A note on graphs with largest total k-domination number

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Abstract: Let $k \geq 1$ be an integer and let G be a graph of order p. A set S of vertices in a graph is a total k-dominating set if every vertex of G is within distance at most k from some vertex of S other than itself. The smallest cardinality of such a set of vertices is called the total k-domination number of the graph and is denoted by $\gamma_k^t(G)$. It is well known that $\gamma_k^t(G) \leq \frac{2p}{2k+1}$ for $p \geq 2k+1$. In this paper, we present a characterization of connected graphs that achieve the upper bound. Furthermore, we characterize the connected graph G with $\gamma_k^t(G) + \gamma_k^t(\overline{G}) = \frac{2p}{2k+1} + 2$.
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1 Introduction

Let $G = (V, E)$ be a simple graph of order p. The degree and neighborhood of a vertex v in the graph G are denoted by $d(v)$ and $N(v)$ respectively. A vertex v is called a *leaf* if $d(v) = 1$. The graph induced by $S \subseteq V$ is denoted by $\langle S \rangle$. For arbitrary two vertices $u, v \in V(G)$, let $u - v$ denote a path between u and v in G. Further, the *distance* $d(u, v)$ between two vertices u and v of G is the length of a shortest $u - v$ path if one exists; otherwise $d(u, v) = \infty$. *Eccentricity* $e(v)$ of a vertex v of a connected graph G is the number $\max_{u \in V(G)} d(u, v)$. The *radius* is defined as $\min_{v \in V(G)} e(v)$, while the *diameter* is defined as $\max_{v \in V(G)} e(v)$. Let $rad(G)$ and $diam(G)$

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denote the radius and diameter of G, respectively. A vertex v is a *central vertex* if $e(v) = rad(G)$.

Let $k \geq 1$ be an integer. A set S of vertices in a graph G is a *k-dominating set* if every vertex of G is within distance at most k from some vertex of S. The smallest cardinality of such a set of vertices is called the *k-domination number* of the graph and is denoted by $\gamma_k(G)$. A set S of vertices in a graph G is a *total k-dominating set* of G if every vertex of G is within distance at most k from some vertex of S other than itself. The smallest cardinality of such a set of vertices is called the *total k-domination number* of the graph and is denoted by $\gamma_k^t(G)$. From now on, for a graph G and a positive integer k we denote by $G \circ k$($G \circ 2k$) the graph obtained by taking one copy of G and $|V(G)|$ copies of the path P_k (P_{2k}, resp.) of length $k-1$ ($2k-1$, resp.), and then joining the ith vertex of G to exactly one leaf in the ith copy of P_k (P_{2k}, resp.). Henning et al. [1] gave the following results:

Lemma 1 [1] *For an integer $k \geq 1$, if G is a connected graph of order p, then $\gamma_k(G) = 1$ if $p \leq k+1$ and $\gamma_k(G) \leq \frac{p}{k+1}$ if $p \geq k+1$.*

Lemma 2 [1] *For an integer $k \geq 2$, if G is a connected graph of order p, then $\gamma_k^t(G) = 2$ if $2 \leq p \leq 2k+1$ and $\gamma_k^t(G) \leq \frac{2p}{2k+1}$ if $p \geq 2k+1$.*

Lemma 3 [1] *For an integer $k \geq 2$, if G and $\overline{G}$ are connected graphs of order p, then $\gamma_k^t(G)+\gamma_k^t(\overline{G}) = 4$ if $p \leq 2k+1$ and $\gamma_k^t(G)+\gamma_k^t(\overline{G}) \leq \frac{2p}{2k+1}+2$ if $p \geq 2k+2$.*

Jerzy Topp and Lutz Volkmann [2] characterize the connected graphs that achieve the upper bound of Lemma 1. They have the following results.

Lemma 4 [2] *Let G be a connected graph of order $(k+1)n$. Then $\gamma_k(G) = n$ if and only if at least one of the following condition holds:*

(1) G is any connected graph of order $k+1$;

(2) $G = C_{2k+2}$;

(3) $G = H \circ k$ for some connected graph H of order n.

In this paper, we characterize the connected graphs that achieve the upper bound of Lemma 2 and Lemma 3.

2 Main results

It follows from the definition that $G\circ 2k$ has exactly $(2k+1)|V(G)|$ vertices. If G has no isolated vertices, then $G \circ 2k$ has exactly $|V(G)|$ leaves. For a vertex u of G, we denote by $\overline{u}$ the only leaf of $G\circ 2k$ which is at distance $2k$ from u. In addition, for a vertex v of $G \circ 2k$, we denote by $t(v)$ the unique vertex of G such that v belongs to the $t(v) - \overline{t(v)}$ path.

Theorem 1 *For any connected graph H of order n, $\gamma_k^t(H \circ 2k) = 2n$.*

Proof Assume $V(H) = \{v_1, v_2, \cdots, v_n\}$ and $V(H') = \{v_i^{'} | v_i^{'}$ belongs to the path $v_i - \overline{v_i}$ and $d(v_i, v_i^{'}) = k$ for $i = 1, 2, \cdots, n\}$. Let D be a total

k-dominating set of $H \circ 2k$ with the smallest cardinality. It follows from the structure of $H \circ 2k$ that at least two vertices of the $v_i - \overline{v_i}$ belong to D. Thus, $|D| \geq 2n$. Since $V(H) \cup V(H')$ is a total k-dominating set of $H \circ 2k$ with cardinality $2n$, it follows that $\gamma_k^t(H \circ 2k) = 2n$.

Theorem 2 *Let T be a tree of order $(2k+1)n$ and $n \geq 2$. If $\gamma_k^t(T) = 2n$, then $diam(T) \geq 4k+1$.*

Proof If $diam(T) \leq 2k$, then $rad(T) \leq k$. Hence, a central vertex of T and any other vertex of T form a total k-dominating set of T. So $\gamma_k^t(T) = 2$, which is a contradiction. Hence, $diam(T) \geq 2k+1$. In order to prove Theorem 2, we only consider the following claims.

Claim 1 $diam(T) \geq 3k+1$.

Otherwise, $2k+1 \leq diam(T) \leq 3k$. Let $diam(T) = d$ and let u and v be two vertices of T such that $d(u,v) = d$. Denote by $P : u = u_0, u_1, \cdots, u_d = v$ the $u - v$ path in T. Denote by T_1, T_2 and T_3 the components of $T - \{u_k u_{k+1}, u_{d-k-1} u_{d-k}\}$ that contain u, v and u_{k+1} respectively. Since $2k + 1 \leq diam(T) \leq 3k$, $d(u_k, u_{d-k}) = d - 2k \leq k$. Moreover, since P is the longest path in T, u_k (u_{d-k}, respectively) is at distance at most k from every vertex in T_1 (T_2, respectively).

case 1 $|V(T_3)| \geq 2k+1$. By Lemma 2, $\gamma_k^t(T_3) \leq \frac{2|V(T_3)|}{2k+1}$. Hence, there is a total k-dominating set D_3 of T_3 with $|D_3| \leq \frac{2|V(T_3)|}{2k+1}$. So, $D_3 \cup \{u_k, u_{d-k}\}$ is a total k-dominating set of T. Thus,

$$\begin{aligned}
\gamma_k^t(T) &\leq |D_3| + 2 \\
&\leq \frac{2|V(T_3)|}{2k+1} + 2 \\
&= \frac{2(|V(T)| - |V(T_1)| - |V(T_2)|)}{2k+1} + 2 \\
&= \frac{2|V(T)|}{2k+1} + 2 - \frac{2(|V(T_1)| + |V(T_2)|)}{2k+1} \\
&\leq \frac{2|V(T)|}{2k+1} + 2 - \frac{2(2k+2)}{2k+1} \\
&< \frac{2|V(T)|}{2k+1} \\
&= 2n
\end{aligned}$$

which is a contradiction.

case 2 $|V(T_3)| \leq 2k$. Then $diam(T_3) \leq 2k-1$ and $rad(T_3) \leq k$. Hence, there is a central vertex w (say) of T_3 such that it is at distance at most k from at least one of u_k or u_{d-k} and from each vertex of T_3. Otherwise, if $d(w, u_k) = d(w, u_{d-k}) = k+1$, either w is on P and $d(u_{k+1}, u_{d-k-1}) = 2(k+1) - 2 = 2k > |V(T_3)| - 1$ or w is not on P and $|V(T_3)| \geq 2k + 1$, both contradictions. Thus, $\gamma_k^t(T) \leq |\{u_k, w, u_{d-k}\}| = 3$, which is a contradiction.

Claim 2 If $n \geq 3$ and for each edge e of T at least one component of $T - e$ is of order less than $2k+1$, then $diam(T) \geq 4k+1$.

Otherwise, $3k+1 \leq diam(T) \leq 4k$ by Claim 1. Let $diam(T) = d$ and let u, v be two vertices of T such that $d(u,v) = d$. Denote by $P : u =$

$u_0, u_1, \cdots, u_d = v$ the $u - v$ path in T. Necessarily there exists an integer i, $1 \leq i \leq d-1$, such that the components of $T - u_{i-1}u_i$ and $T - u_iu_{i+1}$ containing u are, respectively, of order less than $2k+1$ and of order at least $2k+1$.

Let T_1' and T_2' be the components of $T - u_i$ containing u and v, respectively. Then T_1' and T_2' are of order less than $2k+1$.

Since $i \leq 2k$, $d-i \leq 2k$ and $3k+1 \leq d \leq 4k$, it follows that $i \geq d-2k \geq k+1$ and $i+1 \leq 2k+1 \leq d-k$. It is obvious that $\{u_k\}$ and $\{u_{d-k}\}$ are k-dominating set of T_1' and T_2', respectively and $d(u_k, u_i) = i - k \leq k$ and $d(u_{d-k}, u_i) = d - k - i \leq k$.

Case 1 $d(u_i) = 2$. The set $\{u_k, u_i, u_{d-k}\}$ is a total k-dominating set of T and $\gamma_k^t(T) \leq 3$, which is a contradiction.

Case 2 $d(u_i) > 2$. Denote by $T_1', T_2', \cdots, T_r'$ the components of $T - u_i$ and by w_i the vertex in T_i' adjacent to u_i for $i = 1, 2, \cdots, r$. We note that $w_1 = u_{i-1}$ and $w_2 = u_{i+1}$. For $j \in \{3, 4, \cdots, r\}$, since the component of $T - u_iw_j$ containing P is of order at least $3k+2$, the component T_j' is of order at most $2k$.

Let I be the set of all indices $j \in \{3, 4, \cdots, r\}$ such that T_j' contains a vertex at distance at least $k+1$ form u_i. If $j \in I$, then since $|V(T_j') \cup \{u_i\}| \leq 2k+1$, T_j' contains a vertex z_j such that $\{z_j\}$is a k-dominating set of T_j' and $d(u_i, z_j) \leq k$. Hence, $D = \{u_k, u_{d-k}, u_i\} \cup \{z_j | j \in I\}$ is a total k-dominating set of T with $\gamma_k^t(T) \leq |D| = 3 + |I|$. Since $|V(T)| \geq d + 1 + (k+1)|I|$ and $d \geq 3k+1$, we have

$$\frac{2|V(T)|}{2k+1} \geq \frac{2(3k+2+(k+1)|I|)}{2k+1} = 3 + |I| + \frac{1+|I|}{2k+1} > 3 + |I|$$

which is a contradiction.

Claim 3 If $n = 2$, then $diam(T) \geq 4k+1$.

Otherwise, $3k+1 \leq diam(T) \leq 4k$. Let i, T_1' and T_2' be defined as in Claim 2. If $|V(T_2')| < 2k+1$, then, with a similar way as Claim 2, $\{u_k, u_{d-k}, u_i\}$ is a total k dominating set of T since $|V(T)| - |V(P)| \leq 4k+2-(3k+2) = k$, which is a contradiction. If $|V(T_2')| = 2k+1$, let T_1 denote the component of $T - \{u_iu_{i+1}\}$ containing u, then $|V(T_1)| = 2k+1$. Hence, $\{u_k\}$ and $\{u_{d-k}\}$ are k-dominating set of T_1 and T_2' respectively. Since $3k+1 \leq diam(T) \leq 4k$, either $d(u_k, u_i) < k$ or $d(u_{d-k}, u_{i+1}) < k$. Assume $d(u_k, u_i) < k$. Then $\{u_k, u_{d-k}, u_{i+1}\}$ is a total k dominating set of T, which is a contradiction.

Claim 4 If T be a tree of order $(2k+1)n$ ($n \geq 3$) with $\gamma_k^t(T) = 2n$, then there exists a subgraph satisfying Claim 2 or Claim 3.

If $n \geq 3$ and there exists an edge e such that both components of $T - e$ are of order at least $2k+1$. Denote by T_1, T_2 the components of $T - e$.

Assume $|V(T_1)| = (2k+1)m + t$, $0 \le t \le 2k$. If $t \ne 0$, then by Lemma 2,

$$\begin{aligned}\gamma_k^t(T) &\le \lfloor \tfrac{2|V(T_1)|}{2k+1} \rfloor + \lfloor \tfrac{2|V(T_2)|}{2k+1} \rfloor \\ &\le 2m + \lfloor \tfrac{2t}{2k+1} \rfloor + 2(n-m-1) + \lfloor 2 - \tfrac{2t}{2k+1} \rfloor \\ &= 2n - 2 + \lfloor \tfrac{2t}{2k+1} \rfloor + \lfloor 2 - \tfrac{2t}{2k+1} \rfloor \\ &= 2n - 1 \\ &< \tfrac{2|V(T)|}{2k+1}\end{aligned}$$

which is a contradiction. Hence, $t = 0$, $\gamma_k^t(T_1) = \frac{2|V(T_1)|}{2k+1}$ and $\gamma_k^t(T_2) = \frac{2|V(T_2)|}{2k+1}$. If T_1 or T_2 satisfy Claim 2 or Claim 3, then the result is true. Otherwise, we replaced T with T_1 and continue until Claim 2 or Claim 3 holds. Since, the number of vertices is limit, it is possible to do so.

By Claim 2-4, the result holds.

Corollary 1 *Let T be a tree of order $4k+2$. If $\gamma_k^t(T) = 4$, then T is isomorphic to P_{4k+2}.*

Theorem 3 *Let T be a tree of order $(2k+1)n$ and $k \ge 2$. Then $\gamma_k^t(T) = 2n$ if and only if at least one of the following conditions holds:*

(1) T is a tree of order $2k+1$;

(2) $T = H \circ 2k$ for some tree H of order $n \ge 2$.

Proof By Lemma 2 and Theorem 1, the sufficiency is obvious. Now we only consider the necessity.

The result is clear for $n = 1$. If $n = 2$, $T = P_{4k+2}$ by Corollary 1. Thus $T = P_2 \circ 2k$. Suppose the result is true for tree of order $(2k+1)n$ with $\gamma_k^t(T) = 2n$ and $n \ge 2$. Let T be a tree of order $(2k+1)(n+1)$ with $\gamma_k^t(T) = 2(n+1)$. Assume $d(u,v) = diam(T) = d$. Denote by $P: u = u_0, u_1, \cdots, u_d = v$ the $u-v$ path in T. By Theorem 2, $d \ge 4k+1$. Let T_1 (T_2, resp.) be the component of $T - u_{2k}u_{2k+1}$ which contains (does not contain resp.) the vertex u_{2k}. Since $|V(T_1)| \ge 2k+1$ and $|V(T_2)| \ge 2k+1$, with a similar way as Claim 4 of Theorem 2, it is easy to prove that $|V(T_1)| = (2k+1)m$ and $|V(T_2)| = (2k+1)(n+1-m)$ for some $1 \le m \le n$. Furthermore, $\gamma_k^t(T_1) = 2m$ and $\gamma_k^t(T_2) = 2(n+1-m)$.

If $m \ge 2$, by induction, then T_1 is isomorphic to $R \circ 2k$ for some tree R of order m. If u_{2k} belongs to R, then there exists an other vertex $w \in R$ such that w is adjacent to u_{2k} since $m \ge 2$. It follows that the length of the path $(\overline{w} - w) \cup \{wu_{2k}\} \cup (u_{2k} - u_d)$ is greater than d, which is a contradiction. If u_{2k} belongs to some path $v_i - \overline{v_i}$ of R and $u_{2k} \ne v_i$, then u_0 belongs to some path $v_j - \overline{v_j}$ and $i \ne j$ Since $d(u_0) = 1$. Let P_{ij} denote one path between v_i and v_j in R. It follows that $(v_i - \overline{v_i}) \cup (u_0 - u_{2k}) \cup (v_j - \overline{v_j}) \cup P_{ij}$ contains a cycle, which is a contradiction.

So, $m = 1$. Then $|V(T_1)| = 2k+1$ and T_1 is isomorphic to P_{2k+1}. Since $|V(T_2)| = (2k+1)n$ and $n \ge 2$, by induction, $T_2 = R \circ 2k$ for some tree of order n. Let $V(R) = \{v_1, v_2, \cdots, v_n\}$ and $V(R') = \{v_1', v_2', \cdots, v_n'\}$ where v_i' belongs to the path $v_i - \overline{v_i}$ and $d(v_i, v_i') = k$ for $i = 1, 2, \cdots, n$.

If u_{2k+1} belongs to some path $v_i - \overline{v_i}$ and $u_{2k+1} \neq v_i$, then $V(R) \cup V(R') \cup \{u_k, u_{2k}\} - \{v_i\}$ is a total k-dominating set of T with cardinality $2n+1$, which is a contradiction. Hence u_{2k+1} belongs to $V(R)$. Then let $H = \langle R \cup \{u_{2k}\} \rangle$. It follows that $T = H \circ 2k$, where H is a tree of order $n+1$.

Theorem 4 *Let G be a connected graph of order $(2k+1)n$ and $k \geq 2$. Then $\gamma_k^t(G) = 2n$ if and only if at least one of the following conditions holds:*

(1) G is any connected graph of order $2k+1$;

(2) $G \cong C_{4k+2}$

(3) $G = H \circ 2k$ for some connected graph H of order n.

Proof The sufficiency is obvious. Now, we only consider the necessity.

By Theorem 2 and Corollary 1, it follows that $G \cong C_{4k+2}$ or $G \cong P_{4k+2}$ for $n = 2$. The proof will be completed by showing that $G = H \circ 2k$ for some connected graph H of order $n \geq 3$. In order to get this, let T be a spanning tree of G. Since $\gamma_k^t(G) \leq \gamma_k^t(T) \leq \frac{2p}{2k+1}$, it follows that $\gamma_k^t(T) = 2n$. By Theorem 3, $T = R \circ 2k$ for some tree R of order n. Let H be the subgraph of G induced by $V(R)$. We claim that $G = H \circ 2k$. Suppose on the contrary that $G \ncong H \circ 2k$.

Let $V(H) = \{v_1, v_2, \cdots, v_n\}$, and let $V(R_i') = \{v_i = v_{i0}', v_{i1}', v_{i2}', \cdots, v_{i(2k)}' = \overline{v_i}\}$ denote the set of vertices that belong to the path $v_i - \overline{v_i}$ in T for $i = 1, 2, \cdots, n$. Let $V(H') = \{v_{1k}', v_{2k}', \cdots, v_{nk}'\}$. Then G contain two vertices $v \in V(G) - V(H)$ and $u \in V(G)$ such that $vu \in E(G) - E(H \circ 2k)$. Since v and u belong to the $t(v) - \overline{t(v)}$ path and the $t(u) - \overline{t(u)}$ path in T, there are two cases to consider.

Case 1 $t(v) = t(u)$. Without loss of generality, assume $t(v) = t(u) = v_i$. Since $k \geq 2$, it follows that vu is a chord of the $v_i - \overline{v_i}$ path.

Case 1.1 either $v = v_{ik}'$ or $u = v_{ik}'$. Without loss of generality, assume $v = v_{ik}'$.

If $u \in \{v_{i0}', v_{i1}', \cdots, v_{i(k-2)}'\}$, then $V(H) \cup V(H') - \{v_{i0}'\}$ is a total k-dominating set of G with cardinality $2n-1$, which is a contradiction.

If $u \in \{v_{i(k+2)}', v_{i(k+2)}', \cdots, v_{i(2k)}'\}$, then $V(H) \cup V(H') \cup \{v_{i(k-1)}'\} - \{v_{i0}', v_{ik}'\}$ is a total k-dominating set of G with cardinality $2n-1$, which is a contradiction.

Case 1.2 $v \in \{v_{i1}', \cdots, v_{i(k-1)}'\}$.

If $u \in \{v_{i0}', v_{i1}', \cdots, v_{i(k-1)}'\}$, then $V(H) \cup V(H') - \{v_{i0}'\}$ is a total k-dominating set of G with cardinality $2n-1$, which is a contradiction.

If $u \in \{v_{i(k+1)}', v_{i(k+2)}', \cdots, v_{i(2k)}'\}$, then $V(H) \cup V(H') \cup \{v\} - \{v_{i0}', v_{ik}'\}$ is a total k-dominating set of G with cardinality $2n-1$, which is a contradiction.

Case 1.3 $v \in \{v_{i(k+1)}', \cdots, v_{i(2k)}'\}$.

If $u \in \{v'_{i0}, v'_{i1}, \cdots, v'_{i(k-1)}\}$, then $V(H) \cup V(H') \cup \{u\} - \{v'_{i0}, v'_{ik}\}$ is a total k-dominating set of G with cardinality $2n-1$, which is a contradiction.

If $u \in \{v'_{i(k+1)}, v'_{i(k+2)}, \cdots, v'_{i(2k)}\}$, then $V(H) \cup V(H') \cup \{v'_{i(k-1)}\} - \{v'_{i0}, v'_{ik}\}$ is a total k-dominating set of G with cardinality $2n-1$, which is a contradiction.

Case 2 $t(v) \neq t(u)$. Without loss of generality, assume $t(v) = v_i$ and $t(u) = v_j$.

Case 2.1 $v \neq v'_{i(2k)}$. Then $V(H) \cup V(H') \cup \{u\} - \{v'_{i0}, v'_{j0}\}$ is a total k-dominating set of G with cardinality at most $2n-1$, which is a contradiction.

Case 2.2 $v = v'_{i(2k)}$

Case 2.2.1 $u \neq v'_{j0}$ and $u \neq v'_{j(2k)}$. Then $V(H) \cup V(H') \cup \{v'_{i(2k)}\} - \{v'_{i0}, v'_{j0}\}$ is a total k-dominating set of G with cardinality $2n-1$, which is a contradiction.

Case 2.2.2 $u = v'_{j0}$. Then $V(H) \cup V(H') - \{v'_{ik}\}$ is a total k-dominating set of G with cardinality $2n-1$, which is a contradiction.

Case 2.2.3 $u = v'_{j(2k)}$. Since $|V(H)| = n \geq 3$, without loss of generality, we can assume v'_{j0} is adjacent to at least one vertex of H other than v'_{i0}. Then $V(H) \cup V(H') \cup \{v'_{i(2k)}\} - \{v'_{i0}, v'_{jk}\}$ is a total k-dominating set of G with cardinality $2n-1$, which is a contradiction.

Since both Case 1 and Case 2 lead to a contradiction, it follows that $G = H \circ 2k$, which completes the proof.

Theorem 5 *Let G and $\overline{G}$ be connected graphs of order $p = (2k+1)n$ and $k \geq 2$ Then $\gamma_k^t(G) + \gamma_k^t(\overline{G}) = \frac{2p}{2k+1} + 2$ if and only if at least one of the following conditions holds:*

(1) Both G and $\overline{G}$ are connected graphs of order $2k+1$;

(2) $G \cong C_{4k+2}$ or $\overline{G} \cong C_{4k+2}$

(3) $G = H \circ 2k$ or $\overline{G} = H \circ 2k$ for some connected graph H of order n.

Proof The sufficiency is obvious by Theorem 4. Now, we only consider the necessity.

If either $diam(G) \geq 3$ or $diam(\overline{G}) \geq 3$, say $diam(G) \geq 3$, then it is obvious that $\gamma_k^t(\overline{G}) = 2$. So, $\gamma_k^t(G) = \frac{2p}{2k+1}$. By Theorem 4, it follows that at least one of the three conditions of the theorem holds.

If both $diam(G) \leq 2$ and $diam(\overline{G}) \leq 2$, then $\gamma_k^t(G) = \gamma_k^t(\overline{G}) = 2$. Hence, $\frac{2p}{2k+1} = 2$. That is $p = 2k+1$. So, the condition 1 of the theorem holds.

References:

[1] M.A. Henning, O.R. Oellermann and H.C. Swart. Bounds on distance domination parameters. *J. Comb. Information & System Sciences* 16(1) (1991), 11-18.

[2] J. Topp, L. Volkmann. On packing and covering numbers of graphs. *Discrete Math.*, 96 (1991), 229-238.

[3] G.H. Fricke, M.A. Henning, O.R. Oellermann, H.C. Swart. An algorithm to find two distance domination parameters. *Discrete Applied Math.*, 68 (1996), 85-91.

[4] J.H. Hattinggh, M.A. Henning, J.L. Walters. Distance irredundance in graphs: complexity issues. *Ars Comb.* , 50 (1998), 3-22.

[5] J.H. Hattinggh, M.A. Henning. The ratio of the distance irredundance and domination numbers of a graph. *J. of Graph Theory* , 18 (1994), 1-9.