Ozeki Polynomials and Jacobi Forms *†

Koichi Betsumiya
betumi@jobu.ac.jp
Jobu University, 634-1
Iaesaki, Japan
and
YoungJu Choie
yjc@postech.ac.kr
Department of Mathematics
Pohang University of Science and Technology
Pohang 790–784, Korea

Abstract

A Jacobi polynomial was introduced by Ozeki. It corresponds to the codes over $\mathbb{F}_2$. Later, Bannai and Ozeki showed how to construct Jacobi forms with various index using a Jacobi polynomial corresponding to the binary codes. It generalizes Broué-Enguehard map. In this paper, we study Jacobi polynomial which corresponds to the codes over $\mathbb{F}_{2^f}$. We show how to construct Jacobi forms with various index over the totally real field. This is one of extension of Broué-Enguehard map.

Keywords Ozeki Polynomial, Clifford group, Jacobi forms over the totally real fields

1 Introduction

A Jacobi polynomial was introduced by Ozeki in [11]. It is a generalization of weight enumerators of codes over $\mathbb{F}_2$. Later, Bannai and Ozeki showed how to construct Jacobi forms with various index using a Jacobi polynomial corresponding to the binary codes [1]. It generalizes Broué-Enguehard map [7, 2, 3]. In this paper, we study Ozeki polynomials as a generalization of

*2000 *Mathematics Subject Classification;* Primary 11F41, 11E10, 15A63, 15A66

†This work was partially supported by KOSEF R01-2003-000-11596-0 and ITRC

the Jacobi polynomials which corresponds to the codes over $\mathbb{F}_{2^f}$. We show how to construct Jacobi forms with various index over a certain totally real field from Type II codes over $\mathbb{F}_{2^f}$, which is defined as even codes in [12]. This is another extension of Broué-Enguehard map.

2 Diagonalized Clifford-Weil Group and its Invariant Ring

2.1 Notations and Definitions

In this section we recall the definition of Clifford Weil group. We follow notations and definitions given in [5, 6, 10].

Let $\mathbb{F} = \mathbb{F}_{2^f}$ be the finite field of order 2^f. Let $B = \{b_1, \ldots, b_f\}$ be an $\mathbb{F}_2$-basis of $\mathbb{F}$ such that $\mathrm{tr}(b_i b_j) := \mathrm{Tr}_{\mathbb{F}/\mathbb{F}_2}(b_i b_j) = \delta_{i,j}$, for all $i, j = 1, \ldots, f$, where $\mathrm{Tr}_{\mathbb{F}/\mathbb{F}_2}(a)$ denotes the trace of $a \in \mathbb{F}$ over $\mathbb{F}_2$. B is called a **self-complementary** basis of $\mathbb{F}$ over $\mathbb{F}_2$. Define $\mathrm{wt}_B \colon \mathbb{F} \to \mathbb{Z}$ by

$$\mathrm{wt}_B(\sum_{j=1}^{f} a_j b_j \mid a_j \in \mathbb{F}_2) := \#\{j \mid a_j = 1, 1 \leq j \leq f\},$$

to be the **Lee weight** with respect to B.

Now, let us take and fix a self-complementary basis $B = \{b_1, \ldots, b_f\}$ of $\mathbb{F}$ over $\mathbb{F}_2$. For an element $r \in \mathbb{F}$, let h, m_r and d_r be the $\mathbb{C}$-algebra endomorphisms of $\mathbb{C}[x_a \mid a \in \mathbb{F}]$ defined by

$$h(x_a) := 2^{-\frac{f}{2}} \sum_{b \in \mathbb{F}} (-1)^{\mathrm{tr}(ab)} x_b, \quad m_r(x_a) := x_{ar}, \quad d_r(x_a) := i^{\mathrm{wt}_B(ar)} x_a$$

for any $a \in \mathbb{F}$. Similarly for an element $r \in \mathbb{F}$, let $h^{(g)}$, $m_r^{(g)}$ and $d_r^{(g)}$ be the $\mathbb{C}$-algebra endomorphisms of $\mathbb{C}[x_{a,j} \mid a \in \mathbb{F}, \quad 1 \leq j \leq g]$ defined by

$$h^{(g)}(x_{a,j}) := 2^{-\frac{f}{2}} \sum_{b \in \mathbb{F}} (-1)^{\mathrm{tr}(ab)} x_{b,j}, \quad m_r^{(g)}(x_{a,j}) := x_{ar,j},$$

$$d_r^{(g)}(x_{a,j}) := i^{\mathrm{wt}_B(ar)} x_{a,j}$$

for any $a \in \mathbb{F}$ and $1 \leq j \leq g$.

Definition 1. *The group*

$$G_f := \langle h, m_r, d_r \mid r \in \mathbb{F}^* \rangle$$

is called the associated **Clifford-Weil** *group. The group*

$$G_f^{(g)} := \langle h^{(g)}, m_r^{(g)}, d_r^{(g)} \mid r \in \mathbb{F}^* \rangle$$

is called the **diagonalized Clifford-Weil** *group.*

Remark 2.1. *1. One can check that the Clifford-Weil group does not depend on the choice of basis.*

2. The Clifford-Weil group and the diagonalized Clifford-Weil group can be identified with the following matrix group;

$$(h)_{uv} = 2^{-\frac{f}{2}}(-1)^{\mathrm{tr}(uv)},$$

$$(m_r)_{uv} = \begin{cases} 1 & \text{for } rv = u \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (d_r)_{uv} = \begin{cases} i^{\mathrm{wt}_B(ur)} & \text{for } v = u \\ 0 & \text{otherwise,} \end{cases}$$

$h^{(g)}$, $m_r^{(g)}$ and $d_r^{(g)}$ are g-multiple diagonalized matrices of h, m_r and d_r respectively;

$$h^{(g)} = \begin{pmatrix} h & & & 0 \\ & h & & \\ & & \ddots & \\ 0 & & & h \end{pmatrix}, m_r^{(g)} = \begin{pmatrix} m_r & & & 0 \\ & m_r & & \\ & & \ddots & \\ 0 & & & m_r \end{pmatrix},$$

$$d_r^{(g)} = \begin{pmatrix} d_r & & & 0 \\ & d_r & & \\ & & \ddots & \\ 0 & & & d_r \end{pmatrix},$$

Example 2.2. *In [4, 5, 10] the group $G_2^{(1)}$ has been explicitly given as follows;*

$$G_f^{(1)} = \left\langle \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \right\rangle$$

3 Type II codes over $\mathbb{F}_{2^f}$ and Invariant Ring

In this subsection we define Ozeki polynomial which is generalization of a Jacobi polynomial studied in [11]. Its name was already appeared in [8].

A code C of length n over $\mathbb{F}_{2^f}$ is a $\mathbb{F}_{2^f}$-subspace of $\mathbb{F}_{2^f}^n$. An element of C is called a codeword. Let $v = (v_1, v_2, \ldots, v_n) \in \{1, 2, \ldots, g\}^n$ be a vector of length n.

Remark 3.1. *Note that the number of such vectors v can be identified as number of partition of a set $\{1,..,g\}$ into n-sets by the following correspondence;*

$$v = (v_1, v_2, \ldots, v_n) \Leftrightarrow (w_1, w_2, \ldots, w_n),$$

where for each $k = 1, \ldots, g$, $w_k = \{j \mid v_j = k\}$.

The **Ozeki polynomial** of C masked by v is

$$OZ_C(v; X) := \sum_{c\in C} \prod_{j=1}^{n} x_{c_j, v_j},$$

which is a polynomial over $\mathbb{Z}$ with $2^f g$ variables.

The inner product on $\mathbb{F}_{2^f}^n$ is given by the usual Euclidean norm;

$$[u, v] := \sum_{j=1}^{n} u_j v_j, u = (u_j), v = (v_j) \in \mathbb{F}_{2^f}^n.$$

So, the dual $C^\perp$ of C is defined as $C^\perp = \{v' \in \mathbb{F}_{2^f}^n \mid [v', v] = 0 \text{ for all } v \in C\}$. A code C is called self-dual if $C = C^\perp$. The Lee weight of a codeword is the sum of Lee weights of codeword entries defined in Section 2.1 with respect to a fixed self-complementary basis.
A self-dual code over $\mathbb{F}_{2^f}$ is said to be **Type II** if the Lee weight of every codeword is a multiple of 4.

We now define invariant polynomial ring; let G be a group.

$$\mathbb{C}[X]^G := \oplus_{\ell\geq 1}\{F(X) \in \mathbb{C}[X] \mid \ F \text{ is homogeneous with}$$
$$\deg(F) = \ell, A \cdot F(X) = F(X), \forall A \in G\},$$

Here the action of a group G on $\mathbb{C}[X]$ is defined as

$$A \cdot F(X) := F(A \cdot X).$$

Theorem 3.2. *1. Let $v = (v_1, v_2, \ldots, v_n) \in \{1, 2, \ldots, g\}^n$ and C be a Type II code of length n over $\mathbb{F}_{2^f}$. $OZ_C(v; X)$ is an element of the invariant polynomial ring*

$$\mathbb{C}[x_{\mathbf{a},j} \mid \mathbf{a} \in \mathbb{F}, \quad 1 \leq j \leq g]^{G_f^{(g)}}$$

of $G_f^{(g)}$.

2. The invariant ring is a graded ring as

$$\mathbb{C}[x_{\mathbf{a},j} \mid \mathbf{a} \in \mathbb{F}, \quad 1 \leq j \leq g]^{G_f^{(g)}} = \oplus_\ell \mathbb{C}[x_{\mathbf{a},j} \mid \mathbf{a} \in \mathbb{F}, \quad 1 \leq j \leq g]_\ell^{G_f^{(g)}}.$$

The each homogeneous degree is divisible by 4, i.e. $\ell \equiv 0 \pmod 4$.

Proof.

1. It suffices to show that $h^{(g)}$ and $d_r^{(g)}$ preserve $OZ_C(v;X)$ for any $r \in \mathbb{F}_{2^f}$.

$$|C|h^{(g)}(OZ_C(v;X)) = \sum_{u\in\mathbb{F}_{2^f}^n} \sum_{c\in C} \prod_{j=1}^{n} (-1)^{\mathrm{tr}(u_j c_j)} x_{c_j,v_j}$$
$$= \sum_{c\in C} \prod_{j=1}^{n} x_{c_j,v_j} \sum_{u\in\mathbb{F}_{2^f}} (-1)^{\mathrm{tr}(u\cdot c)}$$

and

$$\sum_{u\in\mathbb{F}_{2^f}} (-1)^{\mathrm{tr}(u\cdot c)} = \begin{cases} |C| & \text{if } u \in C^\perp \\ 0 & \text{if } u \notin C^\perp. \end{cases}$$

Therefore

$$|C|h^{(g)}(OZ_C(v;X)) = |C^\perp| \sum_{c\in C} \prod_{j=1}^{n} x_{c_j,v_j}$$
$$h^{(g)}(OZ_C(v;X)) = \sum_{c\in C} \prod_{j=1}^{n} x_{c_j,v_j}.$$

Hence $h^{(g)}$ preserves $OZ_C(v;X)$.

Moreover it is clear that

$$d_r^{(g)}(OZ_C(v;X)) = \sum_{c\in C} i^{\mathrm{wt}(rc)} \prod_{j=1}^{n} x_{c_j,v_j}.$$

Then result follows.

2. This is clear since the diagonal matrix iI is an element of $G_f^{(g)}$([10]).

□

4 Jacobi Forms over the Totally Real Field K

We recall the definition of Jacobi forms over the totally real field K and theta-functions. Here, K denotes the totally real extension field of $\mathbb{Q}$ with extension degree f and $\mathcal{O}_K$ denotes its ring of integers. We follow the definition given in [5].

4.1 Jacobi Group

The Jacobi group of the totally real field K will be denoted by

$$\Gamma_1^J(\mathcal{O}_K) := \mathrm{SL}_2(\mathcal{O}_K) \ltimes \mathcal{O}_K^2.$$

This group acts on $\mathcal{H}^f \times \mathbb{C}^f$, where $\mathcal{H}$ denotes the complex upper half plane. Variables of this space will be listed as,

$$(\tau, z) := (\tau_1, \dots, \tau_f, z_1, \dots, z_f).$$

The action of $\Gamma_1^J(\mathcal{O}_K)$ on the space $\mathcal{H}^f \times \mathbb{C}^f$ are given by,

$$\begin{aligned}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\cdot(\tau,z) := (&\frac{\alpha^{(1)}\tau_1+\beta^{(1)}}{\gamma^{(1)}\tau_1+\delta^{(1)}}, \dots, \frac{\alpha^{(f)}\tau_f+\beta^{(f)}}{\gamma^{(f)}\tau_f+\delta^{(f)}}, \frac{z_1}{\gamma^{(1)}\tau_1+\delta^{(1)}},\\ &\dots, \frac{z_f}{\gamma^{(f)}\tau_f+\delta^{(f)}})\\ [\lambda,\mu]\cdot(\tau,z) := (&\tau_1, \tau_2, z_1+\lambda^{(1)}\tau_1+\mu^{(1)}, \dots, z_f+\lambda^{(f)}\tau_f+\mu^{(f)})\end{aligned}$$

for

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_K) \text{ and } [\lambda,\mu] \in \mathcal{O}_K^2.$$

Here $\alpha^{(j)}$ denotes an algebraic conjugate of $\alpha^{(1)}$.

Remark 4.1. *It is known that* $\mathrm{SL}_2(\mathcal{O}_K)$ *is generated by the matrices*

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad \begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix} \quad (\beta \in \mathcal{O}_K).$$

4.2 Jacobi Forms

We first introduce the following notations; for $\tau \in \mathcal{H}^f, z \in \mathbb{C}^f, \gamma, \delta \in \mathcal{O}_K$, denote

$$\mathcal{N}(\gamma\tau+\delta) := \prod_{j=1}^{f}(\gamma^{(j)}\tau_j+\delta^{(j)}),$$

$$e^{2\pi i\,\mathrm{Tr}(m\frac{\gamma z^2}{\gamma\tau+\delta})} := \prod_{j=1}^{f} e^{2\pi i m^{(j)}\frac{\gamma^{(j)}z_j^2}{\gamma^{(j)}\tau_j+\delta^{(j)}}},$$

$$e^{-2\pi i\,\mathrm{Tr}(m(\lambda^2\tau+2\lambda z))} := \prod_{j=1}^{f} e^{-2\pi i m^{(j)}(\lambda^{(j)2}\tau_j+2\lambda^{(j)}z_j)}.$$

Definition 2. *Given $k \in \frac{1}{2}\mathbb{Z}$ and $m \in \mathcal{O}_K$, a function $g\colon \mathcal{H}^f \times \mathbb{C}^f \to \mathbb{C}$ is said to be a Jacobi forms of weight k and index m for a totally real field K if it is analytic function satisfying*

1.

$$\begin{aligned}(g\,|_{k,m}\, M)(\tau, z) &:= \mathcal{N}(\gamma\tau+\delta)^{-k} e^{-2\pi i \operatorname{Tr}(m\frac{\gamma z^2}{\gamma\tau+\delta})} g(M\cdot(\tau,z)) \\ &= g(\tau, z),\end{aligned}$$

for any

$$M = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_K),$$

2.

$$\begin{aligned}(g\,|_m\, [\lambda,\mu])(\tau, z) &:= e^{-2\pi i \operatorname{Tr}(m(\lambda^2\tau+2\lambda z))} g(\tau, [\lambda,\mu]\cdot z) \\ &= g(\tau, z), \forall [\lambda,\mu] \in \mathcal{O}_K,\end{aligned}$$

and

3. it has a Fourier expansion:

$$g(\tau, z) = \sum_{n, r\in\delta_K^{-1}, n\geq 0} c(n,r) e^{2\pi i \operatorname{Tr}(n\tau + rz)}.$$

Here δ_K^{-1} is the inverse different of K.

The $\mathbb{C}$-vector space of Jacobi forms of weight k and index m for the field K is denoted by $\mathcal{J}_{k,m}(SL_2(\mathcal{O}_K))$.

4.3 Theta series

To state the main theorem, we need the transformation formula for theta series. In this section we state, without proof, the useful properties of theta series studied in [5].

Let us assume that a field K is totally real with an inert ideal (2). Then, the reduction map

$$h : \mathcal{O}_K \to \mathbb{F}_{2^f}$$

is an homomorphism with $Ker(h) = (2)$. So, each element $a \in \mathcal{O}_K$ modulo the ideal (2) can be regarded as an element $\mathbf{a} \in \mathbb{F}_{2^f}$. Now, consider the following theta series ; for each $\mathbf{a}$,

$$\theta_{\mathbf{a}}(\tau, z) := \sum_{r\in\delta_K^{-1}, r\in\Psi^{-1}(a)} e^{2\pi i \operatorname{Tr}(\frac{r^2\tau}{4} + rz)}. \tag{1}$$

Then, by the Poisson summation formula, the theta-series satisfies the following transformation formula[5];

Lemma 4.2. *1.*

$$(\theta_\mu |_{\frac{1}{2},1} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix})(\tau, z) = e^{2\pi i \operatorname{Tr}(\frac{\mu^2 \beta}{4})} \theta_\mu(\tau, z), \quad \textit{for any } \beta \in \mathcal{O}_K.$$

2.

$$(\theta_\mu |_{\frac{1}{2},1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})(\tau, z) = \frac{1}{2^{\frac{f}{2}}} \chi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sum_{\nu \in \mathcal{O}_K/2\mathcal{O}_K} e^{2\pi i \operatorname{Tr}(\frac{\mu\nu}{2})} \theta_\nu(\tau, z)$$

with $\chi^4(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) = 1.$

3. $(\theta_\mu|_1[\lambda, \kappa])(\tau, z) = \theta_\mu(\tau, z).$

The following result is known in [5].

Theorem 4.3. *For each* $\beta \in \mathcal{O}_K$, *define the diagonal matrix* A_β, *indexed by* $\mathbb{F}_{2^f} \times \mathbb{F}_{2^f}$, $(A_\beta)_{uv} = e^{\frac{2\pi i Tr(-\beta u)}{4}} \delta_{u,v}$. *Then the Clifford group is generated by*

$$G_f = < h, A_\beta \,|\, \forall \beta \in \mathcal{O}_K >,$$

and so the diagonalized Clifford group is

$$G_f^{(g)} = < h^{(g)}, A_\beta^{(g)} \,|\, \forall \beta \in \mathcal{O}_K >$$

5 Main Theorem and Example

The following theorem gives a relation between the invariant polynomial space of Clifford group and that of Jacobi forms with various index.

Theorem 5.1. *For nonnegative integers* $i_1, i_2, .., i_g$, *with* $\sum_{j=1}^{g} i_j = \ell$, *let* $\mathbb{C}[x_{a,j} | a \in \mathbb{F}, 1 \le j \le g]_{i_1,i_2,..,i_g}^{G_f^{(g)}}$ *as the subspace of* $\mathbb{C}[x_{a,j} | a \in \mathbb{F}, 1 \le j \le g]_\ell^{G_f^{(g)}}$ *consisting of the polynomials*

$$h(x_{a_1 1}, x_{a_2,1}, .., x_{a_{2^f},1}, x_{a_1,2}, x_{a_2,2}, .., x_{a_{2^f},2}, ..., x_{a_1,g}, .., x_{a_{2^f},g})$$

whose degrees with respect to $x_{a_1,1}, x_{a_2,1}, x_{a_3,1}, .., x_{a_{2^f},1}$, *are* i_1, *degrees with respect to* $x_{a_1,2}, x_{a_2,2}, x_{a_3,2}, .., x_{a_{2^f},2}$ *are* i_2, *so on. In general, degrees with respect to* $x_{a_1,j}, x_{a_2,j}, x_{a_3,j}, .., x_{a_{2^f},j}$ *are* i_j. *Now, take an arbitrary* $m_1, m_2, .., m_g \in \mathcal{O}_K$. *Then, the map*

$$\Psi : \oplus_\ell \mathbb{C}[x_{a_1,1}, x_{a_2,1}, .., x_{a_{2^f},1}, x_{a_1,2}, x_{a_2,2}, .., x_{a_1,g}, .., x_{a_{2^f},g}]_{i_1,i_2,..,i_g}^{G_f^{(g)}}$$
$$\rightarrow \oplus_\ell J_{\frac{\ell}{2}, Tr(\sum_{j=1}^{g} i_j (m_j)^2)}(SL_2(\mathcal{O}_K))$$

given by, for any

$h \in \mathbb{C}[x_{\mathbf{a}_1,1}, x_{\mathbf{a}_2,1}, .., x_{\mathbf{a}_{2^f},1}, x_{\mathbf{a}_1,2}, x_{\mathbf{a}_2,2}, .., x_{\mathbf{a}_1,g}, .., x_{\mathbf{a}_{2^f},g}]^{G_f^{(g)}}_{i_1,i_2,..,i_g}$, *the following substitution map*

$$\begin{cases} x_{\mathbf{a}_1,1} \to \theta_{\mathbf{a}_1}(2\tau, 2m_1 z), x_{\mathbf{a}_2,1} \to \theta_{\mathbf{a}_1}(2\tau, 2m_1 z), ..., x_{\mathbf{a}_{2^f},1} \to \theta_{\mathbf{a}_1}(2\tau, 2m_1 z), \\ x_{\mathbf{a}_1,2} \to \theta_{\mathbf{a}_1}(2\tau, 2m_2 z), x_{\mathbf{a}_2,2} \to \theta_{\mathbf{a}_2}(2\tau, 2m_2 z), ..., x_{\mathbf{a}_{2^f},2} \to \theta_{\mathbf{a}_2}(2\tau, 2m_2 z), \\ \cdots \\ x_{\mathbf{a}_1,g} \to \theta_{\mathbf{a}_1}(2\tau, 2m_g z), x_{\mathbf{a}_2,g} \to \theta_{\mathbf{a}_2}(2\tau, 2m_g z), ..., x_{\mathbf{a}_{2^f},g} \to \theta_{\mathbf{a}_{2^f}}(2\tau, 2m_g z) \end{cases}$$

is an algebra homomorphism.

Proof. Let

$$\begin{aligned} H(\tau, z) &= h(\theta_{\mathbf{a}_1,1}(2\tau, 2m_1 z), ..., .., \theta_{\mathbf{a}_{2^f},1}(2\tau, 2m_1 z), \theta_{\mathbf{a}_1,2}(2\tau, 2m_2 z), ..., \\ &\quad .., \theta_{\mathbf{a}_{2^f},2}(2\tau, 2m_2 z), ..., \theta_{\mathbf{a}_1,g}(2\tau, 2m_g z), .., \theta_{\mathbf{a}_{2^f},g}(2\tau, 2m_g z)). \end{aligned}$$

Since $SL_2(\mathcal{O}_K)$ is generated by two types of elements, say, $S_b := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, for each $b \in \mathcal{O}_K$, and $T := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, it is enough to check that

1. $H(\tau + \beta, z) = H(\tau, z), \forall \beta \in \mathcal{O}_K$,
2. $\mathcal{N}(\tau)^{-\frac{\ell}{2}} e^{-2\pi i Tr((\sum_{j=1}^{g} i_j m_j^2)\frac{z^2}{\tau})} H(-\frac{1}{\tau}, \frac{z}{\tau}) = H(\tau, z)$,
3. $e^{-2\pi i Tr((\sum_{j=1}^{g} i_j m_j^2)(\lambda^2\tau + 2\lambda z))} H(\tau, [\lambda, \mu] \cdot z) = H(\tau, z)$.

The first relation follows from the transformation formula of theta series given in Lemma4.2 and Theorm4.3. Next, note that

$$\begin{aligned} \theta_\mu(-\frac{1}{\tau}, \frac{mz}{\tau}) = \quad & \chi\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{N}(\tfrac{\tau}{2I})^{\frac{1}{2}} e^{2\pi i Tr(\frac{m^2 z^2}{\tau})} \\ & \{\textstyle\sum_{\nu \in \mathcal{O}_K/2\mathcal{O}_K} e^{2\pi i \frac{Tr(\mu\nu)}{2}} \theta_\nu(\tau, mz)\}. \end{aligned}$$

So, again, the second relation follows from Lemma4.3 and Theorem3.2. Last transformation formula is immediate from an elliptic property of theta series given in Lemma4.2. Finally, the property of Fourier expansion can be derived directly from theta series expansion. This completes the proof. □

Remark 5.2. *Banni and Ozeki first studied Ozeki polynomial to construct Jacobi forms of various index, which can be realized as the special case, i.e., $f = 1$ in Theorem5.1.*

5.1 Example; case when $f = 2$ and $g = 2$

Let $\mathcal{O}_K$ be the ring of integer of the field $K = \mathbb{Q}(\sqrt{5})$. One can check that

$$\mathcal{O} = \mathbb{Z}\alpha + \mathbb{Z}\alpha^2, \alpha^2 - \alpha - 1 = 0.$$

So, the reduction map modulo (2)

$$h : \mathcal{O} \to \mathbb{F}_{2^2}$$

given by $h(a_1\alpha + a_2\alpha^2) = a_1 \pmod{(2)}\alpha + a_2 \pmod{(2)}\alpha^2$ is a homomorphism with $Ker(h) = (2)$. We present each element a in $\mathbb{F}_{2^2}$ as $\mathrm{a} = (a_1 a_2) \in \mathbb{F}_{2^2}$

Example 5.3. (Theta series)
In this case we have the following four types of theta series;

$$\theta_{(00)}(\tau, z) = \sum_{a,b\in\mathbb{Z}} q^{3a^2+7b^2+8ab}\zeta^{2a+6b},$$

$$\theta_{(10)}(\tau, z) = q^{\frac{3}{4}} \sum_{a,b\in\mathbb{Z}} q^{3a^2+7b^2+8ab+3a+4b}\zeta^{2a+6b+1},$$

$$\theta_{(01)}(\tau, z) = q^{\frac{7}{4}} \sum_{a,b\in\mathbb{Z}} q^{3a^2+7b^2+8ab+4a+6b}\zeta^{2a+6b+3},$$

$$\theta_{(01)}(\tau, z) = q^{\frac{9}{2}} \sum_{a,b\in\mathbb{Z}} q^{3a^2+7b^2+14ab+7a+10b}\zeta^{2a+6b+4},$$

It turns out that we have three distinct Ozeki polynomial for every possible $v \in \{1,2\}^2$;

Example 5.4. (Ozeki Polynomial) *Let C be a code given*
$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & w & w^2 \end{pmatrix}$

1. *For* $v \in \{(1,1,2,2), (2,2,1,1), (1,2,1,2), (2,1,2,1), (1,2,2,1), (2,1,1,2)\}$,

$$OZ_C(v; x_{01}, x_{02}, x_{11}, x_{12}, x_{w1}, x_{w2}, x_{w^21}, x_{w^22})$$

$$= x_{01}^2 x_{02}^2 + 2x_{01}x_{11}x_{w2}x_{w^22} + 2x_{01}x_{w1}x_{12}x_{w^22} + 2x_{01}x_{w^21}x_{12}x_{w2} + x_{11}^2 x_{12}^2$$
$$+2x_{11}x_{w1}x_{02}x_{w^22} + 2x_{11}x_{w^21}x_{02}x_{w2}$$
$$+x_{w1}^2 x_{w2}^2 + 2x_{w1}x_{w^21}x_{02}x_{12} + x_{w^21}^2 x_{w^22}^2$$

In this case, for instance,

$$OZ_C(v;\theta_{(00)}(\tau,\tau,z,z),\theta_{(0,0)}(\tau,\tau,0,0),\theta_{(11)}(\tau,\tau,z,z),\theta_{(11)}(\tau,\tau,0,0),$$

$$\theta_{(1,0)}(\tau,\tau,z,z),\theta_{(10)}(\tau,\tau,0,0),\theta_{(01)}(\tau,\tau,z,z),\theta_{(01)}(\tau,\tau,0,0))$$

is an elliptic Jacobi form of weight 4 *with index* 4.

2. *For* $v \in \{(1,1,1,2),(1,1,2,1),(1,2,1,1),(2,1,1,1)\}$

$$OZ_C(v;x_{01},x_{02},x_{11},x_{12},x_{w1},x_{w2},x_{w^21},x_{w^22})$$

$$= x_{01}^3x_{02} + 3x_{01}x_{11}x_{w1}x_{w^22} + 3x_{01}x_{11}x_{w^21}x_{w2}$$

$$+3x_{01}x_{w1}x_{w^21}x_{12} + x_{11}^3x_{12} + 3x_{11}x_{w1}x_{w^21}x_{02} + x_{w1}^3x_{w2} + x_{w^21}^3x_{w^22}$$

In this case, for instance,

$$OZ_C(v;\theta_{(00)}(\tau,\tau,0,0),\theta_{(0,0)}(\tau,\tau,z,z),\theta_{(11)}(\tau,\tau,0,0),\theta_{(11)}(\tau,\tau,z,z),$$

$$\theta_{(1,0)}(\tau,\tau,0,0),\theta_{(10)}(\tau,\tau,z,z),\theta_{(01)}(\tau,\tau,0,0),\theta_{(01)}(\tau,\tau,z,z))$$

Here, $E_{4,2}(\tau,z)$ *is an elliptic Jacobi Eisenstein series of weight* 4 *with index* 2 *studied in [9].*

3. *For* $v \in \{(2,2,2,1),(2,2,1,2),(2,1,2,2),(1,2,2,2)\}$,

$$OZ_C(v;x_{01},x_{02},x_{11},x_{1,2},x_{w,1},x_{w2},x_{w^21},x_{w^22})$$

$$= x_{01}x_{02}^3 + 3x_{01}x_{12}x_{w2}x_{w^22} + 3x_{11}x_{02}x_{w2}x_{w^22}$$

$$+x_{11}x_{12}^3 + 3x_{w1}x_{02}x_{12}x_{w^22} + x_{w1}x_{w2}^3 + 3x_{w^21}x_{02}x_{12}x_{w2} + x_{w^21}x_{w^22}^3$$

In this case, for instance,

$$OZ_C(v;\theta_{(00)}(\tau,\tau,z,z),\theta_{(0,0)}(\tau,\tau,0,0),\theta_{(11)}(\tau,\tau,z,z),\theta_{(1,0)}(\tau,\tau,z,z),$$

$$\theta_{(1,0)}(\tau,\tau,0,0),\theta_{(01)}(\tau,\tau,z,z),\theta_{(01)}(\tau,\tau,0,0)) = 16E_{4,2}(\tau,z),$$

References

[1] E. Bannai and M. Ozeki, Construction of Jacobi forms from certain combinatorial polynomials, *Proc. Japan Acad. Ser. A Math. Sci.*, 72(1):12–15, 1996.

[2] E.Bannai, E.Bannai, M.Ozeki, Y.Teranishi, On the ring of simultaneous invariants for the Gleason-MacWilliams group. European J. Combin. 20 (1999), no. 7, 619–627.

[3] E.Bannai, Modular invariance property of association schemes, type II codes over finite rings and finite abelian groups, and reminiscences of Francois Jaeger (a survey). Symposium a la Memoire de Francois Jaeger (Grenoble, 1998). Ann. Inst. Fourier (Grenoble) 49 (1999), no. 3, 763–782.

[4] K. Betsumiya and Y. Choie, Codes over $\mathbb{F}_4$, Jacobi forms and Hilbert-Siegel modular forms over $\mathbb{Q}(\sqrt{5})$. to appear in European Journal of Combinatorics, 2004.

[5] K. Betsumiya and Y. Choie, Type II code over $\mathbb{F}_{2^f}$, invariant ring of Clifford-Weil group, and Jacobi forms over totally real field, preprint, 2003.

[6] K. Betsumiya, M. Harada, and A. Munemasa, Type II codes over $\mathbb{F}_{2^r}$, In *Applied algebra, algebraic algorithms and error-correcting codes (Melbourne, 2001)*, volume 2227 of *Lecture Notes in Comput. Sci.*, pages 102–111. Springer, Berlin, 2001.

[7] M. Broué and M. Enguehard, Polynômes des poids de certains codes et fonctions thêta de certains réseaux, *Ann. Sci. École Norm. Sup. (4)*, 5:157–181, 1972.

[8] Y. Choie and P. Solé. A Gleason formula for Ozeki polynomials. *J. Combin. Theory Ser. A*, 98(1):60–73, 2002.

[9] M. Eichler and D. Zagier, Theory of Jacobi forms, Birkhäuser, 1985.

[10] G. Nebe, H.-G. Quebbemann, E. M. Rains, and N. J. A. Sloane. Complete weight enumerators of generalized doubly-even self-dual codes. preprint, 2002.

[11] M. Ozeki. On the notion of Jacobi polynomials for codes. *Math. Proc. Cambridge Philos. Soc.*, 121(1):15–30, 1997.

[12] H.-G. Quebbemann. On even codes. *Discrete Math.*, 98(1):29–34, 1991.