On a generalization of signed total dominating functions of graphs *

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June 1, 2005

Abstract

Let $G = (V, E)$ be a simple graph. For any real valued function $f : V \to \mathbf{R}$, the weight of f is defined as $f(V) = \sum f(v)$, over all vertices $v \in V$. For positive integer k, a total k-subdominating function (TkSF) is a function of $f : V \to \{-1, 1\}$ such that $f(N(v)) \geq 1$ for at least k vertices v of G. The total k-subdomination number $\gamma_{ks}^t(G)$ of a graph G equals the minimum weight of a TkSF on G. In the special case where $k = |V|$, γ_{ks}^t is the signed total domination number [5]. We research total k-subdomination numbers of some graphs and obtain a few lower bounds of $\gamma_{ks}^t(G)$.

1 Introduction

Let $G = (V, E)$ be a simple graph and v be a vertex in V. The open neighborhood of v, denoted by $N(v)$, is the set of vertices adjacent to v, i.e., $N(v) = \{u \in V | uv \in E\}$. The closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. Let $S \subseteq V$, $G[S]$ denotes the subgraph of G induced by S. The degree of v in G is $d_G(v) = |N(v)|$, a vertex v is called even (odd) vertex if $d_G(v)$ is even (odd). A vertex v of a tree T is called a *leaf* of T if

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$d_T(v) = 1$. $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of the vertices of G. When no ambiguity can occur, we often simply write $d(v)$, δ, Δ instead of $d_G(v)$, $\delta(G)$ and $\Delta(G)$, respectively.

2 Definition of total k-subdomination

For any real-valued function $f : V \to \mathbf{R}$ and $S \subseteq V$, let $f(S) = \sum_{v \in S} f(v)$. The weight of f is defined as $f(V)$. A function $f : V \to \{0,1\}$ is said to be a *total dominating function* (TDF) of G if $f(N[v]) \geq 1$ for every $v \in V$. The *total domination number* $\gamma_t(G) = \min\{f(V) | f$ is a TDF on $G\}$.

A *signed total dominating function* (STDF) of G is defined in [5] as a function $f : V \to \{-1,1\}$ such that $f(N(v)) \geq 1$ for every $v \in V$. The *signed total domination number* $\gamma_{st}(G) = \min\{f(V) | f$ is a STDF on $G\}$.

A *k-subdominating function* (kSF) of G is defined in [3] as a function $f : V \to \{-1,1\}$ such that $f(N[v]) \geq 1$ for at least k vertices v of G. The k-subdomination number, denoted by $\gamma_{ks}(G)$, of G is equal to $\min\{f(V) | f$ is a kSF on $G\}$.

In this paper, we develop an analogous theory for total k-subdomination.

Definition. For $k \in \mathbf{Z}^+$, a function $f : V \to \{-1,1\}$ is said to be a *total k-subdominating function* (TkSF) on G if $f(N(v)) \geq 1$ for at least k vertices v of G. The total k-subdomination number, denoted by $\gamma^t_{ks}(G)$, of G is equal to $\min\{f(V) | f$ is a TkSF on $G\}$. A TkSF f is minimal if no $g < f$ is also a TkSF on G. The upper total k-subdomination number, denoted by $\Gamma^t_{ks}(G)$, of G is equal to $\max\{f(V) | f$ is a minimal TkSF on $G\}$.

To ensure existence of TkSF, we henceforth restrict our attention to graphs without isolated vertices.

We use the following notation. Let f be a TkSF of $G = (V, E)$, we say $v \in V$ is covered by f if $f(N(v)) \geq 1$ and denote the set of vertices covered by f, by C_f. Let $P_f = \{v \in V | f(v) = 1\}$, $M_f = \{v \in V | f(V) = -1\}$, and $B_f = \{v \in V | f(N(v)) \in \{1,2\}\}$. For $A, B \subseteq V$, we say *A totally dominates B*, denoted by $A \succ_t B$, if for each $b \in B$, $N(b) \cap A \neq \emptyset$. If $A \succ_t V$, then A is a *total dominating set* of G.

Theorem 1. *A TkSF f on a graph G is minimal if and only if for each k-subset K of C_f, $K \cap B_f \succ_t P_f$.*

Proof. Suppose f is a TkSF satisfying the above condition but f is not minimal. Then there exists a TkSF $g < f$ with k-subset $K' \subseteq C_g \subseteq C_f$. Thus there exists a vertex $v \in V$ with $g(v) < f(v)$, i.e., $g(v) = -1$ and $f(v) = 1$. By assumption $B_f \cap K' \succ_t \{v\}$, i.e., there exists $w \in B_f \cap K' \cap N(v)$. Now, $f(N(w)) \in \{1,2\}$ and $v \in N(w)$, hence $g(N(w)) < 1$, a contradiction which shows that f is minimal.

Conversely, suppose that f is a minimal TkSF and there exists a k-subset $K \subseteq C_f$ with $B_f \cap K \not\succ_t \{v\}$, where $v \in P_f$. Let $h : V \to \{-1,1\}$ be defined by $h(v) = -1$ and $h(w) = f(w)$ for $w \in V - \{v\}$. If $w \in K \cap B_f$, then $w \notin N(v)$ so that $v \notin N(w)$ and $h(N(w)) = f(N(w)) \geq 1$. For $w \in K - B_f$, $f(N(w)) \geq 3$. It is possible that $v \in N(w)$; However, $h(N(w)) \geq f(N(w)) - 2 \geq 1$. Thus h is a TkSF, contrary to the minimality of f. ∎

3 Total k-subdomination numbers of some graphs

Theorem 2. *For any complete graph K_n $(n \geq 2)$,*

$$\gamma_{ks}^t(K_n) = \begin{cases} 0 & \text{if } n \text{ is even and } k \leq \frac{n}{2}, \\ 1 & \text{if } n \text{ is odd and } k < \frac{n}{2}, \\ 2 & \text{if } n \text{ is even and } \frac{n}{2} < k \leq n, \\ 3 & \text{if } n \text{ odd and } \frac{n}{2} < k \leq n. \end{cases}$$

Proof. Let f be a minimum TkSF on $K_n = (V, E)$.

Case 1. $k \leq \frac{n}{2}$.

Since there exists at least one vertex $v \in V$ with $f(N(v)) = f(V) - f(v) \geq 1$, it follows that $f(V) \geq f(v) + 1 \geq 0$. Especially, if n is odd, then $f(V)$ is odd. Then $f(V) \geq 1$.

On the other hand, define $g : V \to \{-1, 1\}$ by

$$g(x) = \begin{cases} 1 & \text{for } \lceil \frac{n}{2} \rceil \text{ vertices } x \text{ in } V, \\ -1 & \text{otherwise.} \end{cases}$$

Then g is a TkSF of K_n of weight $g(V) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$

So $\gamma_{ks}^t(K_n) \leq g(V)$.

Consequently, $\gamma_{ks}^t(K_n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$

Case 2. $\frac{n}{2} < k \leq n$.

Similar to Case 1, we have $|P_f| - |M_f| = f(V) \geq 0$. Since $|P_f| + |M_f| = n$, then $|P_f| \geq \frac{n}{2}$. Since there exist k vertices $v \in V$ such that $f(N(v)) \geq 1$, and $|C_f| \geq k > \frac{n}{2}$, it follows that there exists at least one vertex $u \in P_f$ such that $f(N(u)) = f(V) - f(u) \geq 1$. Then $f(V) \geq f(u) + 1 = 2$. Especially, if n is odd, then $f(V) \geq 3$.

On the other hand, define $g : V \to \{-1, 1\}$ by

$$g(x) = \begin{cases} 1 & \text{for } \lceil \frac{n}{2} \rceil + 1 \text{ vertices } x \text{ in } V, \\ -1 & \text{otherwise.} \end{cases}$$

Then g is a TkSF of K_n of weight $g(V) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$

So $\gamma_{ks}^t(K_n) \leq g(V)$.

Consequently, $\gamma^t_{ks}(K_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$ ∎

Theorem 3. *For any complete bipartite graph $K_{m,n}$ $(n \geq m \geq 1)$,*

$$\gamma^t_{ks}(K_{m,n}) = \begin{cases} 1-n & \text{if } m \text{ is odd and } k \leq n, \\ 2-n & \text{if } m \text{ is even and } k \leq n, \\ 2 & \text{if } m, n \text{ is odd and } n < k \leq m+n, \\ 3 & \text{if } m+n \text{ is odd and } n < k \leq m+n, \\ 4 & \text{if } m, n \text{ is even and } n < k \leq m+n, \end{cases}$$

Proof. Let $K_{m,n} = (V, E)$ and let U and W be the partite sets of $K_{m,n}$ with $|U| = m$ and $|W| = n$. Among all the minimum TkSF on $K_{m,n}$, let f be one that assigns the value -1 to as many vertices of W as possible. Let U^+ and U^- be the sets of vertices in U that are assigned the value $+1$ and -1 under f, respectively. Let W^+ and W^- be defined analogously. Then $\gamma^t_{ks}(K_{m,n}) = f(V) = f(U) + f(W) = |U^+| - |U^-| + |W^+| - |W^-|$.

Case 1. $k \leq n$.

We show that $W = W^-$, i.e., each vertex of W is assigned the value -1 under f. Assume, to the contrary, that $W^+ \neq \emptyset$.

If $f(U) \geq 1$, then let $f_1 : V \rightarrow \{-1, 1\}$ be defined as follows: Let $f_1(v) = -1$ if $v \in W^+$ and $f_1(v) = f(v)$ if $v \notin W^+$. Since $f_1(N(w)) = f(U) \geq 1$ for each $w \in W$, it follows that f_1 is a TkSF on $K_{m,n}$ of weight less than that of f, a contradiction.

If $f(U) \leq 0$, then $|U^+| \leq |U^-|$. Since there exist k vertices v of V such that $f(N(v)) \geq 1$, it follows that $f(W) \geq 1$, i.e., $|W^+| > |W^-|$, then $|W^+| > \frac{1}{2}|W| \geq \frac{1}{2}|U|$. Let $f_2 : V \rightarrow \{-1, 1\}$ be defined as follows: Let $f_2(v) = -1$ for $\lceil \frac{|U|+1}{2} \rceil$ vertices of v of W^+, $f_2(v) = 1$ for $\lceil \frac{|U|+1}{2} \rceil$ vertices u of U and $f_2(v) = f(v)$ for all remaining vertices v of V. Since $f_2(N(w)) = f_2(U) \geq 1$ for each $w \in W$, it follows that f_2 is a TkSF on $K_{m,n}$ of weight $f_2(V) \leq f(V)$. However, f_2 assigns the value -1 to more vertices of W than does f, contrary to our choice of f. We deduce, therefore, that $W = W^-$.

Now let w be a vertex in W for which $f(N(w)) \geq 1$. Then $|U^+| - |U^-| = f(U) = f(N(w)) \geq 1$. Note that $|U^+| - |U^-| = f(N(w)) \geq 2$ if m is even.

Thus $\gamma^t_{ks}(K_{m,n}) = |U^+| - |U^-| + |W^+| - |W^-| \geq \begin{cases} 1-n & \text{if } m \text{ is odd,} \\ 2-n & \text{if } m \text{ is even.} \end{cases}$

On the other hand, define $g : V \rightarrow \{-1, 1\}$ by

$$g(x) = \begin{cases} 1 & \text{for } \lceil \frac{m+1}{2} \rceil + 1 \text{ vertices } x \text{ of } W, \\ -1 & \text{otherwise.} \end{cases}$$

Then g is a TkSF of K_n of weight $g(V) = \begin{cases} 1-n & \text{if } m \text{ is odd,} \\ 2-n & \text{if } m \text{ is even.} \end{cases}$

So $\gamma^t_{ks}(K_n) \leq g(V)$.

Consequently, if $k \leq n$, $\gamma^t_{ks}(K_{m,n}) = \begin{cases} 1-n & \text{if } m \text{ is odd,} \\ 2-n & \text{if } m \text{ is even.} \end{cases}$

Case 2. $n < k \leq m + n$.

In this case, there exist $w \in W$, and $u \in U$ such that $f(N(w)) \geq 1$ and $f(N(u)) \geq 1$. Then $f(U) = f(N(w)) \geq 1$ and $f(W) = f(N(u)) \geq 1$. Note that $f(U) \geq 2$ if m is even and $f(W) \geq 2$ if n is even.

Thus $\gamma_{ks}^t(K_{m,n}) = f(U) + f(W) \geq \begin{cases} 2 & \text{if } m, n \text{ are odd,} \\ 3 & \text{if } m+n \text{ is odd,} \\ 4 & \text{if } m, n \text{ are even.} \end{cases}$

On the other hand, define $g : V \to \{-1, 1\}$ by

$$g(x) = \begin{cases} 1 & \text{for } \lceil \frac{m+1}{2} \rceil \text{ vertices } x \text{ of } W \text{ and } \lceil \frac{n+1}{2} \rceil \text{ vertices } x \text{ of } U, \\ -1 & \text{otherwise.} \end{cases}$$

Then g is a TkSF of $K_{m,n}$ of weight $g(V) = \begin{cases} 2 & \text{if } m, n \text{ are odd,} \\ 3 & \text{if } m+n \text{ is odd,} \\ 4 & \text{if } m, n \text{ are even.} \end{cases}$

So $\gamma_{ks}^t(K_n) \leq g(V)$.

Consequently, if $n < k \leq m+n$, $\gamma_{ks}^t(K_n) = \begin{cases} 2 & \text{if } m, n \text{ are odd,} \\ 3 & \text{if } m+n \text{ is odd,} \\ 4 & \text{if } m, n \text{ are even.} \end{cases}$

The result now follows. ∎

Corollary 1. *For any star $K_{1,n-1}$ $(n \geq 2)$,*

$$\gamma_{ks}^t(K_{1,n-1}) = \begin{cases} 2-n & \text{if } k \leq n-1 \\ 2 & \text{if } k = n \text{ and } n \text{ is even,} \\ 3 & \text{if } k = n \text{ and } n \text{ is odd.} \end{cases}$$

Lemma 1. *For any tree $T = (V, E)$ on n vertices $(n \geq 2)$, $\gamma_{st}(T) \geq 2$ with equality if and only if each vertex v of T is an odd vertex and v is at least adjacent to $\frac{d_T(v)-1}{2}$ leaves of T.*

Proof. Let $f : V \to \{-1, 1\}$ be any minimum signed total dominating function (STDF) of T. Let $P_f = \{v \in V | f(v) = 1\}$, $M_f = \{v \in V | f(v) = -1\}$. If $M_f = \emptyset$, then $\gamma_{st}(T) = n \geq 2$. Therefore, we may assume there exists a vertex $v \in M_f$, else there is nothing left to prove. Let T be rooted at v. Since $f(N(v)) \geq 1$, at least one child x of vertex v is assigned the value 1 under f. On the other hand, $f(N(x)) \geq 1$ and $f(v) = -1$, at least two children x_1, x_2 of vertex x are assigned the value 1 under f. If $M_f = \{v\}$, we have $\gamma_{st}(T) = |P_f| - |M_f| \geq 3 - 1 = 2$. If $M_f - \{v\} \neq \emptyset$, let $w_1 \in M_f - \{v\}$ and w_1 be a child of vertex w. Since $f(N(w)) \geq 1$, that is, $|N(w) \cap P_f| - |N(w) \cap M_f| \geq 1$, then there at least exists another child w_2 of w with $f(w_2) = 1$, i.e., there at least exists one brother w_2 of w_1 that belongs to the set P_f. Hence, we can conclude that $|P_f| \geq |M_f| + 2$. Thus $\gamma_{st}(T) = |P_f| - |M_f| \geq 2$.

It remains for us to show that $\gamma_{st}(T) = 2$ if and only if each vertex v of T is one odd vertex and v is at least adjacent to $\frac{d_T(v)-1}{2}$ leaves of T. If T is a tree of order $n = 2$, it is trivial. So, in the following proof we assume that T is a tree of order $n \geq 3$.

Let $T_1 = (V_1, E_1)$ be the tree obtained from T removing all the leaves of T, let $L = \{v \in V | d_T(V) = 1\}$.

We first prove the sufficiency. For any vertex $v \in V$, the degree $d_T(v)$ of v is odd and v is at least adjacent to $\frac{d_T(v)-1}{2}$ leaves of T. Let g be a function of T such that for any $v \in V - L$, $\frac{d_T(v)-1}{2}$ leaves, which are adjacent to vertex v, of T are assigned to the value -1 and the else vertices are assigned the value 1. Therefore, for any $v \in V - L$ $f(N(v)) = \frac{d_T(v)+1}{2} - \frac{d_T(v)-1}{2} = 1$; for any $v \in L$. Obviously, $f(N(v)) = 1$. Thus g be a STDF of T, the weight $g(V) = \sum_{v \in V-L}(f(N[v]) - d_{T_1}(T)) = \sum_{v \in V-L}(f(N[v]) - \sum_{v \in V-L} d_{T_1}(v) = \sum_{v \in V-L} 2 - 2|E_1| = 2(|V_1| - |E_1|) = 2$. So $\gamma_{st}(T) \leq g(V) = 2$. On the other hand, $\gamma_{st}(T) \geq 2$. Thus $\gamma_{st}(T) = 2$.

To prove the necessity, suppose $\gamma_{st}(T) = 2$, then $|P_f| = |M_f| + 2$. Let $f : V \to \{-1, 1\}$ be any minimum STDF of T. Since $n \geq 3$, hence $M_f \neq \emptyset$. We show that $M_f \subseteq L$, i.e., $d_T(v) = 1$ for every $v \in M_f$. Assume the contrary, there exists one vertex $v \in M_f$ with $d_T(v) \geq 2$, let T be rooted at v, since $f(N(v)) \geq 1$, at least two children x, y of v are assigned the value 1 under f. Furthermore, since $f(N(x)) \geq 1$, $f(N(y)) \geq 1$ and $f(v) = -1$, at least two children x_1, x_2 of x and two children y_1, y_2 of y are assigned the value 1 under f, i.e., there at least exist two children and four grandchildren of v in set P_f. Now consider any vertex $w \in M_f - \{v\}$. If w is adjacent to v, similarly, it follows that at least two children and four grandchildren of w in set P_f. If w is adjacent to another vertex $u (u \neq v)$, since $f(N(u)) \geq 1$, at least one child w_1 of u are assigned the value 1 under f, i.e., there exists one brother w_1 of w with $f(w_1) = 1$. So $|P_f| \geq |M_f| + 5$, it contradicts the fact that $|P_f| = |M_f| + 2$.

Therefore, we have $V - L \subseteq P_f$, i.e., for any $v \in V - L$, $f(v) = 1$.

Now we prove that $f(N(v)) = 1$ for every $v \in V$. If $v \in L$, obviously, $f(N(v)) = 1$. Thus assume that there exists one vertex $u \in V - L$ with $f(N(u)) > 1$, i.e., $f(N[u]) > 2$. Furthermore, we have $f(N[v]) \geq 2$ for any $v \in V - L$. Thus $\gamma_{st}(T) = f(V) = \sum_{v \in V-L}(f(N[v]) - d_{T_1}(T)) = \sum_{v \in V-L} f(N[v]) - \sum_{v \in V-L} d_{T_1}(v) > \sum_{v \in V-L} 2 - 2|E_1| = 2(|V_1| - |E_1|) = 2$, a contradiction.

Since $f(N(v)) = 1$ for every $v \in V$, it follows that $d_T(V)$ is odd and v is adjacent to $\frac{d_T(v)-1}{2}$ vertices in M_f. Further, since $M_f \subseteq L$, thus v is at least adjacent to $\frac{d_T(v)-1}{2}$ leaves of T. This completes the proof. ∎

Corollary 2. *For any tree $T = (V, E)$ on n vertices ($n \geq 2$),*

$$\gamma_{ks}(T) \geq \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By Lemma 1, $\gamma_{st}(T) \geq 2$. Especially, if n is odd, note that there exists at least one even vertex in T, then $\gamma_{st}(T) \geq 3$. ∎

By Corollary 1 and Corollary 2, we have

Theorem 4. *For any tree T on n vertices ($n \geq 2$),*

$$\gamma_{ks}^t(T) \geq \begin{cases} 2-n & \text{if } k \leq n-1, \\ 2 & \text{if } k = n \text{ and } n \text{ is even}, \\ 3 & \text{if } k = n \text{ and } n \text{ is odd}. \end{cases}$$

with equality for $T = K_{1,n-1}$.

Theorem 5. *For any cycle* C_n $(n \geq 3)$,

$$\gamma_{ks}^t(C_n) = \begin{cases} 0 & \text{if } n \text{ is even and } k = \frac{n}{2}, \\ n & \text{if } k = n, \\ 2k-n+2 & \text{otherwise}. \end{cases}$$

Proof. Let $C_n : v_1v_2 \ldots v_nv_1$ be a cycle on n vertices and f be a minimum TkSF on $C_n = (V, E)$. Let I denote the set of all isolated vertices in $C_n[C_f]$.

If $k = n$, it is obvious that $\gamma_{ks}^t(C_n) = n$.

If $k \leq n-1$, clearly, for any $v \in C_f$, $N(v) \subseteq P_f$ and $|N(v)| = 2$. Thus $|P_f| \geq |C_f| \geq k$, $|M_f| \leq n-k$, i.e., $\gamma_{ks}^t(C_n) = |P_f| - |M_f| \geq 2k-n$. Especially, if there exists $i(i \geq 2)$ consecutive vertices in $C_n[C_f]$, without loss of generality, let $v_1, v_2, \ldots, v_i \in C_f$, then $v_n, v_1, v_2, \ldots, v_i, v_{i+1} \in P_f$. In this case, $|P_f| \geq k+1$, $|M_f| \leq n-k-1$, i.e., $\gamma_{ks}^t(C_n) = |P_f| - |M_f| \geq 2k-n+2$.

From the above analysis, we have the conclusions as follows:

Case 1. n is even and $k = \frac{n}{2}$.

Then $\gamma_{ks}^t(C_n) \geq 2k-n = 0$.

Case 2. $k \leq \frac{n-1}{2}$.

Case 2.1. $I = C_f$.

This is to say, all vertices in $C_n[C_f]$ are the isolated vertices. Since $N(v) \subseteq P_f$ and $|N(v)| = 2$ for any $v \in C_f$, and $k \leq \frac{n-1}{2}$, it follows that $|P_f| \geq |C_f|+1 \geq k+1$. Then $|M_f| \leq n-k-1$. Thus $\gamma_{ks}^t(C_n) \geq 2k-n+2$.

Case 2.2. $I \subset C_f$.

In this case, there exist $i(i \geq 2)$ consecutive vertices in $C_n[C_f]$. Thus $\gamma_{ks}^t(C_n) \geq 2k-n+2$.

Case 3. $\frac{n}{2} < k \leq n-1$.

Then there exist $i(i \geq 2)$ consecutive vertices in $C_n[C_f]$. Thus $\gamma_{ks}^t(C_n) \geq 2k-n+2$.

On the other hand, define the function $g : V \to \{-1, 1\}$ as follows:

Case 1. n is even and $k = \frac{n}{2}$.

Define $g(v_i) = \begin{cases} 1 & \text{if } i \text{ is odd}, \\ -1 & \text{otherwise}. \end{cases}$

Then g is a TkSF of C_n of weight $g(V) = 0$. So $\gamma_{ks}^t(C_n) \leq g(V) = 0$.

Case 2. $k \leq \frac{n-1}{2}$.

Define $g(v_i) = \begin{cases} 1 & \text{if } i \text{ is odd and } i \leq 2k+1, \\ -1 & \text{otherwise}. \end{cases}$

Then g is a TkSF of C_n of weight $g(V) = 2k-n+2$. So $\gamma_{ks}^t(C_n) \leq g(V) = 2k-n+2$.

Case 3. $\frac{n}{2} < k \leq n-2$.

Define $g(v_i) = \begin{cases} 1 & \text{if } i \text{ is odd or } i \le 2(k - \lceil \frac{n}{2} \rceil) + 2, \\ -1 & \text{otherwise.} \end{cases}$

Then g is a TkSF of C_n of weight $g(V) = 2k - n + 2$. So $\gamma_{ks}^t(C_n) \le g(V) = 2k - n + 2$.

Case 4. $k = n - 1$.

Define $g(v_i) = 1$ for $1 \le i \le n$. Then g is a TkSF of C_n of weight $g(V) = n$. So $\gamma_{ks}^t(C_n) \le g(V) = n = 2k - n + 2$.

The result now follows. ∎

Theorem 6. *For any path P_n $(n \ge 2)$,*

$$\gamma_{ks}^t(P_n) = \begin{cases} -1 & \text{if } n \text{ is odd and } k = \frac{n+1}{2}, \\ 2k - n & \text{otherwise.} \end{cases}$$

Proof. Let $P_n : v_1 v_2 \dots v_n$ be a path on n vertices, and f be a minimum TkSF on $P_n = (V, E)$. Thus $\gamma_{ks}^t(P_n) = |P_f| - |M_f|$. Let I denote the set of all isolated vertices in $P_n[C_f]$.

Case 1. $I = C_f$.

This is to say, all vertices in $P_n[C_f]$ are the isolated vertices. If n is odd, $k \le |C_f| \le \frac{n+1}{2}$; If n is even, $k \le |C_f| \le \frac{n}{2}$. Clearly, for any $v \in C_f$, $N(v) \subseteq P_f$ and $|N(v)| = \begin{cases} 1 & \text{if } v = v_1 \text{ or } v_n, \\ 2 & \text{otherwise.} \end{cases}$

Case 1.1 n is even.

Obviously, $|P_f| \ge |C_f| \ge k$, then $|M_f| \le n - k$. Thus $\gamma_{ks}^t(P_n) \ge 2k - n$.

Case 1.2. n is odd.

If $k = \frac{n+1}{2}$, then $|P_f| \ge |C_f| - 1 \ge k - 1$, then $|M_f| \le k$. Thus $\gamma_{ks}^t(P_n) \ge -1$; If $k < \frac{n+1}{2}$, then $|P_f| \ge |C_f| \ge k$, then $|M_f| \le n - k$. Thus $\gamma_{ks}^t(P_n) \ge 2k - n$.

Case 2. $I \subset C_f$.

Clearly, $C_f - I \subset P_f$. (1)

Furthermore, for every vertex $v \in I$, $N(v) \subseteq P_f$ and $|N(v)| \in \{1, 2\}$. (2)

It follows from (1) and (2) that $|P_f| \ge |C_f| \ge k$. Then $|M_f| \le n - k$. Thus $\gamma_{ks}^t(P_n) \ge 2k - n$.

Consequently, $\gamma_{ks}^t(P_n) \ge \begin{cases} -1 & \text{if } n \text{ is odd and } k = \frac{n+1}{2}, \\ 2k - n & \text{otherwise.} \end{cases}$

On the other hand, define a function $g : V \to \{-1, 1\}$ as follows:

Case 1. $k \le \frac{n}{2}$.

Define $g(v_i) = \begin{cases} 1 & \text{if } i \text{ is even and } 2 \le i \le 2k, \\ -1 & \text{otherwise.} \end{cases}$

Then g is a TkSF of P_n with weight $g(V) = 2k - n$. So $\gamma_{ks}^t(P_n) \le 2k - n$.

Case 2. n is odd and $k = \frac{n+1}{2}$.

Define $g(v_i) = \begin{cases} 1 & \text{if } i \text{ is even,} \\ -1 & \text{otherwise.} \end{cases}$

Then g is a TkSF of P_n with weight $g(V) = -1$. So $\gamma_{ks}^t(P_n) \le -1$.

Case 3. n is even and $\frac{n}{2} < k \le n$.

Define $g(v_i) = \begin{cases} 1 & \text{if } i \text{ is odd or } i \le 2k-n, \\ -1 & \text{otherwise.} \end{cases}$

Then g is a TkSF of P_n with weight $g(V) = 2k-n$. So $\gamma^t_{ks}(P_n) \le 2k-n$.

Case 4. n is odd and $\frac{n+1}{2} < k \le n$.

Define $g(v_i) = \begin{cases} 1 & \text{if } i \text{ is even or } i \le 2k-n, \\ -1 & \text{otherwise.} \end{cases}$

Then g is a TkSF of P_n with weight $g(V) = 2k-n$. So $\gamma^t_{ks}(P_n) \le 2k-n$.

Consequently, $\gamma^t_{ks}(P_n) \ge \begin{cases} -1 & \text{if } n \text{ is odd and } k = \frac{n+1}{2}, \\ 2k-n & \text{otherwise.} \end{cases}$

The result now follows. ∎

On the upper bounds of a tree T, we have the following conjecture.

Conjecture 1. *For any tree T of order $n (n \ge 2)$,*

$\gamma^t_{ks}(T) \le \gamma^t_{ks}(P_n) = \begin{cases} -1 & \text{if } n \text{ is odd and } k = \frac{n+1}{2}, \\ 2k-n & \text{otherwise.} \end{cases}$

4 Lower bounds on total k-subdomination number

Theorem 7. *For any graph G of order n and maximum Δ, minimum degree $\delta \ge 1$,*

$$\gamma^t_{ks}(G) \ge \frac{(\delta - 3\Delta)n + 2(\Delta+1)k}{\Delta + \delta} n.$$

Proof. Let f be a minimum TkSF on $G = (V, E)$. Let P_f and M_f be the sets of vertices in G that are assigned the values +1 and −1 under f, respectively. Let $P_f = P_\Delta \cup P_\delta \cup P_\Theta$ where P_Δ and P_δ are sets of all vertices of P_f with degree equal to Δ and δ, respectively, and P_Θ contains all other vertices in P_f, if any. Let $M_f = M_\Delta \cup M_\delta \cup M_\Theta$ where P_Δ, P_δ and P_Θ are defined similarly. Further, for $i \in \{\Delta, \delta, \Theta\}$, let V_i be defined by $V_i = P_i \cup M_i$. Thus $n = |V_\Delta| + |V_\delta| + |V_\Theta|$.

Since for at least k vertices $v \in V$, $f(N(v)) \ge 1$, we have

$$\sum_{v \in V} f(N(v)) \ge k - \Delta(n-k) = (\Delta+1)k - \Delta n.$$

The sum $\sum_{v \in V} f(N(v))$ counts the value $f(v)$ exactly $d(v)$ times for each vertex $v \in V$, i.e., $\sum_{v \in V} f(N(v)) = \sum_{v \in V} f(v)d(v)$. Thus

$$\sum_{v \in V} f(v)d(v) \ge (\Delta+1)k - \Delta n.$$

Breaking the sum up into the six summations and replacing $f(v)$ with the corresponding value of 1 or −1 yields

$\sum_{v\in P_\Delta} d(v)+\sum_{v\in P_\delta} d(v)+\sum_{v\in P_\Theta} d(v)-\sum_{v\in M_\Delta} d(v)-\sum_{v\in M_\delta} d(v)-\sum_{v\in M_\Theta} d(v)\geq(\Delta+1)k-\Delta n.$

We know that $d(v)=\Delta$ for all v in P_Δ or M_Δ, and $d(v)=\delta$ for all v in P_δ or M_δ. For any vertex v in either P_Θ or M_Θ, $\delta+1\leq d(v)\leq\Delta-1$. Therefore, we have

$\Delta|P_\Delta|+\delta|P_\delta|+(\Delta-1)|P_\Theta|-\Delta|M_\Delta|-\delta|M_\delta|-(\delta+1)|M_\Theta|\geq(\Delta+1)k-\Delta n.$

For $i\in\{\Delta,\delta,\Theta\}$, we replace $|P_i|$ with $|V_i|-|M_i|$ in the above inequality. Therefore, we have

$$\Delta|V_\Delta|+\delta|V_\delta|+(\Delta-1)|V_\Theta| \geq(\Delta+1)k-\Delta n+2\Delta|M_\Delta|+2\delta|M_\delta|+(\Delta+\delta)|M_\Theta|.$$

It follows that

$$\begin{aligned} &2\Delta n-(\Delta+1)k\\ &\geq 2\Delta|M_\Delta|+2\delta|M_\delta|+(\Delta+\delta)|M_\Theta|+(\Delta-\delta)|V_\delta|+|V_\Theta|\\ &=2\Delta|M_\Delta|+2\delta|M_\delta|+(\Delta+\delta)|M_\Theta|+(\Delta-\delta)(|P_\delta|+|M_\delta|)+(|P_\Theta|+|M_\Theta|)\\ &=2\Delta|M_\Delta|+(\Delta+\delta)|M_\delta|+(\Delta+\delta+1)|M_\Theta|+(\Delta-\delta)|P_\delta|+|P_\Theta|\\ &\geq(\Delta+\delta)|M_\Delta|+(\Delta+\delta)|M_\delta|+(\Delta+\delta)|M_\Theta|\\ &=(\Delta+\delta)|M_f|. \end{aligned}$$

Therefore,

$$|M_f|\leq\frac{2\Delta n-(\Delta+1)k}{\Delta+\delta}n.$$

So

$$\gamma_{ks}^t(G)=|P_f|-|M_f|=n-2|M_f|\geq n-2\frac{2\Delta n-(\Delta+1)k}{\Delta+\delta} =\frac{(\delta-3\Delta)n+2(\Delta+1)k}{\Delta+\delta}.$$ ∎

Theorem 8. *If a graph G has no isolated vertices and every vertex in G is an even vertex, then*

$$\gamma_{ks}^t(G)\geq\frac{(\delta-3\Delta)n+2(\Delta+2)k}{\Delta+\delta}.$$

Proof. If every vertex in G is an even vertex, then there exist k vertices in V such that $f(N(v))\geq 2$. Then $\sum_{v\in V} f(N(v))\geq 2k-\Delta(n-k)=(\Delta+2)k-\Delta n$. Similar to the proof of Theorem 7, we can finish the proof of Theorem 8. ∎

Theorem 9. *For every r-regular graph G of order n,*

$$\gamma_{ks}^t(G)\geq\begin{cases}\frac{(r+1)k-rn}{r} & \text{if } r \text{ is odd,}\\ \frac{(r+2)k-rn}{r} & \text{if } r \text{ is even.}\end{cases}$$

Proof. If r is odd, by Theorem 7, we have $\gamma_{ks}^t(G)\geq\frac{(r+1)k-rn}{r}$. If k is even, by Theorem 8, we have $\gamma_{ks}^t(G)\geq\frac{(r+2)k-rn}{r}$. ∎

Corollary 3. [5] *For every r-regular graph G of order n,*

$$\gamma_{ks}^t(G)\geq\begin{cases}\frac{n}{r} & \text{if } r \text{ is odd,}\\ \frac{2n}{r} & \text{if } r \text{ is even.}\end{cases}$$

We have recently learned from the referee that some of the results of this paper have been obtained independently by Harris, Hattingh, and Henning in [6] and by Henning in [7]. Lemma 1 appears in [7] in a slightly different form. Theorems 5, 6, and 7 appear in [6]. Conjecture 1 is proved in [6].

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