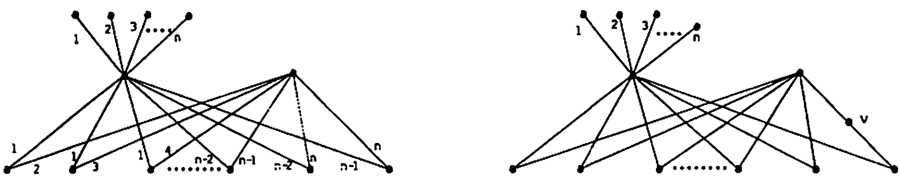
$$\phi(f) = \begin{cases} k+1 & \text{if} \quad f = xy; \\ \varphi(f) & \quad \text{otherwise.} \end{cases}$$

Then ϕ is a point-distinguishing $(k+1)$-coloring of G which contradicts $\chi_0(G) = n$.
Case 2. Both $deg_{G_e^s}(x)$ and $deg_{G_e^s}(y)$ are greater or equal to 2.

Let $z \in N(y)$ such that $\overline{\varphi(z)} \neq \overline{\varphi(y)} - \{\varphi(yv)\} \bigcup \{\varphi(xv)\}$. We can get a point-distinguishing $(k + 1)$-coloring ϕ of G as follows.

$$\phi(f) = \begin{cases} \varphi(xv) & \text{if } f = xy; \\ k+1 & \text{if } f = yz; \\ \varphi(f) & \text{otherwise.} \end{cases}$$

Then ϕ is a point-distinguishing $(k+1)$-coloring of G which contradicts $\chi_0(G) = n$. So $\chi_0(G_e^s) \geq n - 1$.



c$_0$ G) = n   c$_0$ (G U v) =n+1

Figure 4: The critical graphs for edge-splittings

On the other hand, suppose that ψ is a point-distinguishing n-coloring of G. We can get a point-distinguishing $(n + 1)$-coloring λ of G_e^s as follows.

$$\lambda(f) = \begin{cases} \psi(xy) & \text{if} \quad f = xv; \\ n+1 & \text{if} \quad f = vy; \\ \psi(f) & \text{otherwise.} \end{cases}$$

Then λ is a point-distinguishing $(n + 1)$-coloring of G_e^s. Hence $\chi_0(G_e^s) \leq n + 1$.

For each $n \geq 3$, Figure 4 illustrates a graph G such that $\chi_0(G) = n$ and $\chi_0(G_e^s) = n+1$. This shows that the upper bound in Theorem 2.4 is best possible. ◇

Unfortunately we can neither improve the lower bound in Theorem 2.4 nor find an example to attain the lower bound.

Let $G = (V, E)$ be a graph, and v be a vertex of G. The vertex-splitting graph is the graph $G_v^s = (V', E')$ where $V' = V \bigcup \{v'\}$ and $E' = E \bigcup \{v'u | vu \in E\}$.

Theorem 2.5. Let $G = (V, E)$ be a graph with $\chi_0(G) = n$, and v be a vertex of G with degree k. Then $n - k \leq \chi_0(G_v^s) \leq n+1$ and both bounds are best possible.

Proof. By Theorem 2.2, $\chi_0(G_v^s) - 1 \leq \chi_0(G) = n \leq \chi_0(G_v^s) + k$. Hence $n - k \leq \chi_0(G_v^s) \leq n+1$.

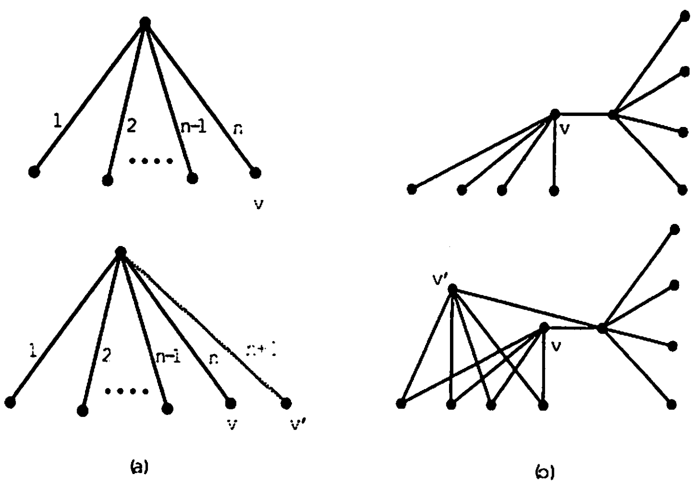


Figure 5: The critical graphs for vertex-splittings

Figure 5(a) is an example for $\chi_0(G) = n$ and $\chi_0(G_v^s) = n+1$. While Figure 5(b) is an example for $\chi_0(G) = n$ and $\chi_0(G_v^s) = n - k$ where $k \leq \binom{n-k}{2}$ and $n-k-1 \geq 3$. These show that both bounds in Theorem 2.5 are best possible. ◇

As for the subgraphs, we need some more preparations.

Lemma 2.6. Let $G = (V, E)$ be a point-distinguishable graph. Then $\chi_0(G) \geq log_2|V(G)|$.

Proof. The total number of distinct subsets of k colors is 2^k, so that if $\chi_0(G) = k$ then $2^k \geq |V(G)|$, and $\chi_0(G) \geq log_2|V(G)|$. ◇

Note that if G is a point-distinguishable graph not only a single isolated vertex, then $2 \leq \chi_0(G)$.

Lemma 2.7. Let $G = (V, E)$ be a point-distinguishable graph. Then $\chi_0(G) \leq |V(G)|$.

Proof. We prove it by induction on $|V(G)|$.

(1). If $|V(G)| = 3$, all of the point-distinguishable graphs are in Fig.6 and the point-distinguishing chromatic indices of these graphs are less than 3.

Figure 6: Point-distinguishable graphs of order 3

(2). Suppose that the theorem holds for all graphs of order k. Let $G = (V, E)$ be a graph of order $k + 1$. Take a vertex v of G. Let $H = G - v$. Suppose $E(G) = E(H) \bigcup \{vv_i | i = 1, 2, \cdots, m\}$. If φ is the point-distinguishing $\chi_0(H)$-coloring of H. We can extend φ to be a point-distinguishing $(\chi_0(H)+1)$-coloring of G as follows.

$$\phi(f) = \begin{cases} \varphi(f) & \text{if} \quad f \in E(H); \\ \chi_0(H) + 1 & \text{if} \quad f \in vv_i, \ i = 1, 2, \cdots, m. \end{cases}$$

Then ϕ is a point-distinguishing $(\chi_0(H)+1)$-coloring of G and $\chi_0(G) \leq \chi_0(H) + 1 \leq k + 1$.

Hence by induction, $\chi_0(G) \leq |V(G)|$ for all point-distinguishable graph G. $\Diamond$

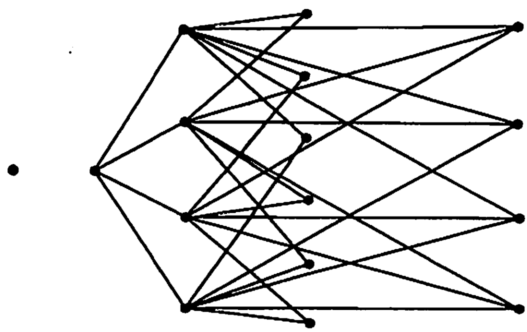


Figure 7: A fully colored point-distinguishing graph of order 2^4

Let $X = \{1, 2, \cdots, n\}$. Define a graph $G = (V, E)$, where $V = \binom{X}{0} \bigcup \binom{X}{1} \bigcup \cdots \bigcup \binom{X}{n}$, and $E = \bigcup_{i=1}^{n} (\bigcup_{j=2}^{n} \{\{i\}P | P \in \binom{X}{j} \text{ and } i \in P\})$ where $\binom{X}{j}$ is the set of all

j-subset of X. Since $|V| = 2^n$, by Lemma 2.6 we know that $\chi_O(G) \geq n$. Define a point-distinguishing n-coloring φ of G as $\varphi(\{i\}P) = i$ for all $\{i\}P \in E$. Then $\varphi(s) = s$ for all $s \in V$. Hence φ is a point-distinguishing n-coloring of G and $\chi_O(G) = n$. We call G a fully colored point-distinguishing graph of order 2^n(See Figure 7).

Theorem 2.8. If G is a point distinguishable graph, then $log_2|V(G)| \leq \chi_O(G) \leq |V(G)|$ and both bounds are best possible.
Proof. The lower bound can be attained by the fully colored point-distinguishing graphs and the upper bound can be attained by the disjoint union of k 3-cycles for $k = 1, 2, \cdots$. ◇

Corollary 2.9. Let $G = (V, E)$ be a graph. Suppose that H is a point-distinguishable subgraph of G with $\chi_O(G) = n$. Then $\chi_O(H) \leq 2^n$.
Proof. Since $\chi_O(G) = n$, by Lemma 2.6 we know that $|V(G)| \leq 2^n$. Then by Lemma 2.7 we know that $\chi_O(H) \leq |V(H)| \leq 2^n$. ◇

Let $G = (V, E)$ be a graph. The complement graph of G is the graph $\overline{G} = (V', E')$, where $V' = V$ and $E' = \{uv | u, v \in V \text{ and } uv \notin E\}$.

Corollary 2.10. Let $G = (V, E)$ be a graph with $\chi_O(G) = n$. Then $log_2|V(G)| \leq \chi_O(\overline{G}) \leq |V(G)|$.

3 Binary Operations On Graphs

Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be two graphs such that $V_1 \bigcap V_2 = \phi$. The union graph of G and H is a graph $G \bigcup H = (V, E)$ where $V = V_1 \bigcup V_2$, and $E = E_1 \bigcup E_2$.

Theorem 3.1. Let $G = (V_1, E_1)$, $H = (V_2, E_2)$ be two point-distinguishable graphs with $\chi_O(G) = m$ and $\chi_O(H) = n$ and not both G and H having an isolated vertex. If $V_1 \bigcap V_2 = \phi$, then $max\{m, n\} \leq \chi_O(G \bigcup H) \leq m + n$.
Proof. Without lost of generality, we may assume that $m \geq n$. It is clear that $G \bigcup H$ is point-distinguishable. Let f be a point-distinguishing coloring of $G \bigcup H$ using $\chi_O(G \bigcup H)$ colors. Then $f|_G$ and $f|_H$ are point-distinguishing colorings of G respectively. Hence $\chi_O(G \bigcup H) \geq max\{m, n\} = m$.

On the other hand, since $\chi_O(G) = m$ and $\chi_O(H) = n$, there is a point-distinguishing coloring f of H using colors $1, 2, \ldots, n$, and a point-distinguish coloring g of G using colors $n+1, n+2, \ldots, n+m$. Then using f and g together, we get a point-distinguishing coloring of $G \bigcup H$ using $m + n$ colors. So $\chi_O(G \bigcup H) \leq m + n$.

If we take $G = K_{1,m}$ and $H = K_{2,l}$ where $l \leq \binom{m}{2}$, then it is clear that $\chi_O(G) = m \geq \chi_O(H)$ and $\chi_O(G \bigcup H) = m$ for all $m \geq 3$. Hence the lower bound is best possible. In Figure 8 we take $G = K_{1,m}$ and $H = (V, E)$ where

$V = \{M, N, u_1, \cdots, u_l, v_1, \cdots, v_n\}$ and $E = \{Mu_j \mid j = 1, \cdots, l\} \bigcup \{Nu_j | j = 1, 2, \cdots, l\} \bigcup \{Mv_i | i = 1, 2, \cdots, n\}$ where $n \leq m$ and $l \leq \binom{n}{2}$. Then it is clear that for $n \geq 3$, $\chi_0(G) = m$, $\chi_0(H) = n$ and $\chi_0(G \bigcup H) = m + n$. Hence the upper bound is also best possible. $\diamond$

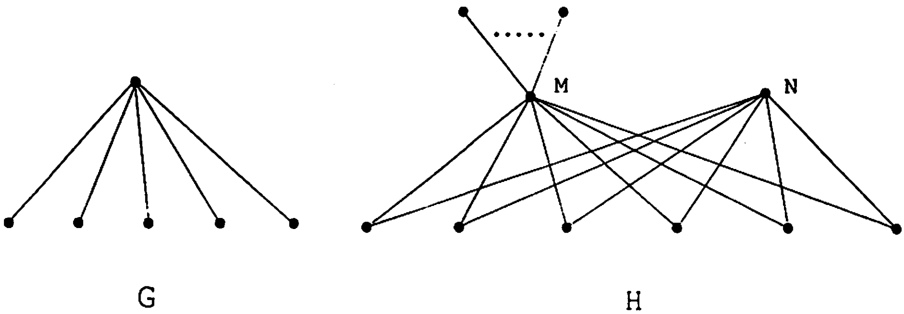


Figure 8: The critical graphs for Union of C and H

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The Cartesian Product of G_1 and G_2 is a graph $G_1 \times G_2 = (V, E)$ where $V = V_1 \times V_2$ and $E = \{(u, x)(v, y) | x = y \text{ and } uv \in E_1 \text{ or } u = v \text{ and } xy \in E_2\}$.

Theorem 3.2. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $\chi_0(G_1) = m$, and $\chi_0(G_2) = n$. Then $\chi_0(G_1 \times G_2) \leq m + n$ and this bound is best possible.

Proof. Let φ_1 be a point-distinguishing m-coloring of G_1 using colors $1, 2, \ldots, m$. And let φ_2 be a point-distinguishing n-coloring of G_2 using colors $m + 1, m + 2, \ldots, m + n$. Define a point-distinguishing $(m + n)$-coloring σ of $G_1 \times G_2$ as follows.

$$\sigma(e) = \begin{cases} \varphi_1(xy) & \text{if } \; e = (x, u)(y, u) \text{ where} x, y \in V_1, u \in V_2 \text{ and } xy \in E_1; \\ \varphi_2(uv) & \text{if } \; e = (x, u)(x, v) \text{ where} u, v \in V_2, uv \in E_2 \text{ and } x \in V_1. \end{cases}$$

Let $X = (u_1, v_1)$, $Y = (u_2, v_2) \in V(G_1 \times G_2)$. Then $\overline{\sigma(X)} = \overline{\varphi_1(u_1)} \bigcup \overline{\varphi_2(v_1)}$ and $\overline{\sigma(Y)} = \overline{\varphi_1(u_2)} \bigcup \overline{\varphi_2(v_2)}$. Since $\overline{\varphi_1(u_1)}, \overline{\varphi_1(u_2)} \subset \{1, 2, \cdots, m\}$ and $\overline{\varphi_2(v_1)}, \overline{\varphi_2(v_2)} \subset \{m+1, m+2, \cdots, m+n\}$, $\overline{\sigma(X)} = \overline{\sigma(Y)}$ implies that $\overline{\varphi_1(u_1)} = \overline{\varphi_1(u_2)}$ and $\overline{\varphi_2(v_1)} = \overline{\varphi_2(v_2)}$; and then implies $u_1 = u_2$ and $v_1 = v_2$, iė; $X = Y$. So that σ is a point-distinguishing $(m + n)$-coloring. Hence $\chi_0(G_1 \times G_2) \leq m + n$.

Suppose that G_1 and G_2 are fully colored point-distinguishing graphs of orders 2^n and 2^m respectively. Then $|V(G_1 \times G_2)| = 2^{m+n}$, so that $\chi_0(G_1 \times G_2) \geq m+n$. On the other hand, by the inequality above, we have $\chi_0(G_1 \times G_2) \leq m+n$. Hence $\chi_0(G_1 \times G_2) = m + n$ and this shows that the bound in Theorem 3.2 is best possible. ◊

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The composition graph of G_1 and G_2 is a graph $G_1[G_2] = (V, E)$ where $V = V_1 \times V_2$ and $E = \{(u,x)(v,y)|uv \in E_1 \text{ or } u = v \text{ and } xy \in E_2\}$.

Theorem 3.3. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $\chi_0(G_1) = m$, and $\chi_0(G_2) = n$. Then $\chi_0(G_1[G_2]) \leq m + n$ and this bound is best possible.

Proof. The proof of this theorem is highly similar to the proof of Theorem 3.2. If we take G_1 and G_2 to be two fully colored point-distinguishing graphs of orders 2^m and 2^n respectively. Then $\chi_0(G_1[G_2]) = m + n$ for the same reason as in Theorem 3.2, and this shows that the bound is best possible. ◊

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \bigcap V_2 = \phi$. The join graph of G_1 and G_2 is a graph $G_1 \vee G_2 = (V, E)$ where $V = V_1 \bigcup V_2$ and $E = E_1 \bigcup E_2 \bigcup \{uv|u \in V_1, v \in V_2\}$.

Theorem 3.4. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \bigcap V_2 = \phi$. If $\chi_0(G_1) = m$, and $\chi_0(G_2) = n$, then $\lceil log_2(|V_1| + |V_2|) \rceil \leq \chi_0(G_1 \vee G_2) \leq \lceil log_2(max\{|V_1|, |V_2|\}) \rceil + 3$.

Proof. By [4], we know that $\lceil log_2(m+n) \rceil \leq \chi_0(K_{m,n}) \leq \lceil log_2(max\{m,n\}) \rceil + 2$. And by the definition we know $E(G_1 \vee G_2) = E(K_{|V_1|,|V_2|}) \bigcup E_1 \bigcup E_2$. Suppose that φ is a point-distinguishing k-coloring of $K_{|V_1|,|V_2|}$ where $k = \chi_0(K_{|V_1|,|V_2|}) \leq \lceil (log_2 max\{m,n\}) \rceil + 2$. We extend φ to be a point-distinguishing $(k+1)$-coloring of $G_1 \vee G_2$ as follows.

$$\phi(f) = \begin{cases} \varphi(f) & \text{if } f \in E(K_{|V_1|,|V_2|}); \text{ and} \\ k+1 & \text{if } f \in E_1 \bigcup E_2 \end{cases}$$

Hence $\chi_0(G_1 \vee G_2) \leq \lceil log_2(max\{|V_1|, |V_2|\}) \rceil + 3$.

On the other hand, by Theorem 2.8 we know that $\lceil log_2(|V_1| + |V_2|) \rceil \leq \chi_0(G_1 \vee G_2)$. ◊

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