On the Cordiality of the t-ply $P_t(u, v)$.

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Abstract: Let G be a simple graph with vertex set V and edge set E. A vertex labeling $f : V \to \{0, 1\}$ induces an edge labeling $\overline{f} : E \to \{0, 1\}$ defined by $\overline{f}(uv) = |f(u) - f(v)|$. Let $v_f(0), v_f(1)$ denote the number of vertices v with $f(v) = 0$ and $f(v) = 1$ respectively. Let $e_f(0)$ and $e_f(1)$ be similarly defined. A graph is said to be **cordial** if there exists a vertex labeling f such that $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

In this paper, we give necessary and sufficient conditions for the cordiality of the $t$ply $P_t(u, v)$, i.e a thread of ply number t.

Introduction

Throughout this paper, all graphs are finite, simple and undirected. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G. A mapping $f : V(G) \to \{0, 1\}$ is called a binary vertex labeling of G and $f(v)$ is called the label of the vertex v under f. For an edge $e = uv$, the induced labeling $\overline{f} : E(G) \to \{0, 1\}$ is given by $\overline{f}(e) = |f(u) - f(v)|$. Let $v_f(0), v_f(1)$ be the number of vertices in G having labels 0 and 1 respectively under f. Let $e_f(0), e_f(1)$ be the number of edges having labels 0 and 1 respectively under $\overline{f}$. A binary labeling f of G is called a **cordial labeling** if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called a **cordial graph** if it admits a cordial labeling.

Cordial labelings were first introduced by Cahit as a weaker version of both graceful and harmonious graphs [5]. In the same paper, Cahit proved the following- owing:

Theorem A: If G is an Eulerian graph with e edges, where $e \equiv 2(mod 4)$, then G has no cordial labeling.

In [1] several families of wheel related graphs were shown to be cordial.

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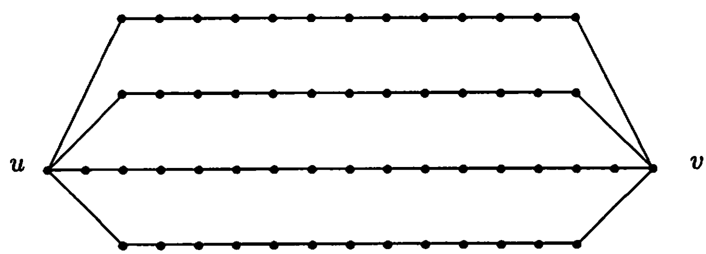
Shee and Ho [7] proved that the one - point union $F_m^{(n)}$ of n copies of the fan (shell) F_m are cordial for all $m \geq 3, n \geq 1$. In [2], this result was generalised by proving cordiality of all multiple shells. In [3], the cordiality of the $t-$uniform homeomorphs $P_t(G)$ of a cordial graph G was investigated while in [4], necessary and sufficient conditions for the cordiality of the $t-$ uniform homeomorphs of complete graphs were

Cordial Labelings of Plys

Definition: A $t-$ply $P_t(u,v)$ is a graph with t paths, each of length at least two and such that no two paths have a vertex in common except for the end vertices u and v.



$P_4(u,v)$

K.R. Parthasarathy [6] uses the term **thread** with ply number t for $P_t(u,v)$. It should be be noted that strictly speaking $P_t(u,v)$ should denote the family of all t-plies having end points u and v. Since we are going to prove cordiality of all the graphs in this family which do not satisfy the conditions of Theorem A, we use this notation.

Let $P \equiv \{u, v_1, \cdots, v_n, v\}$ be a typical path with end points u and v in the t-ply $P_t(u,v)$. The length $l(P)$ of this path is $n+1$. We say that the path P is of type i if $l(P) \equiv i(\text{mod}4)), i = 1,2,3,4$. Denote by t_i, the number of paths of the type $i, i = 1,2,3,4$. Then

$$t = t_1 + t_2 + t_3 + t_4 \cdots\cdots (\text{I})$$

If $e = \mid E(P_t(u,v)) \mid$, then

$$e \equiv (t_1 + 2\, t_2 + 3\, t_3)(\text{mod}4) \cdots\cdots (\text{II})$$

Further, let $t_1 = 4\,s_1 + x_1, t_3 = 4\,s_3 + x_3, t_2 = 2\,s_2 + x_2, t_4 = 2\,s_4 + x_4, 0 \leq x_1,\, x_3 \leq 3$ and $0 \leq x_2,\, x_4 \leq 1$. By (II), it follows that

$$e \equiv x_1 + 2\,x_2 + 3\,x_3 (\text{mod}4) \cdots\cdots (\text{III})$$

Since x_1, x_3 take 4 values each and x_2, x_4 take 2 values each, there will in all be 64 cases to be considered for cordiality. In the following 8 cases, $P_t(u,v)$ is Eulerian and by (III), $e \equiv 2$ (mod4), and hence by Theorem A, $P_t(u,v)$is not cordial.

1.$x_1 = x_3 = 0; x_2 = x_4 = 1.$

2: $x_1 = x_2 = x_4 = 0, x_3 = 2.$

3: $x_1 = x_2 = x_3 = x_4 = 1.$

4: $x_1 = 1, x_2 = x_4 = 0, x_3 = 3.$

5: $x_1 = 2, x_2 = x_4 = x_3 = 0.$

6: $x_1 = 2 = x_3, x_2 = x_4 = 1.$

7: $x_1 = 3, x_2 = x_4 = 0, x_3 = 1.$

8: $x_1 = x_3 = 3, x_2 = x_4 = 1.$

In this paper, we prove the following:

Theorem: Except the eight cases mentioned earlier, the $t-$ply $P_t(u,v)$ is cordial.

Proof: We make use of two types of labelings of the end vertices u, v. We call a vertex labeling, a labeling of Type A if $f(u) = 1, f(v) = 0$ and of Type B if $f(u) = 0 = f(v)$. For each type, the labeling will be done in two stages. Let $v_f'(0), v_f'(1)$ be the number of vertices labeled in the first stage with the labels 0 and 1 respectively and let $v_f''(0), v_f''(1)$ be the number of vertices labeled in the second stage with the labels 0 and 1 respectively. Let $e_f'(0), e_f'(1), e_f''(0), e_f''(1)$ be defined similarly.Then

$v_f(0) = v_f'(0) + v_f''(0), v_f(1) = v_f'(1) + v_f''(1),$

$e_f(0) = e_f'(0) + e_f''(0), e_f(1) = e_f'(1) + e_f''(1).$

Whenever we find that the number of vertices (respectively edges) being labeled 0 is the same as the number of vertices (respectively edges) being labeled 1, we say that the vertices (respectively edges)are equitably labeled.

Labelings of Type A

Let f be a binary labeling of $P_t(u,v)$ defined in two stages as follows:
Stage 1: For a path $P \equiv \{u, v_1, \cdots, v_n, v\}$ of $P_t(u,v)$, let $n = 4\,q + r, r = 1,2,3,4$. Note that n and hence q and r take different values for different paths.

Label the first $4q$ vertices on each path as $0,0,1,1,\cdots,0,0,1,1$, i.e for $1 \leq j \leq 4q$, let $f(v_j) = 0, j \equiv 1,2 \pmod 4$ and $f(v_j) = 1, j \equiv 0,3 \pmod 4$. Observe that for each path, out of $4q$ edges which have received a label so far, exactly $2q$ edges have received label 0 and remaining $2q$ edges have received label 1. Similarly, exactly half the vertices labeled so far have received the label 0 and the remaining half have received the label 1.

On each path of type 1, the last four intermediate vertices, on each path of type 2, only the last intermediate vertex, on each path of type 3, the last 2 intermediate vertices and on each path of type 4, the last 3 intermediate vertices remain to be labeled. Moreover, the labeling so far is equitable on vertices as well as on edges.

Out of $4s_1 + x_1$ paths of type 1, label the last 4 vertices as $0,0,1,1$ in that order on $3s_1$ paths and for the next s_1 paths of type 1, label the last four vertices as $1,1,0,0$. Then of the $20s_1$ edges which have been just labeled, $10s_1$ will have received the label 0 and $10s_1$ will have received the label 1. Hence so far, the labeling is equitable on the vertices as well as on the edges.There now remain x_1 paths of type 1, on each of which the last 4 intermediate vertices need to be labeled.

Out of $2s_2 + x_2$ paths of type 2, label the last unlabeled vertex as 1 on s_2 paths and on the next s_2 paths of type two, label the last unlabeled vertex as 0. Then of the $4s_2$ edges that have been just labeled, exactly half have received the label 0 and half have received the label 1. Thus so far,the labeling is equitable on vertices as well as on edges.

For $4s_3 + x_3$ paths of type 3, label the last 2 vertices as $1,0$ in that order on $3s_3$ paths and for the next s_3 paths of type 3, label the last two vertices as $0,1$. Then of the $12s_3$ edges which have just been labeled, $6s_3$ will have received the label 0 and $6s_3$ will have received the label 1. Hence so far,the labeling is equitable on vertices as well as on edges.

Out of the $2s_4 + x_4$ paths of type 4, label the last 3 unlabeled vertices as 1, 0, 1 on s_4 paths and on the next s_4 paths of type 4, label the last three unlabeled vertices as 1, 0, 0. Then of the $8s_4$ edges that have been just labeled, exactly half have received the label 0 and half have received the label 1. Thus so far,the labeling is again equitable on vertices as well as on edges.

There now remain x_1 paths of type 1 on each of which the last 4 intermediate vertices remain to be labeled, x_2 paths of type 2 on each of which the last one intermediate vertex remains to be labeled, x_3 paths of type 3 on each of which the last 2 intermediate vertices remain to be labeled and finally, x_4 paths of type 4 on each of which the last 3 intermediate vertices remain to be labeled.

Since the labeling so far is equitable on vertices as well as on edges $| v_f(0) - v_f(1) |=| v_f''(0) - v_f''(1) |$ and $| e_f(0) - e_f(1) |=| e_f''(0) - e_f''(1) |$.

Stage2: If $x_i = 0$ for some i, then no further paths of type i remain incompletely labeled. For the remaining values of $x_i, i = 1, 2, 3, 4$, we first construct a set of labelings for the unlabeled vertices and use them as need arises.

If $x_1 = 1$, there is only one path left of type 1 on which the last four intermediate vertices need to be labeled. For these vertices define the labeling a_{11} as follows: $a_{11}(v_{n-3}) = a_{11}(v_{n-2}) = 0, a_{11}(v_{n-1}) = a_{11}(v_n) = 1$. This labeling is equitable on the vertices. Of the 5 edges which now receive a label, 2 receive the label 0 and 3 edges receive the label 1.

If $x_1 = 2$, there are 2 paths left of type 1 on which the last four intermediate vertices remain to be labeled. We keep ready two different labelings a_{21}, a_{22} for these paths:

Define $a_{21}(v_{n-3}) = a_{21}(v_{n-2}) = 0; a_{21}(v_{n-1}) = a_{21}(v_n) = 1$

for both the paths. Now define

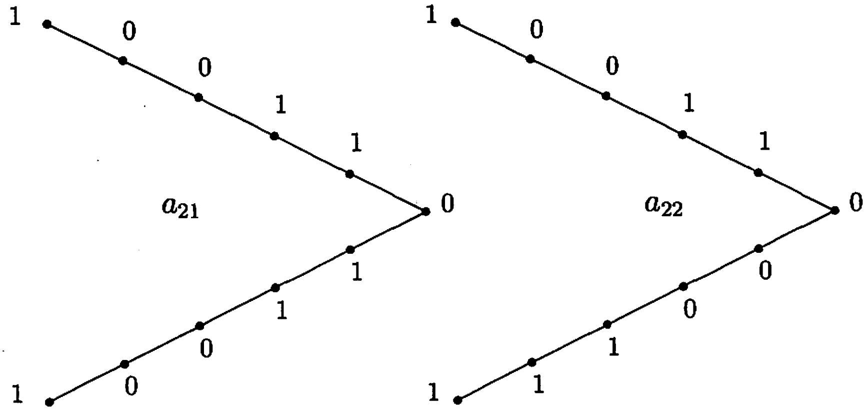
$a_{22}(v_{n-3}) = a_{22}(v_{n-2}) = 0; a_{22}(v_{n-1}) = a_{22}(v_n) = 1$

for one of the paths and

$a_{22}(v_{n-3}) = a_{22}(v_{n-2}) = 1; a_{22}(v_{n-1}) = a_{22}(v_n) = 0$

for the other path.

Then $v_{a_{21}}(0) = v_{a_{21}}(1) = 4; v_{a_{22}}(0) = v_{a_{22}}(1) = 4$. For the labeling a_{21}, 4 edges receive the label 0 and 6 the label 1, while for the labeling a_{22}, 6 edges receive the label 0 and 4 the label 1.



If $x_1 = 3$, there will be 3 paths of type 1 left. In this case we keep ready two labelings a_{31}, a_{32}, for the last four intermediate unlabeled vertices on these 3 paths defined as follows:

$a_{31}(v_{n-3}) = a_{31}(v_{n-2}) = 0; a_{31}(v_{n-1}) = a_{31}(v_n) = 1$

for all three paths and

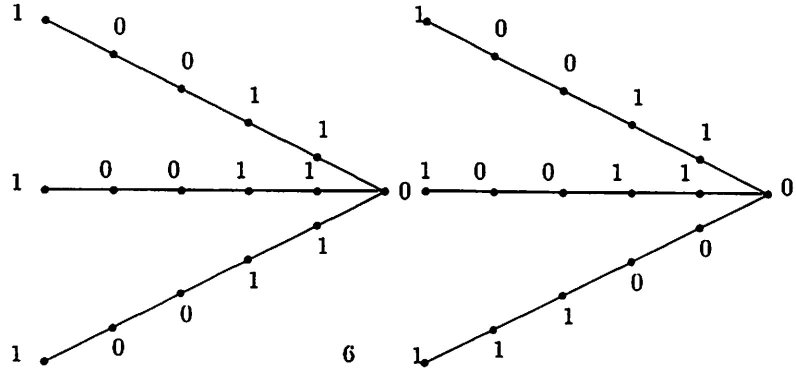
$a_{32}(v_{n-3}) = a_{32}(v_{n-2}) = 0; a_{32}(v_{n-1}) = a_{32}(v_n) = 1$

for two of the three paths and

$a_{32}(v_{n-3}) = a_{32}(v_{n-2}) = 1; a_{32}(v_{n-1}) = a_{32}(v_n) = 0$

for the third path. Clearly,

$v_{a_{31}}(0) = v_{a_{31}}(1) = v_{a_{32}}(0) = v_{a_{32}}(1) = 6.$



a_{31} a_{32}

For the labeling a_{31}, 6 edges receive the label 0 and 9 receive the label 1. For the labeling a_{32}, 8 edges receive the label 0 and 7 edges receive the label 1.

If $x_2 = 1$, there is one path of type 2 on which the last one intermediate vertex remains to be labeled.In this case we keep ready two labelings b_{11}, b_{12}. Define the labeling b_{11} by

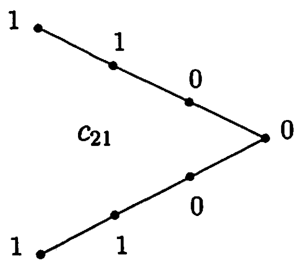
$b_{11}(v_n) = 0$. This will produce one additional vertex with label 0. Moreover, of the new edges labeled now, one receives the label 0 and one receives the label 1.

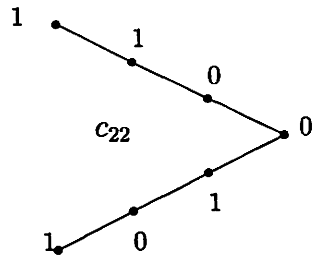
Define the labeling b_{12} by

$b_{12}(v_n) = 0$. This will produce one additional vertex with label 1. Moreover, of the new edges labeled now, one receives the label 0 and one receives the label 1.

If $x_3 = 1$, there is one path of type 3 on which the last two intermediate vertices remain unlabeled. For these vertices, define the labeling c_{11} by $c_{11}(v_{n-1}) = 1, c_{11}(v_n) = 0$. This means that vertices are labeled equitably so far but out of three edges labeled at this stage, 2 receive the label 0 and one receives the label 1.

If $x_3 = 2$, there are two paths of type 3 on each of which the last two intermediate vertices remain unlabeled. In this case again, we keep ready two labelings, c_{21}, c_{22} defined by $c_{21}(v_{n-1}) = 1, c_{21}(v_n) = 0$, for both the paths, and $c_{22}(v_{n-1}) = 1, c_{22}(v_n) = 0$ for one path and $c_{22}(v_{n-1}) = 0, c_{22}(v_n) = 1$ for the other path. Clearly c_{21} and c_{22} label the vertices equitably.





Out of the 6 new edges which now receive labels, 4 are assigned the label 0 and 2 are assigned the label 1 by c_{21}. On the other hand, c_{22} labels 2 of these edges 0 and the remaining 4 the label 1.

If $x_3 = 3$, there are 3 paths of type 3 left on each of which the last two intermediate vertices remain unlabeled. Define the labeling c_{31} for these vertices as $c_{31}(v_{n-1}) = 1, c_{31}(v_n) = 0$ for two of these paths and $c_{31}(v_{n-1}) = 0, c_{31}(v_n) = 1$ for the third path. Again the vertices are labeled equitably. However, out of 9 additional edges at this stage, 4 edges receive the label 0 and 5 edges receive the label 1.

Finally, let $x_4 = 1$. There will be one path of type 4 on which the last 3 intermediate vertices remain to be labeled.We keep ready two labelings d_{11}, d_{12} for these vertices, where

$$d_{11}(v_{n-2}) = d_{11}(v_n) = 1, d_{11}(v_{n-1}) = 0,$$

and

$$d_{12}(v_{n-2}) = d_{12}(v_{n-1}) = 1, d_{12}(v_n) = 0.$$

Then d_{11}, d_{12} both assign label 1 to one extra vertex. Out of the 4 new edges labeled by them, 1 edge is assigned label 0 and 3 edges are assigned the label 1 by d_{11} whereas 3 edges are assigned the label 0 and one is assigned the label 1 by d_{12}.

That f is a cordial labeling is indicated by the following tables. Note that in some cases of Stage 2, there is a choice of labelings for f. The choice made, in each case, is indicated by writing the appropriately chosen labeling function in parenthesis. Note also that, as the tables indicate, 45 cases are covered by labelings of Type A and f is cordial in these cases.

TABLE I: $x_1 = 0$.

		$x_2 = 0$				$x_2 = 1$				
x_3	x_4	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$	x_2	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
0	0	0	0	0	0	$1(b_{12})$	1	0	1	1
$1(c_{11})$	0	1	1	2	1	$1(b_{12})$	2	1	3	2
$1(c_{11})$	$1(d_{11})$	2	3	3	4	$1(b_{11})$	3	3	4	5
$2(c_{21})$	$1(d_{11})$	3	4	5	5	$1(b_{11})$	4	4	6	6
$3(c_{31})$	0	3	3	4	5	$1(b_{12})$	4	3	5	6
$3(c_{31})$	$1(d_{12})$	4	5	7	6	$1(b_{11})$	5	5	8	7

TABLE II: $x_1 = 1$. For x_1 we use the labeling a_{11}.

		$x_2 = 0$				$x_2 = 1$				
x_3	x_4	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$	x_2	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
0	0	2	2	2	3	$1(b_{12})$	3	2	3	4
0	$1(d_{12})$	3	4	5	4	$1(b_{11})$	4	4	6	5
$1(c_{11})$	0	3	3	4	4	$1(b_{12})$	4	3	5	5
$2(c_{21})$	0	4	4	6	5	$1(b_{12})$	5	4	7	6
$2(c_{21})$	$1(d_{11})$	5	6	7	8	$1(b_{11})$	6	6	8	9
$3(c_{31})$	$1(d_{12})$	6	7	9	9	$1(b_{11})$	7	7	10	10

TABLE III: $x_1 = 2$. (Labeling used for x_1 indicated in the first column).

			$x_2 = 0$				$x_2 = 1$				
x_1	x_3	x_4	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$	x_2	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
$2(a_{21})$	0	$1(d_{12})$	5	6	7	7	$1(b_{11})$	6	6	8	8
$2(a_{21})$	$1(c_{11})$	0	5	5	6	7	$1(b_{12})$	6	5	7	8
$2(a_{21})$	$1(c_{11})$	$1(d_{12})$	6	7	9	8	$1(b_{11})$	7	7	10	9
$2(a_{21})$	$2(c_{21})$	0	6	6	8	8	$1(b_{12})$	7	6	9	9
$2(a_{22})$	$3((c_{31})$	0	7	7	10	9	$1(b_{12})$	8	7	11	10
$2(a_{21})$	$3((c_{31})$	$1(d_{12})$	8	9	11	12	$1(b_{11})$	9	9	12	13

TABLE IV: $x_1 = 3$. (Labeling used for x_1 indicated in the first column).

			$x_2 = 0$				$x_2 = 1$				
x_1	x_3	x_4	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$	x_2	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
$3(a_{32})$	0	0	6	6	8	7	$1(b_{12})$	7	6	9	8
$3(a_{31})$	0	$1(d_{12})$	7	8	9	10	$1(b_{11})$	8	8	10	11
$3(a_{31})$	$2(c_{21})$	0	8	8	10	11	$1(b_{12})$	9	8	11	12
$3(a_{31})$	$2(c_{21})$	$1(d_{12})$	9	10	13	12	$1(b_{11})$	10	10	14	13
$3(a_{32})$	$3((c_{31})$	0	9	9	12	12	$1(b_{12})$	10	9	13	13
$3(a_{32})$	$1((c_{11})$	$1(d_{12})$	8	9	11	11	$1(b_{11})$	9	9	12	12

The only cases which now remain for consideration are as follows:

(1)$x_1 = x_2 = x_3 = 0, x_4 = 1$

(2)$x_1 = x_4 = 0, x_2 = 1, x_3 = 2$

(3)$x_1 = x_3 = x_4 = 1, x_2 = 0$

(4)$x_1 = x_2 = 1, x_3 = 3, x_4 = 0$

(5)$x_1 = x_3 = 2, x_2 = 0, x_4 = 1$

(6)$x_1 = 2, x_2 = 1, x_3 = x_4 = 0$

(7)$x_1 = x_3 = 3, x_2 = 0, x_4 = 1$

(8)$x_1 = 3, x_2 = x_3 = 1, x_4 = 0.$

For these eight cases, $e \equiv 0(\text{mod}4)$. It can be proved that in each of these cases, if we use labelings of Type A, then $e_f(0)$ will always be odd, hence such a labeling will not be cordial. We therefore show that $P_t(u,v)$ is cordial, in each of these cases, by using labelings of Type B as described below. Recall that for a labeling of Type B, $f(u) = 0 = f(v.)$

Labelings of Type B:

Let f be a binary labeling of $P_t(u,v)$ defined in two stages as follows:
Stage 1: $f(u) = 0 = f(v)$. For each path in $P_t(u,v)$, let n, q, r be as described earlier and label the first $4q$ vertices on each path as $1,1,0,0,\cdots,1,1,0,0$, i.e for $1 \leq j \leq 4q$, let $f(v_j) = 1, j \equiv 1,2 \ (\text{mod}4)$ and $f(v_j) = 0, j \equiv 0,3 \ (\text{mod}4)$. Then, of the $4q$ edges which have received a label so far, exactly $2q$ edges will have received the label 0 and $2q$ edges the label 1. Similarly, of all the intermediate vertices labeled so far,exactly half will have received the label 0 and half the label 1.

For $3s_1$ of the paths of type 1, label the last four intermediate vertices by f as $1,1,0,0$ and for s_1 paths of type 1, label the last four intermediate vertices as $1,0,0,1$. Then out of the $20s_1$ edges which have just been labeled, $10s_1$ receive the label 0 and $10s_1$ the label 1. Hence, so far, edges are equitably labeled. It is easily seen that the intermediate vertices are also equitably labeled.There now remain x_1 paths of type 1, on each of which the last four intermediate vertices need to be labeled.

Out of $2s_2 + x_2$ paths of type 2, assign the label 1 to the last unlabeled vertex on s_2 paths and assign the label 0 to the last intermediate vertex of the next s_2 paths. Then of the $4s_2$ edges that have been just labeled, $2s_2$ will have received the label 0 and $2s_2$ the label 1. For these $2s_2$ paths also, the edges and intermediate vertices are equitably labeled. There now remain x_2 paths of type 2, on each of which the last vertex remains to be labeled.

Out of $4s_3 + x_3$ paths of type 3, assign labels $1, 0$ to the last two intermediate vertices on $2s_3$ paths, assign labels $1, 1$ to the last two intermediate vertices on the next s_3 paths and finally assign labels $0, 0$ to the last two intermediate vertices of the next s_3 paths.Then of the $12s_3$ edges which have just been labeled, $6s_3$ will have received the label 0 and $6s_3$ will have received the label 1. For these $4s_3$ paths of type 3, the edges as well as intermediate vertices are equitably labeled.There now remain x_3 paths of type 3, on each of which the last two vertices remain to be labeled.

Out of the $2s_4 + x_4$ paths of type 4, assign labels $1, 1, 0$ for the last three intermediate vertices of s_4 paths and assign labels $1, 0, 0$to the last three intermediate vertices of the next s_4 paths. Out of the $8s_4$ edges that have received a label so far, $4s_4$ will have received the label 0 and $4s_4$ the label 1. Hence, for these $2s_4$ paths of type 4, the intermediate vertices as well as edges are equitably labeled. There now remain x_4 paths of type 4, on each of which the last three intermediate vertices remain to be labeled.

Clearly, $v'_f(0) = v'_f(1) + 2$ and $e'_f(0) = e'_f(1)$. Thus:
$| v_f(0) - v_f(1) | = | v''_f(0) + 2 - v''_f(1) |$ and $| e_f(0) - e_f(1) | = | e''_f(0) - e''_f(1) |$.

Stage 2: If $x_i = 0$, for some i, no path of type i is incompletely labeled. For the remaining values of $x_i, i = 1, 2, 3, 4$, we first construct a set of labelings for the unlabeled vertices and use them as need arises, in the sequel.

If $x_1 = 1$, only one path of type 1 will be left, on which the last four intermediate vertices need to be labeled. Define the labelings j_{11}, j_{12} for these vertices by: $j_{11}(v_{n-3}) = j_{11}(v_{n-2}) = 1, j_{11}(v_{n-1}) = j_{11}(v_n) = 0$. Then j_{11} labels the vertices equitably. Also, of the 5 new edges which receive a label due to j_{11}, 3 get the label 0 and 2 the label 1. Define j_{12} by:

$$j_{12}(v_{n-3}) = j_{12}(v_{n-2}) = j_{12}(v_{n-1}) = 1, j_{11}(v_n) = 0.$$

Then j_{12} labels one of the intermediate vertices by the label 0 and 3 of the intermediate vertices by the label 1. Of the 5 edges which now receive a label, 3 receive the label 0 and 2 the label 1.

If $x_1 = 2$, there will be 2 paths of type 1 left on each of which the last 4 intermediate vertices remain to be labeled. We keep ready the following labelings, to be used as need arises: Define the labeling j_{21} as:

$$j_{21}(v_{n-3}) = j_{21}(v_{n-2}) = 1, j_{21}(v_{n-1}) = j_{21}(v_n) = 0$$

for both the paths. Here the vertices are equitably labeled and of the 10 edges which have received a label, 6 receive the label 0 and 4 the label 1. Define the labeling j_{22} by:

$$j_{22}(v_{n-3}) = j_{22}(v_{n-2}) = j_{22}(v_{n-1}) = j_{22}(v_n) = 1$$

for one path and

$$j_{22}(v_{n-3}) = j_{22}(v_{n-2}) = j_{22}(v_{n-1}) = 0, j_{22}(v_n) = 1$$

for the other path.Of the 8 vertices which are labeled thus, 3 receive the label 0 and 5 the label 1, while of the 10 edges that receive a label, 6 receive the label 0 and 4 the label 1.

If $x_1 = 3$, there will be three paths of type 1 left.We define two labelings j_{31}, j_{32} for the vertices as follows:

$$j_{31}(v_{n-3}) = j_{31}(v_{n-2}) = 1, j_{31}(v_{n-1}) = j_{31}(v_n) = 0$$

for all three paths. j_{31} labels the vertices equitably. Of the 15 edges so labeled, 9 receive the label 0 and 6 receive the label 1. The labeling j_{32} is defined as follows:

$$j_{32}(v_{n-3}) = j_{32}(v_{n-2}) = j_{32}(v_{n-1}) = 1, j_{32}(v_n) = 0$$

for the first path.

$$j_{32}(v_{n-3}) = j_{32}(v_{n-2}) = 1, j_{32}(v_{n-1}) = j_{32}(v_n) = 0$$

for the second path and

$$j_{32}(v_{n-3}) = j_{32}(v_n) = 1, j_{32}(v_{n-2}) = j_{32}(v_{n-1}) = 0$$

for the third path. Of the 12 intermediate vertices that get labeled by j_{32}, 5 get the label 0 and 7 the label 1. Of the edges that receive labels, 7 get the label 0 and 8 get the label 1.

If $x_2 = 1$, there will be left only one path of type 2 on which the last intermediate vertex requires labeling. For this vertex, we keep two labelings ready viz: k_{11} and k_{12} where $k_{11}(v_n) = 1, k_{12}(v_n) = 0$. Then $v_{k_{11}}(0) = 0, v_{k_{11}}(1) = 1$ and $v_{k_{12}}(0) = 1, v_{k_{12}}(1) = 0$. Of the two edges which now receive a label, the labeling k_{11} assigns the label 1 to both the edges and the labeling k_{12} assigns the label 0 to both the edges.

If $x_3 = 1$, there will be one path of type 3 left on which the last two intermediate vertices have to be labeled. For these two vertices, define l_{11} by $l_{11}(v_{n-1}) = l_{11}(v_n) = 1$. Then two extra vertices have received label 1. Of the 3 edges that have received a label, 1 edge receives the label 0 and 2 edges receive the label 1.

If $x_3 = 2$, there will be left only two paths of type 2, on each of which the last two intermediate vertices need to be labeled. For these two paths, we keep ready two labelings l_{21}, l_{22} ready where $l_{21}(v_{n-1}) = l_{21}(v_n) = 1$ for one path and $l_{21}(v_{n-1}) = 1, l_{21}(v_n) = 0$ for the other path, whereas $l_{22}(v_{n-1}) = 1 = l_{22}(v_n)$ for both the paths.

Then l_{21} labels one of the intermediate vertices as 0 and labels 3 of the intermediate vertices as 1. Of the edges now labeled, 2 get the label 0 and 4 get the label 1. The labeling l_{22} assigns the label 1 to all 4 intermediate vertices on the two paths, while 2 of the edges receive a label 2 and 4 receive the label 1.

If $x_3 = 3$, three paths of type 3 are left, on which the last 2 intermediate vertices need to be labeled. Define the labeling l_{31} for these vertices by $l_{31}(v_{n-1}) = 0 = l_{31}(v_n)$ for one path, $l_{31}(v_{n-1}) = 1 = l_{31}(v_n)$ for the second path and $l_{31}(v_{n-1}) = 1, l_{31}(v_n) = 0$ for the third path.The intermediate vertices are equitably labeled, while 5 of the edges thus labeled receive the label 0 and 4 receive the label 1. Define the labeling l_{32} by $l_{32}(v_{n-1}) = 1 = l_{32}(v_n)$ for one path, $l_{32}(v_{n-1}) = 1, l_{32}(v_n) = 0$ for the other two paths.Of the intermediate vertices which now receive labels, 2 receive the label 0 and 4 the label 1, while of the edges which receive labels, 3 receive the label 0 and 6 receive the label 1.

Finally, if $x_4 = 1$, we see that there is one path of type 4 left, on which the last three intermediate vertices need to be labeled. For these vertices we keep ready the labelings m_{11}, m_{12} defined as follows: $m_{11}(v_{n-2}) = m_{11}(v_{n-1}) = 1, m_{11}(v_n) = 0$ and $m_{12}(v_{n-2}) = m_{12}(v_{n-1}) = m_{12}(v_n) = 1$. The labeling m_{11} labels one of the intermediate vertices as 0 and the other two intermediate vertices as 1. Of the edges which are thus labeled, 2 edges receive the label 0 while 2 receive the label 1. The labeling m_{12} labels all three intermediate vertices as 1. Of the edges which are thus labeled, 2 edges receive the label 0 while 2 receive the label 1.

TABLE V:

x_1	x_2	x_3	x_4	$v_f''(0)$	$v_f''(1)$	$e_f''(0)$	$e_f''(1)$
0	0	0	$1(m_{12})$	0	3	2	2
0	$1(k_{12})$	$2(l_{22})$	0	1	4	4	4
$1(j_{11})$	0	$1(l_{11})$	$1(m_{11})$	3	6	6	6
$1(j_{12})$	$1(k_{11})$	$3(l_{31})$	0	4	7	8	8
$2(j_{21})$	0	$2(l_{21})$	$1(m_{11})$	6	9	10	10
$2(j_{22})$	$1(k_{11})$	0	0	3	6	6	6
$3(j_{31})$	0	$3(l_{32})$	$1(m_{11})$	9	12	14	14
$3(j_{32})$	$1(k_{12})$	$1(l_{11})$	0	6	9	10	10

That f is a cordial labeling is indicated by the above table. As before, whenever an option for the labeling has been exercised, it is indicated in the parenthesis.

Remark: In a sequel to this paper, we have investigated cordiality of elongated plys, which do not satisfy the hypothesis of Theorem A.

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