Matricial construction of $k - colourings$ on square lattice *

J.M. Marín †, A. Márquez ‡ and M.P. Revuelta §
Departamento de Matemática Aplicada I.
Universidad de Sevilla (Spain).

December 4, 2003

Abstract

The paper contains two main results. First, we obtain the chromatic polynomial on the $n \times m$ section of the square lattice solving a problem propose by Read and Tutte [5], the chromatic polynomial of the *bracelet* square lattice and we find a recurrent- constructive process for the matrices of the $k - colourings$. The key concept for obtaining the inductive method is the *compatible matrix*.

Our second main result deals with the compatible matrix as the adjacency matrix of a graph. This represents a family of graphs, which is described.

1 Introduction

For positive integer k, a $k - colouring$ of a graph G is a mapping of the vertex set $V(G)$ into the set $I_k = \{1, 2, ..., k\}$ such that if $e = (i, j)$ is any edge of G, $\phi(i) \neq \phi(j)$. The members of I_k are the k colours. The number of such mappings ϕ is now commonly known as the *chromatic polynomial* of G and it is denoted by $\chi(G; k)$. Therefore the *chromatic polynomial* is a function which gives the number of ways of colouring a graph G when k colours are available.

Let $L(n, m)$ be a graph having as vertices the set

$$\{(i, j) \in Z^+ \times Z^+ : 1 \leq i \leq n, 1 \leq j \leq m\}$$

*Partially supported by projects BFM2001-2474-ORI and PAI FQM-164
†jmarin@us.es
‡almar@us.es
§pastora@us.es

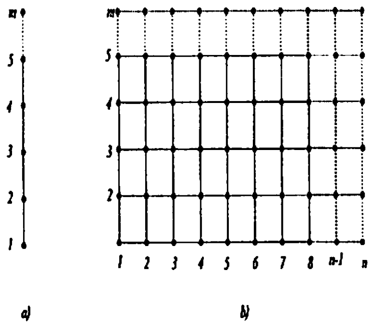


Figure 1: $L(8,5)$

such that the vertices (i,j), (i',j') are adjacent if and only if $|i-i'|+|j-j'|=1$. $L(n,m)$ is an $n\times m$ section of the square lattice (see Figure 1). We will say that $L(n,n)$, denoted by simplicity like L_n, is the square lattice having n^2 vertices and $2n(n-1)$ edges. Let $\chi(L_n;k)$ be the number of $k-$*colourings* of the square lattice L_n.

Let L_n^T be the graph obtained from the square lattice L_{n+1} by identifying the boundary vertices $(i,1)$ and $(i,n+1)$, for $1\le i\le n+1$, and the vertices $(1,j)$ and $(n+1,j)$, for $1\le j\le n+1$, and deleting any parallel edge. So L_n^T can be embedded in the torus and every vertex has degree four. This is often referred to as the toroidal square lattice. Let $\chi(L_n^T;k)$ be the number of $k-$*colourings* of L_n^T. It is known [1] that for a fixed integer $k\ge 3$ the limits of the sequences $\{(\chi(L_n^T;k))^{1/n^2}\}$ and $\{(\chi(L_n;k))^{1/n^2}\}$ are equal and this limit is denoted by $\hat{\chi}(k)$. Besides its intrinsic mathematical interest, the importance in computing $\hat{\chi}(k)$ is mainly because of the significance of the square lattice in statistical physics. Biggs and Meredith in [3] obtained the estimate

$$\hat{\chi}(k)\sim\frac{1}{2}(k-3+\sqrt{k^2-2k+5}),$$

where $g(k)\sim h(k)$ has its usual meaning that the limit as $k\to\infty$ of $g(k)/h(k)$ is 1.

Lower and upper bounds for $\hat{\chi}(k)$ were given by Biggs in [2]. He used the transfer matrix technique to obtain

$$\frac{k^2-3k+3}{k-1}\le\hat{\chi}(k)\le\frac{1}{2}(k-2+\sqrt{k^2-4k+8}).$$

This paper contains two main results. First, we obtain the chromatic polynomial on the $n\times m$ section of the square lattice solving the Problem 8.1 propose by Read and Tutte in [5], the chromatic polynomial of the

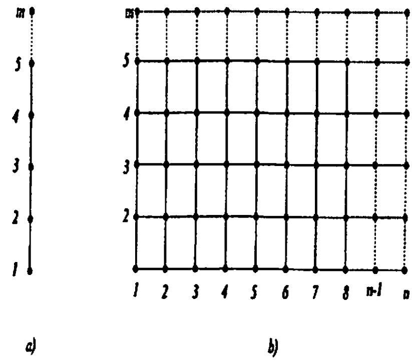


Figure 2: *a)* $L(1,m)$ *b)* $L(n,m)$

bracelet square lattice proving in addition that the behavior of chromatic polynomial on the square lattice is the same that on the cylindrical square lattice. The key concept for obtaining the chromatic polynomial on the square lattice is the *compatible matrix* finding a recurrent- constructive process for the matrices of the $k-$ *colourings*.

Our second main result has to do with the compatible matrix as the adjacency matrix of a graph. This represents a family of graphs, which is described in the last section.

We conclude the paper with some remarks and open problems.

2 Compatible matrix of $k-$*colourings* on square lattice

We consider the problem of counting $k-$ *colourings* on the $n \times m$ section of the square lattice denoted by $L(n,m)$.

If we want to obtain the number of colourings of $L(n,m)$ from the number of colourings of $L(n-1,m)$, then it is sufficient to study how many colourings of $L(1,m)$ we can add to $L(n-1,m)$ such that a colouring of $L(n,m)$ will be obtained. The fact that we can add or not a colouring to $L(n-1,m)$ will depend only of its end pattern of colouring, i.e. the colouring of a graph of type $L(1,m)$ which it is obtaining if we take in $L(n-1,m)$ the points $\{(n-1,i) : 1 \leq i \leq m\}$. Therefore, we have to study the compatibility between colourings of $L(1,m)$, solving in this way the problem (see Figure 2).

For $m = 1$ the problem is easy due to each k-colouring of $L(1,1)$ is compatible with $(k-1)$ k-colourings of $L(1,1)$. Hence, $\chi(L(1,m);k) =$

$k(k-1)^{m-1}$. The same situation is repeated for $m=2$, each k-colouring of $L(1,2)$ is compatible with k^2-3k+3 k-colourings of $L(1,2)$. Therefore $\chi(L(2,m);k) = k(k-1)(k^2-3k+3)^{m-1}$. For $m=3$ the situation is different. If we hope to obtain the chromatic polynomial of $L(3,m)$, with a direct method of counting, then the situation becomes much harder. Therefore, we will use the compatible matrix technique (see[4]).

Two $k-$ *colourings*, a_i and a_j with $i \neq j$ of $L(1,m)$ are *compatible* if $a_i(v) \neq a_j(v)$ for each vertex v of $L(1,m)$. We denote by $C_m(k)$ to a *compatible matrix*, whose rows and columns correspond to the $k-$ *colourings* of $L(1,m)$, in the following way:

$$(C_m(k))_{i,j} = \left\{ \begin{array}{ll} 1 & \text{if } a_i \text{ and } a_j \text{ are compatible} \\ 0 & \text{otherwise} \end{array} \right\}$$

Firstly, we will see a particular case of 3-colourings, to help the reader to understand the process better:

Let a_1^{m-1}, a_2^{m-1} and a_3^{m-1} be all proper 3-colourings of $L(1,m-1)$ such that its end patterns (i.e. the colour assigned to the finish vertex of $L(1,m-1)$) are a_1, a_2, a_3 respectively.

It is easy to see that the following expression is verified:

$$\begin{bmatrix} a_1^m \\ a_2^m \\ a_3^m \end{bmatrix} = \begin{bmatrix} a_2^{m-1}+a_3^{m-1} \\ a_1^{m-1}+a_3^{m-1} \\ a_1^{m-1}+a_2^{m-1} \end{bmatrix} = C^{m-1} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

where $C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is the compatible matrix of $L(1,m)$ for 3-colourings.

Let us see the general case now for k-colourings:

Theorem 1 *The chromatic polynomial of $L(n,m)$ is given by*
$\chi(L(n,m);k) = \sum_{i,j} (C_m(k))_{i,j}^{n-1}$.

Proof. Let $a_1^n, a_2^n, \ldots, a_r^n$ be all proper $k-$ *colourings* of the graph $L(n,m)$ such that its end patterns are $a_1, a_2, \ldots, a_r$ respectively.

We denote by $C = C_m(k)$.

We have: $\begin{bmatrix} a_1^n \\ a_2^n \\ \vdots \\ a_r^n \end{bmatrix} = C \begin{bmatrix} a_1^{n-1} \\ a_2^{n-1} \\ \vdots \\ a_r^{n-1} \end{bmatrix} = C^{n-1} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix}$

Since, $(a_i^n) = C^{n-1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ due to $a_i^1 = 1$ for each i.

then,

$\chi(L(n,m);k) = \sum_{i=1}^{r}(a_i^n) = \sum_{i,j}(C)_{i,j}^{n-1}$

Thus, we obtain the result. ■

We obtain next the number of $3-colourings$ of the graph $L(1,m)$ which can not be closed. It is denoted by *bracelet*. Let $\overset{\mho}{\chi}(L(1,m);k)$ be the number of $k-colourings$ of the graph $L(1,m)=\{v_1,v_2,...,v_m\}$ where the colour associate to v_1 is the same to the colour associate to v_m.

Theorem 2 $\overset{\mho}{\chi}(L(1,m);k) = 2^{m-1} - 2(-1)^m$.

Proof. First, we consider $\begin{bmatrix} a_1^1 \\ a_2^1 \\ a_3^1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

and the compatible matrix $C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Using elementary tools for the compute of the m-th power of a matrix we obtain the following: $C^m = \begin{bmatrix} a_m & b_m & b_m \\ b_m & a_m & b_m \\ b_m & b_m & a_m \end{bmatrix}$

where $\left\{ \begin{array}{c} a_m = \frac{1}{3}(2^m + 2(-1)^m) \\ b_m = \frac{1}{3}(2^m - (-1)^m) \end{array} \right\}$

$trace(C^m) = 2^m + 2(-1)^m$,

and

$\sum_{i,j}(C^{m-1})_{i,j} = 3a_{m-1} + 6b_{m-1} = (2^{m-1} + 2(-1)^{m-1}) + 6\frac{1}{3}(2^{m-1} - (-1)^{m-1}) = 3 \times 2^{m-1}$.

Thus,

$\overset{\mho}{\chi}(L(1,m);3) = \sum_{i,j}(C^{m-1})_{i,j} - trace(C^m) = 3\times 2^{m-1} - (2^m + 2(-1)^m) = = 2^{m-1} - 2(-1)^m$. ■

We can generalize the previous result for $L(n,m)$, obtaining the chromatic polynomial on the $n\times m$ section of the *bracelet*, denoted by $\overset{\mho}{\chi}(L(n,m);k)$.

Theorem 3 $\overset{\mho}{\chi}(L(n,m);k) = \sum_{i,j}(C^{n-1})_{i,j} - trace(C^n)$

Proof. It is clear that the number of $k-colourings$ on the $n \times m$ section of the square lattice which can not be closed will be equal to the sum of all elements of the matrix C^{n-1}(i.e. $\sum_{i,j}(C^{n-1})_{i,j} = k(k-1)^{n-1}$)

minus the number of proper $k-colourings$ on the $n \times m$ section of the square lattice when we identify the boundary vertices $(i,1)$ and (i,n), for $1 \le i \le n$, and deleting any parallel edge, i.e. $trace(C^n)$. ∎

It is well-known that the behavior of chromatic polynomial on the infinite square lattice is the same that on the toroidal square lattice (see [1]). We show now that the behavior of chromatic polynomial on the cylindrical square lattice is the same to both.

Let L_n^C be the graph obtained from the square lattice L_{n+1} by identifying the boundary vertices $(i,1)$ and $(i,n+1)$, for $1 \le i \le n+1$, and deleting any parallel edge. So L_n^C will be referred to as the cylindrical square lattice. Let $\chi(L_n^C;k)$ be the number of $k-colourings$ of L_n^C.

Proposition 4 $\lim_{n\to\infty} \chi(L_n^C;k))^{1/n^2} = \lim_{n\to\infty} \chi(L_n;k))^{1/n^2}$

Proof. Clearly, $\chi(L_n;k) \ge \chi(L_n^C;k) \ge \chi(L_n^T;k)$.
It is know (see [1]) that $\lim_{n\to\infty} \chi(L_n;k))^{1/n^2} = \lim_{n\to\infty} \chi(L_n^T;k))^{1/n^2}$.
Therefore, we have

$$\lim_{n\to\infty} \chi(L_n;k))^{1/n^2} = \lim_{n\to\infty} \chi(L_n^C;k))^{1/n^2} = \lim_{n\to\infty} \chi(L_n^T;k))^{1/n^2}.$$

∎

3 Construction of the compatible matrix

In this section, we give a constructive process to obtain a compatible matrix. We begin with $3-colourings$ to better follow the method and after that, we generalize the inductive method for the construction of the compatible matrix of $k-colourings$ on the $n \times m$ section of the square lattice.

Between $L(1,m-1)$ and $L(1,m)$, the compatible matrix for the $3-colourings$ is given by:

	a_1	a_2	a_3
a_1	0	1	1
a_2	1	0	1
a_3	1	1	0

In $L(2,m)$ the patterns of colours used in the ends are the following:

$$a_1a_2, a_1a_3, a_2a_1, a_2a_3, a_3a_1 \text{ and } a_3a_2.$$

Therefore, between $L(2,m-1)$ and $L(2,m)$, the compatible matrix is given by:

	a_1a_2	a_1a_3	a_2a_1	a_2a_3	a_3a_1	a_3a_2
a_1a_2	0	0	1	1	1	0
a_1a_3	0	0	1	0	1	1
a_2a_1	1	1	0	0	0	1
a_2a_3	1	0	0	0	1	1
a_3a_1	1	1	0	1	0	0
a_3a_2	1	0	1	1	0	0

Surprisingly, we can obtain a relationship between the compatible matrices on $L(1,m)$ and $L(2,m)$ and the most important is that this relation gives an inductive method for the construction of the compatible matrix on the $L(n,m)$. For instance, the submatrix 2×2 in the position $(1,2)$ of the compatible matrix on $L(2,m)$ is the submatrix obtained eliminating the $1-th$ row and the $2-nd$ column in the compatible matrix on $L(1,m)$. This process can be generalized except for the diagonal elements where it is clear that submatrices 2×2 with zeros are obtained. In general, we can describe the process in the following way:

$$C_m = \begin{pmatrix} \bar{0} & \bar{x} & \bar{y} \\ \bar{x}^t & \bar{0} & \bar{z} \\ \bar{y}^t & \bar{z}^t & \bar{0} \end{pmatrix}$$

$$C_{m+1} = \begin{pmatrix} \boxed{\begin{matrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{matrix}} & \boxed{\begin{matrix} \bar{x}^t & \bar{z} \\ \bar{y}^t & \bar{0} \end{matrix}} & \boxed{\begin{matrix} \bar{x}^t & \bar{0} \\ \bar{y}^t & \bar{z}^t \end{matrix}} \\ \boxed{\begin{matrix} \bar{x} & \bar{y} \\ \bar{z}^t & \bar{0} \end{matrix}} & \boxed{\begin{matrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{matrix}} & \boxed{\begin{matrix} \bar{0} & \bar{x} \\ \bar{y}^t & \bar{z}^t \end{matrix}} \\ \boxed{\begin{matrix} \bar{x} & \bar{y} \\ \bar{0} & \bar{z} \end{matrix}} & \boxed{\begin{matrix} \bar{0} & \bar{y} \\ \bar{x}^t & \bar{z} \end{matrix}} & \boxed{\begin{matrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{matrix}} \end{pmatrix}$$

$$C_{m+2} = \begin{pmatrix} \boxed{\begin{matrix} \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{matrix}} & \boxed{\begin{matrix} \bar{x} & \bar{y} & \bar{0} & \bar{x} \\ \bar{z}^t & \bar{0} & \bar{y}^t & \bar{z}^t \\ \bar{x} & \bar{y} & \bar{0} & \bar{0} \\ \bar{0} & \bar{z} & \bar{0} & \bar{0} \end{matrix}} & \boxed{\begin{matrix} \bar{x} & \bar{y} & \bar{0} & \bar{0} \\ \bar{z}^t & \bar{0} & \bar{0} & \bar{0} \\ \bar{x} & \bar{y} & \bar{0} & \bar{y} \\ \bar{0} & \bar{z} & \bar{x}^t & \bar{z} \end{matrix}} \\ \boxed{\begin{matrix} \bar{x}^t & \bar{z} & \bar{x}^t & \bar{0} \\ \bar{y}^t & \bar{0} & \bar{y}^t & \bar{z}^t \\ \bar{0} & \bar{y} & \bar{0} & \bar{0} \\ \bar{x}^t & \bar{z} & \bar{0} & \bar{0} \end{matrix}} & \boxed{\begin{matrix} \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{matrix}} & \boxed{\begin{matrix} \bar{0} & \bar{0} & \bar{x}^t & \bar{z} \\ \bar{0} & \bar{0} & \bar{y}^t & \bar{0} \\ \bar{x} & \bar{y} & \bar{0} & \bar{y} \\ \bar{0} & \bar{z} & \bar{x}^t & \bar{z} \end{matrix}} \\ \boxed{\begin{matrix} \bar{x}^t & \bar{z} & \bar{x}^t & \bar{0} \\ \bar{y}^t & \bar{0} & \bar{y}^t & \bar{z}^t \\ \bar{0} & \bar{0} & \bar{0} & \bar{x} \\ \bar{0} & \bar{0} & \bar{y}^t & \bar{z}^t \end{matrix}} & \boxed{\begin{matrix} \bar{0} & \bar{0} & \bar{x}^t & \bar{0} \\ \bar{0} & \bar{0} & \bar{y}^t & \bar{z}^t \\ \bar{x} & \bar{y} & \bar{0} & \bar{x} \\ \bar{z}^t & \bar{0} & \bar{y}^t & \bar{z}^t \end{matrix}} & \boxed{\begin{matrix} \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{matrix}} \end{pmatrix}$$

All the results described previously are generalizable for k-colourings. For them, the compatible matrix of the $k-colourings$ on the $1 \times m$ section of the square lattice is given by:

$$\begin{pmatrix} & \mid & a_1 & a_2 & \cdots & a_k \\ - & - & - & - & - & - \\ a_1 & \mid & 0 & 1 & \cdots & 1 \\ a_2 & \mid & 1 & 0 & \cdots & 1 \\ \vdots & \mid & \vdots & \vdots & \cdots & \vdots \\ a_k & \mid & 1 & 1 & \cdots & 0 \end{pmatrix}.$$

Following an inductive method as previous process defined, we can obtain the compatible matrices of the $k - colourings$ on the $n \times m$ section of the square lattice.

Bounds for the eigenvalues of $C_m(k)$

We obtain next the same bounds for the eigenvalues of $C_m(k)$ that Biggs in [2]. But, it is important to remark that we can prove that with our inductive process of construction of compatible matrix. This fact, can be relevant and from now on, due to the compatible matrices are known and these can be constructed, we will be able to try to find better bounds for the eigenvalues or exact formula for them in a future work.

Since $C_m(k)$ is a matrix with non-negative entries, the classical theory of Perron and Frobenius may be applied. Therefore, there is a unique one eigenvalue, $\lambda_m(k)$ with greatest absolute value of the $C_m(k)$ and such that:

1. It is real.
2. It has multiplicity one.
3. It is not greater than the maximum row sum $M_m(k)$ of $C_m(k)$.
4. It is not less than the mean row sum $m_m(k)$ of $C_m(k)$.

Lemma 5 $M_{m+1}(k) \leq (k-2)M_m(k) + M_{m-1}(k)$.

Proof. By the construction of the compatible matrices, we obtain that each block row in $C_{m+1}(k)$ may be considered in the following way:

Let $B_{i,.}$ be $i-th$ block row in the matrix $C_{m+1}(k)$, where

$B_{i,.} = B_{i,1} + ... + B_{i,k}$

Let f_i^m be $i-th$ block row in $C_m(k)$, let c_j^m be $j-th$ block column in $C_m(k)$ and let $e_{i,j}^m$ be the block element in the position (i,j) in the matrix $C_m(k)$. Hence, we have:

$B_{i,1} = C_m(k) - f_i^m - c_1^m + e_{i,1}^m$

$B_{i,2} = C_m(k) - f_i^m - c_2^m + e_{i,2}^m$

$B_{i,3} = C_m(k) - f_i^m - c_3^m + e_{i,3}^m$

....

$B_{i,k} = C_m(k) - f_i^m - c_k^m + e_{i,k}^m$
Therefore,
$B_{i,.} = kC_m(k) - kf_i^m - C_m(k) + f_i^m$

Considering an $i-th$ block row in the matrix $C_{m+1}(k)$ minus a diagonal block (due to in the compatible matrix a diagonal block is a null block), we obtain:

$(kC_m(k) - kf_i^m - C_m(k) + f_i^m) - (C_m(k) - f_i^m - c_i^m + e_{i,i}^m) =$
$= (k-2)(C_m(k) - f_i^m) + (c_i^m - e_{i,i}^m)$

Hence, the sum of each block rows in $C_{m+1}(k)$ is equal to $(k-2)$ times the compatible matrix $C_m(k)$ minus $i-th$ block rows in the matrix $C_m(k)$ in addition $i-th$ block columns in the matrix $C_m(k)$ minus the block element in the position (i,i) in the matrix $C_m(k)$.

In conclusion, we obtain, $M_{m+1}(k) \leq (k-2)M_m(k) + M_{m-1}(k)$. ■

Lemma 6 $m_{m+1}(k) = \frac{k^2-3k+3}{k-1} m_m(k)$

Proof. We consider now, the sum of all block rows in $C_{m+1}(k)$:

$$
\begin{array}{l}
(k-2)(C_m(k) - f_1^m) + (c_1^m - e_{1,1}^m) \\
(k-2)(C_m(k) - f_2^m) + (c_2^m - e_{2,2}^m) \\
(k-2)(C_m(k) - f_3^m) + (c_3^m - e_{3,3}^m) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(k-2)(C_m(k) - f_k^m) + (c_k^m - e_{k,k}^m) \\
\hline
(+) := \ (k^2 - 3k + 1)C_m(k)
\end{array}
$$

$m_{m+1}(k) = \frac{(k^2-3k+3)M_m(k)}{k(k-1)} = \frac{k^2-3k+3}{k-1} m_m(k)$ ■

For instance, in the case $k = 3$, as a consequence of two previous lemmas, and applying the classical theory of Perron and Frobenius is obtained:

$$m_m(3) \leq \lambda_m \leq M_m(3)$$
$$\frac{3}{2} \leq (\lambda_m)^{\frac{1}{m}} \leq \frac{1+\sqrt{5}}{2}$$

In the general case, we obtain the following result for $k-$ *colourings*.

By Lemma 5, we consider the equation $r^2 = (k-2)r + 1$. One solution is $r^* = \frac{(k-2)+\sqrt{(k-2)^2+4}}{2}$. So, $(\lambda_m)^{\frac{1}{m}} \leq r^*$.

As a consequence of two previous lemmas, we have:

$$\frac{(k^2-3k+3)}{(k-1)} \leq (\lambda_m)^{\frac{1}{m}} \leq \frac{(k-2)+\sqrt{(k-2)^2+4}}{2}$$

4 $C_m(k)$ as the adjacency matrix of a family of graphs

It is convenient to regard $C_m(k)$ as the adjacency matrix of a graph whose vertices are the $k-$ *colourings* of $L(1,m)$.

For instance, we can see, from the point of view of geometry, the following:

For $k=3$, the compatible matrix of the $1\times m$ section of the square lattice represents the adjacency matrix of a complete graph K_3,

$$\begin{array}{ccccc} & \mid & a_1 & a_2 & a_3 \\ - & - & - & - & - \\ a_1 & \mid & 0 & 1 & 1 \\ a_2 & \mid & 1 & 0 & 1 \\ a_3 & \mid & 1 & 1 & 0 \end{array} \Rrightarrow K_3 \equiv \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

For $k=4$, the compatible matrix of the $1\times m$ section of the square lattice represents the adjacency matrix of a complete graph K_4.

In general, for $k=n$, the compatible matrix of the $1\times m$ section of the square lattice represents the adjacency matrix of a complete graph K_n.

Definition 7 *We consider the adjacency matrix of a complete graph K_n. We define the adjoint matrix associate to K_n and its will be denoted by $A(K_n)$ in the following way: Each element (i,j) of the adjoin matrix is obtained as the submatrix when we eliminate the i-th row and the j-th column in the adjacency matrix of K_n.*

Example:

$$K_4 \equiv \begin{pmatrix} \boxed{0} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{0} & \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{1} & \boxed{0} & \boxed{1} \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{0} \end{pmatrix}$$

$$A(K_4) \equiv \left(\begin{array}{c|c|c|c} \begin{matrix}0&1&1\\1&0&1\\1&1&0\end{matrix} & \begin{matrix}1&1&1\\1&0&1\\1&1&0\end{matrix} & \begin{matrix}1&0&1\\1&1&1\\1&1&0\end{matrix} & \begin{matrix}1&0&1\\1&1&0\\1&1&1\end{matrix} \\ \hline \begin{matrix}1&1&1\\1&0&1\\1&1&0\end{matrix} & \begin{matrix}0&1&1\\1&0&1\\1&1&0\end{matrix} & \begin{matrix}0&1&1\\1&1&1\\1&1&0\end{matrix} & \begin{matrix}0&1&1\\1&1&0\\1&1&1\end{matrix} \\ \hline \begin{matrix}1&1&1\\0&1&1\\1&1&0\end{matrix} & \begin{matrix}0&1&1\\1&1&1\\1&1&0\end{matrix} & \begin{matrix}0&1&1\\1&0&1\\1&1&0\end{matrix} & \begin{matrix}0&1&1\\1&0&1\\1&1&1\end{matrix} \\ \hline \begin{matrix}1&1&1\\0&1&1\\1&0&1\end{matrix} & \begin{matrix}0&1&1\\1&1&1\\1&0&1\end{matrix} & \begin{matrix}0&1&1\\1&0&1\\1&1&1\end{matrix} & \begin{matrix}0&1&1\\1&0&1\\1&1&0\end{matrix} \end{array}\right)$$

The adjoint matrices are the compatible matrices except by one little different, in the diagonal boxes, in the compatible matrices, we have boxes of zeros. We will use the adjoin matrices for the proof the following result:

Theorem 8 *The compatible matrices of the n – colourings are $(n-1)$-power of K_n.*

Before of begin the proof of the Theorem 8, we can see particular cases.

1. $k = 3$

The compatible matrix of $L(1, m)$:

	a_1	a_2	a_3
a_1	0	1	1
a_2	1	0	1
a_3	1	1	0

represents as adjacency matrix to

$$K_3 \equiv \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The compatible matrix of $L(2, m)$:

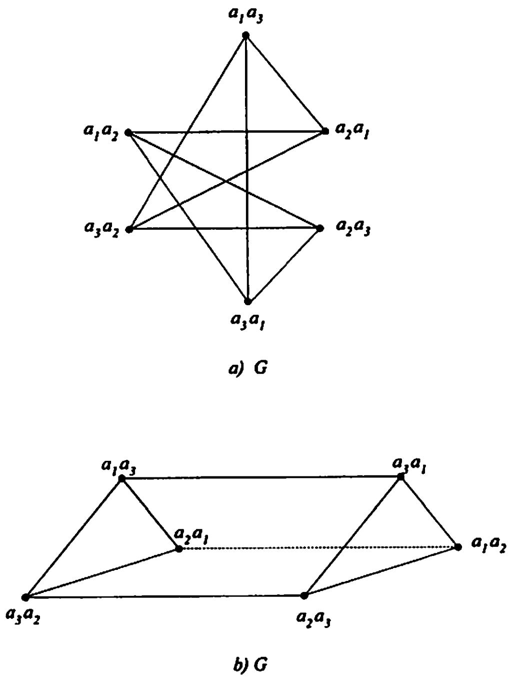


Figure 3: *a*) G, graph associate to the compatible matrix of $L(2,m)$ *b*) Other representation of G

	a_1a_2	a_1a_3	a_2a_1	a_2a_3	a_3a_1	a_3a_2
a_1a_2	0	0	1	1	1	0
a_1a_3	0	0	1	0	1	1
a_2a_1	1	1	0	0	0	1
a_2a_3	1	0	0	0	1	1
a_3a_1	1	1	0	1	0	0
a_3a_2	1	0	1	1	0	0

represents as adjacency matrix to two K_3. One of them is $\{a_1a_3, a_2a_1, a_3a_2\}$ and the other is $\{a_1a_2, a_2a_3, a_3a_1\}$ (see Figure 3).

2. $k = 4$

$$
K_4 = \left[\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
1 & 0 & 1 & 1 & 1 \\
2 & 1 & 0 & 1 & 1 \\
3 & 1 & 1 & 0 & 1 \\
4 & 1 & 1 & 1 & 0
\end{array}\right]
$$

Using the adjoint matrix we obtain the following matrix denoted by $A(K_4)$

$$
\left[\begin{array}{c|ccc|ccc|ccc|ccc}
 & \boxed{2^{(1)}} & 3 & 4 & 1 & \boxed{3^{(1)}} & 4 & 1 & 2 & \boxed{4^{(1)}} & \boxed{1^{(1)}} & 2 & 3 \\
2^{(1)} & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
3^{(2)} & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
4^{(3)} & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
\hline
 & 2 & \boxed{3^{(2)}} & 4 & 1 & 3 & \boxed{4^{(2)}} & \boxed{1^{(2)}} & 2 & 4 & 1 & \boxed{2^{(2)}} & 3 \\
1^{(3)} & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
3^{(1)} & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
4^{(2)} & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
\hline
 & 2 & 3 & \boxed{4^{(3)}} & \boxed{1^{(3)}} & 3 & 4 & 1 & \boxed{2^{(3)}} & 4 & 1 & 2 & \boxed{3^{(3)}} \\
1^{(2)} & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
2^{(3)} & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
4^{(1)} & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
\hline
 & 2 & 3 & 4 & 1 & 3 & 4 & 1 & 2 & 4 & 1 & 2 & 3 \\
1^{(1)} & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
2^{(2)} & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
3^{(3)} & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}\right]
$$

In $A(K_4)$, as an adjacency matrix, we have 12 vertices: $v_1, ..., v_{12}$, but only 4 different types, due to we have in the matrix 4 independent rows and 4 independent columns. We must observe that we obtain 3 different K_4 joint between themselves. If we eliminate the diagonal boxes, which in the compatible matrix are null blocks, only we are eliminating edges between the 3 different K_4. Therefore, in the compatible matrix we have 3 different K_4. Following the inductive process for the construction the compatible matrices we will obtain 3^2 different K_4.

Proof. For simplicity of notation, we will denote the matrices by its rows and its columns.

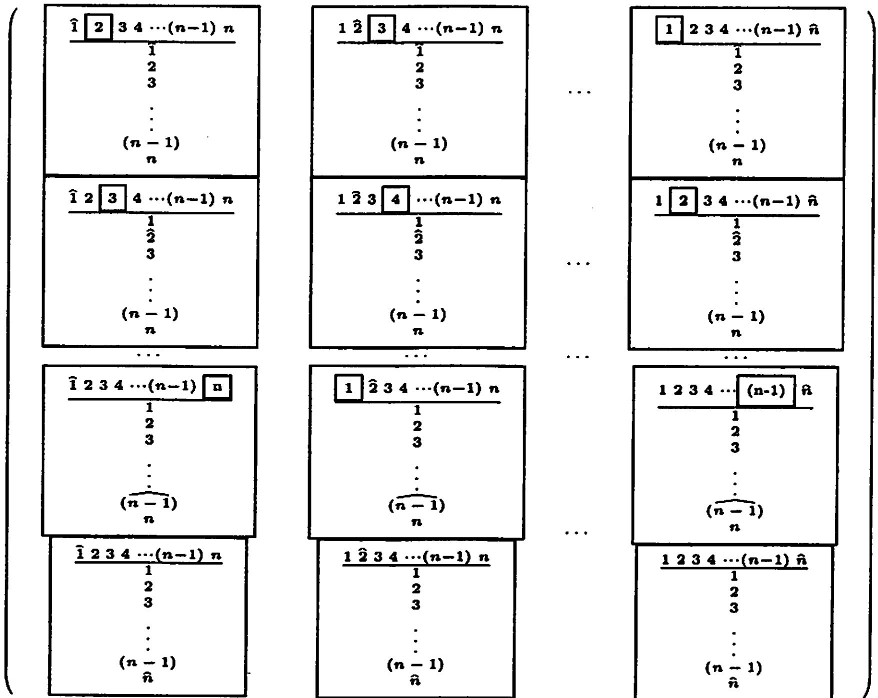
Let $12...\hat{i}...n$ be n columns in a matrix minus the $i-th$ column.

Let $\begin{matrix} 1 \\ 2 \\ \vdots \\ \hat{j} \\ \vdots \\ n \end{matrix}$ be n rows in a matrix minus the $j-th$ row.

We consider now,

$$K_n = \begin{pmatrix} & | & 1 & 2 & \cdots & n \\ - & - & - & - & - & - \\ 1 & | & 0 & 1 & \cdots & 1 \\ 2 & | & 1 & 0 & \cdots & 1 \\ \vdots & | & \vdots & \vdots & \cdots & \vdots \\ n & | & 1 & 1 & \cdots & 0 \end{pmatrix}$$

Using the adjoint matrix we obtain the following matrix denoted by $A(K_n)$



In $A(K_n)$, as an adjacency matrix, we have $(n-1)n$ vertices: $v_1, \ldots, v_{(n-1)n}$, but only n different types, because we have in the matrix n independent rows and n independent columns. Clearly, we can see that we obtain $(n-1)$ different K_n joint between themselves. If we eliminate the diagonal boxes, which in the compatible matrix are null blocks, only we are eliminating edges between the $(n-1)$ different K_n. Therefore, in the compatible matrix we have $(n-1)$ different K_n. Following the inductive process for the construction the compatible matrices we will obtain $(n-1)^2$ different K_n. ■

5 Conclusion and Open problems

We have obtained the chromatic polynomial on the $m \times n$ section of the square lattice and the *bracelet* square lattice and we have found a recurrent-constructive process for the matrices of the *k – colourings*. In addition, the compatible matrix as the adjacency matrix of a graph, represents a family of graphs, which is described.

As it was mentioned previously, we will work to find exact formulas for the eigenvalues of the compatible matrix. On the other hand, it is well-known that if G is a planar graph then the Flow Polynomial of G is essentially the Chromatic Polynomial of a dual graph of G. Therefore, in the future work we will try to get the flow polynomial on square lattice through compatible matrix and obtain a relation between the compatible matrices to the chromatic and the flow polynomials, so that a relationship between the family of graphs which are represented by compatible matrices.

Acknowledgements

We are very grateful to an anonymous referee for carefully reviewing the manuscript and for making useful suggestions.

References

[1] N.L. Biggs, Chromatic and thermodynamic limits, J. Phys. A **8** (1975) L 110–112.

[2] N.L. Biggs, Colouring square lattice graphs, Bull. London Math. Soc. **9** (1977) 54–56.

[3] N.L. Biggs and G.H.J. Meredith, Approximations for chromatic polynomials, J. Comb. Theory, Ser. B **20** (1) (1976) 5–19.

[4] N.L. Biggs, A matrix method for chromatic polynomials, LSE Report LSE-CDAM-99-03 http://www.cdam.lse.ac.uk/Reports/

[5] R.C. Read and W.T. Tutte, Chromatic Polynomials, Selected Topics in Graph Theory 3 (edited by L.W. Beineke and R.J. Wilson). Academic Press, London, 1988.