Applications Related to the Generalized Seidel Matrix

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Abstract

Let α, β be any numbers. Given an initial sequence $a_{0,m}$ ($m = 0,1,2,\cdots$), define the sequences $a_{n,m}$ ($n \geq 1$) recursively by

$$a_{n,m} = \alpha a_{n-1,m} + \beta a_{n-1,m+1}, \qquad \text{for } n \geq 1, m \geq 0.$$

We call the matrix $(a_{n,m})_{n,m\geq 0}$ as a generalized Seidel matrix with a parameter pair (α, β). If $\alpha = \beta = 1$, then this matrix is the classical Seidel matrix. For various different parameter pairs (α, β) we will impose some evenness or oddness conditions on the exponential generating functions of the initial sequence $a_{0,m}$ and the final sequence $a_{n,0}$ of a genaralized Seidel matrix (i.e., we require that these generating functions or certain related functions are even or odd). These conditions imply that the initial sequences and final sequences are equal to well-known classical sequences such as those of the Euler numbers, the Genocchi numbers, and the Springer numbers.

As applications, we give a straightforward proof of the continued fraction representations of the ordinary generating functions of the sequence of Genocchi numbers. And we also get the continued fractions representations of the ordinary generating functions of the Genocchi polynomials, Bernoulli polynomials, and Euler polynomials. Lastly, we give some applications of congruences for the Euler polynomials.

Key Words: Seidel matrix, continued fraction, Springer numbers, Genocchi numbers, Euler numbers.

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1 Introduction to Generalized Seidel Matrices

Let α, β be any complex numbers. Given an initial sequence $a_{0,m}$ ($m = 0, 1, 2, \cdots$), define the sequences $a_{n,m}$ ($n \geq 1$) recursively by

$$a_{n,m} = \alpha a_{n-1,m} + \beta a_{n-1,m+1}, \qquad \text{for } n \geq 1, m \geq 0.$$

We call the matrix $(a_{n,m})_{n,m\geq 0}$ a generalized Seidel matrix with a parameter pair (α, β). If $\alpha = \beta = 1$, then this matrix is the classical Seidel matrix (ref. [3]).

Proposition 1. *Given an initial sequence $a_{0,m}$ ($m = 0, 1, 2, \cdots$), define the sequences $a_{n,m}$ ($n \geq 1$) recursively by*

$$a_{n,m} = \alpha a_{n-1,m} + \beta a_{n-1,m+1}, \qquad (n \geq 1, m \geq 0). \tag{1}$$

Then the sequences $a_{n,m}$ can be expressed as

$$a_{n,m} = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \beta^k a_{0,m+k}. \tag{2}$$

The sequence $a_{n,0}$ is called a final sequence. From Eq. 2 the final sequence $a_{n,0}$ has the relation with the initial sequnece $a_{0,n}$:

$$a_{n,0} = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \beta^k a_{0,k}. \tag{3}$$

Proposition 2. *Let $A(x) = \sum_{n=0}^{\infty} \frac{a_{0,n}x^n}{n!}$ be the exponential generating function (egf) of $a_{0,n}$ and $\bar{A}(x)$ the egf of $a_{n,0}$. Then*

$$\bar{A}(x) = e^{\alpha x} A(\beta x), \quad \text{or} \quad A(x) = e^{-\alpha x/\beta} \bar{A}(\frac{x}{\beta}). \tag{4}$$

Proposition 3. *Let $a(x) = \sum_{n=0}^{\infty} a_{0,n} x^{n+1}$ be the ordinary generating function (ogf) of $a_{0,n}$ and $\bar{a}(x)$ be the ogf of $a_{n,0}$. Then*

$$a(x) = \beta \bar{a}(\frac{x}{\alpha x + \beta}), \quad \text{or} \quad \bar{a}(x) = \frac{1}{\beta} a(\frac{\beta x}{1 - \alpha x}). \tag{5}$$

Remark 1. The ogf $a(x)$ (resp. $\bar{a}(x)$) is, in a formal sense, the Laplace transform of $A(x)$ (resp. $\bar{A}(x)$), that is,

$$\int_0^{\infty} A(t) e^{-t/x} dt = a(x).$$

When we impose certain evenness or oddness conditions on the egf $A(x)$ and $\bar{A}(x)$ of some generalized Seidel matrix, its initial and final sequence will be equal to some classical sequnece such as that of the Euler numbers, tangent numbers, Genocchi numbers, etc..

Case A: For $\alpha = \beta = 1$, Dumont [2] imposed the following conditions on the egf $A(x)$ and $\bar{A}(x)$.

A1. $A(x)$ is an even function and $A(0) = 1$.

A2. $\bar{A}(x) - 1$ is an odd function.

These conditions determine uniquely the initial sequence $a_{0,n} = E_n$ and the final sequence $a_{n,0} = T_n$, i.e. the egf are

$$A(x) = \operatorname{sech} x = \sum_{n=0}^{\infty} \frac{E_n x^n}{n!}, \tag{6}$$

$$\bar{A}(x) = 1 + \tanh x = \sum_{n=0}^{\infty} \frac{T_n x^n}{n!}, \tag{7}$$

where the numbers E_n and T_n is the classical Euler numbers and tangent numbers.

This resulting Seidel matrix is the "Euler-Bernoulli triangle" (ref. [2], [7])

1	0	−1	0	5	0	−61	...
1	−1	−1	5	5	−61	...	
0	−2	4	10	−56	...		
−2	2	14	−46	...			
0	16	−32	...				
16	−16	...					
0	...						

Case B: For $\alpha = -1/2$ and $\beta = 1/2$, we impose the following conditions on the egf $A(x)$ and $\bar{A}(x)$:

B1. $A(x)$ is an odd function.

B2. $\bar{A}(x) - x$ is an even function.

Then we have

$$A(x) = 2x \operatorname{sech} x = \sum_{n=1}^{\infty} \frac{2nE_{n-1}x^n}{n!}, \quad \text{and} \tag{8}$$

$$\bar{A}(x) = \frac{2x}{e^x + 1} = \sum_{n=0}^{\infty} \frac{G_n x^n}{n!}, \tag{9}$$

where G_n is the classical sequence of Genocchi numbers.

In section 2, we use the above conditions **B1** and **B2** to give a straightforward proof of the continued fraction representations of the ogf of the Genocchi numbers.

The Springer numbers (ref. [2]) are defined by

$$S(x) = e^x \operatorname{sech} 2x = \sum_{n=0}^{\infty} \frac{S_n x^n}{n!}. \tag{10}$$

The even (resp. odd) part of the Springer numbers is what Glaisher (ref. [2]) called the numbers P_n (resp. Q_n). That is to say,

$$\frac{\cosh x}{\cosh 2x} = \sum_{n=0}^{\infty} \frac{S_{2n} x^{2n}}{(2n)!}, \qquad \frac{\sinh x}{\cosh 2x} = \sum_{n=0}^{\infty} \frac{S_{2n+1} x^{2n+1}}{(2n+1)!}. \tag{11}$$

Springer introduced these numbers for a problem about root systems, and Arnold showed these numbers as counting various types of snakes (ref. [6]). In the following we give some generalized Seidel matrices related to these numbers.

Case C: For $\alpha = -1$ and $\beta = 2$, we impose the following conditions on the egf $A(x)$ and $\bar{A}(x)$:

C1. $A(x) - \tanh x$ is an even function.

C2. $\bar{A}(x)$ is an odd function.

Then we have $A(x) = 1 + \tanh x - \operatorname{sech} x$ and $\bar{A}(x) = \dfrac{2 \sinh x}{\cosh 2x}$.

Case D: For $\alpha = 1$ and $\beta = 2$, we impose the following conditions on the egf $A(x)$ and $\bar{A}(x)$:

D1. $A(x) + \tanh x$ is an even function.

D2. $\bar{A}(x)$ is an even function.

Then we have $A(x) = 1 - \tanh x + \operatorname{sech} x$ and $\bar{A}(x) = \dfrac{2 \cosh x}{\cosh 2x}$.

Case E: For $\alpha = \beta = 1/2$, we impose the following conditions on the egf $A(x)$ and $\bar{A}(x)$:

E1. $A(x) - \dfrac{\cosh x}{\cosh 2x}$ is an odd function.

E2. $\bar{A}(x)$ is an even function.

Then we have $A(x) = \frac{e^{-x}}{\cosh 2x} = S(-x)$ and $\bar{A}(x) = \operatorname{sech} x$.

Case F: For $\alpha = \beta = 1/2$, we impose the following conditions on the egf $A(x)$ and $\bar{A}(x)$:

F1. $e^{-x}A(x)$ is an even function.

F2. $\bar{A}(x) - 1$ is an odd function.

Then we have $A(x) = S(x)$ and $\bar{A}(x) = 1 + \tanh x$.

Case G: For $\alpha = 1$ and $\beta = 2$, we impose the following conditions on the egf $A(x)$ and $\bar{A}(x)$:

G1. $A(x)$ is an even function.

G2. $e^{x}\bar{A}(x) - 1$ is an odd function.

Then we have $A(x) = \operatorname{sech} x$ and $\bar{A}(x) = S(x)$.

In section 3–5, we give the continued fraction representations of the ogf of the Genocchi polynomials, Bernolli polynomials, and Euler polynomials. In section 6, we give some applications of congruences for the Euler polynomials.

2 The continued fractions of the ogf of the Genocchi numbers

In the following we discuss some applications on the continued fraction representation of the corresponding ordinary generating functions.

We need a lemma from [2]

Lemma 1. *The following representations of a series $f(x)$ are equivalent:*

$$f(x) = \frac{x}{1+}\frac{c_1x}{1+}\frac{c_2x}{1+}\frac{c_3x}{1+}\cdots,$$

$$= \frac{x}{1+c_1x-}\frac{c_1c_2x^2}{1+(c_2+c_3)x-}\frac{c_3c_4x^2}{1+(c_4+c_5)x-}\cdots, \tag{12}$$

$$= x - \frac{c_1x^2}{1+(c_1+c_2)x-}\frac{c_2c_3x^2}{1+(c_3+c_4)x-}\frac{c_4c_5x^2}{1+(c_5+c_6)x-}\cdots. \tag{13}$$

Here we use the notation

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots\frac{a_n}{b_n+}\cdots$$

to represent

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots\, b_{n-1} + \cfrac{a_n}{\ddots}}}}.$$

Theorem 1. *The continued fraction representations for the ogf of the Genocchi numbers are*

$$\sum_{n=0}^{\infty} G_n x^{n+1}$$

$$= \frac{x^2}{1+}\frac{x}{1-}\frac{x}{1+}\frac{x}{1-}\frac{x}{1+}\cdots+\frac{nx}{1}-\frac{nx}{1}+\frac{nx}{1}-\frac{nx}{1}+\cdots \tag{14}$$

$$= \frac{x^2}{1+x+}\frac{x^2}{1}+\frac{x^2}{1+x+}\cdots+\frac{(nx)^2}{1}+\frac{(nx)^2}{1+x+}\cdots \tag{15}$$

$$= x^2-\frac{x^2}{1+}\frac{x^2}{1+}\frac{2x^2}{1+}\cdots+\frac{(nx)^2}{1}+\frac{n(n+1)x^2}{1}+\cdots. \tag{16}$$

Proof. Consider the above Case **B**, i.e. the matrix $GS(-1/2, 1/2)$ with conditions **B1, B2**. Since $\bar{A}(x)$ is the egf of G_n, $\bar{a}(x)$ is the ogf of G_n. Assume

$$\begin{aligned}\bar{a}(x) &= \frac{x^2}{1+}\frac{c_1x}{1+}\frac{c_2x}{1+}\cdots \\ &= x^2-\frac{c_1x^3}{1+(c_1+c_2)x-}\frac{c_2c_3x^2}{1+(c_3+c_4)x-}\cdots\end{aligned}$$

Since $\bar{a}(x)-x^2$ is an odd function,

$$c_{2n-1}+c_{2n}=0, \qquad \text{for } n\geq 1. \tag{17}$$

From Proposition 3 and Equation 12 we have

$$\begin{aligned}a(x) &= -\bar{a}(\frac{-2x}{x+2}) \\ &= \frac{-(\frac{-2x}{x+2})^2}{1+c_1(\frac{-2x}{x+2})-}\frac{c_1c_2(\frac{-2x}{x+2})^2}{1+(c_2+c_3)(\frac{-2x}{x+2})-}\frac{c_3c_4(\frac{-2x}{x+2})^2}{1+(c_4+c_5)(\frac{-2x}{x+2})-}\cdots \\ &= \frac{-4x^2}{(1-2c_1)x^2+(4-4c_1)x+4-}\frac{4c_1c_2x^2}{1+(c_2+c_3)(\frac{-2x}{x+2})} \\ &\qquad -\frac{4c_3c_4x^2}{(1-2c_4-2c_5)x^2+(4-4c_4-4c_5)x+4-}\cdots.\end{aligned}$$

Since $a(x)$ is an even function, this gives

$$\begin{aligned} &1 - c_1 = 0, \\ &c_{4n-2} + c_{4n-1} = 0, \qquad 1 - c_{4n} - c_{4n+1} = 0, \quad \text{for } n \geq 1. \end{aligned} \tag{18}$$

Combining with Eq. 17 we obtain

$$c_{4n-3} = c_{4n-1} = n, \quad c_{4n-2} = c_{4n} = -n, \quad \text{for } n \geq 1.$$

This completes our proof □

3 The continued fractions of the ogf of the Genocchi polynomials

Let the initial sequence be the Genocchi numbers G_n in a generalized Seidel matrix with $\alpha = \delta$ and $\beta = 1$. Then the egf of the final sequence becomes

$$\bar{A}(x) = e^{\delta x} A(x) = \frac{2xe^{\delta x}}{e^x + 1} = \sum_{n=1}^{\infty} \frac{G_n(\delta) x^n}{n!}. \tag{19}$$

Here $G_n(\delta)$ is a polynomial in δ, the so-called n-th Genocchi polynomial (ref. [4]).

Now since the continued fraction representation for the ogf of the Genocchi numbers is (see Eq.14)

$$\begin{aligned} a(x) &= \sum_{n=0}^{\infty} G_n x^{n+1} \\ &= \frac{x^2}{1+}\frac{x}{1-}\frac{x}{1+}\frac{x}{1-}\frac{x}{1+}\cdots\frac{nx}{+1-}\frac{nx}{1+}\frac{nx}{1-}\frac{nx}{1+}\cdots. \end{aligned}$$

From Proposition 3

$$\begin{aligned} \bar{a}(x) &= \sum_{n=0}^{\infty} G_n(\delta) x^{n+1} = a\left(\frac{x}{1-\delta x}\right) \\ &= \frac{x^2}{(1-\delta x)^2+}\frac{x(1-\delta x)}{1}\frac{x}{-1-\delta x+}\frac{x}{1-}\frac{x}{1-\delta x+}\frac{2x}{1-}\frac{2x}{1-\delta x} \\ &\quad \frac{2x}{+1-}\frac{2x}{1-\delta x+}\cdots\frac{nx}{+1-}\frac{nx}{1-\delta x+}\frac{nx}{1-}\frac{nx}{1-\delta x+}\cdots. \end{aligned} \tag{20}$$

Hence we get the continued fraction representation for the ogf of the Genocchi polynomials.

4 Analogies with Bernoulli polynomials

The Bernoulli polynomials $B_n(X)$ are defined by

$$\frac{te^{Xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(X)\frac{t^n}{n!}, \tag{21}$$

the Bernoulli numbers can be defined by $B_n = B_n(0)$.

Let the initial sequence be the Bernoulli numbers B_n in a generalized Seidel matrix with $\alpha = \delta$ and $\beta = 1$. Then the egf of the final sequence becomes

$$\bar{A}(x) = e^{\delta x}A(x) = \frac{xe^{\delta x}}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(\delta)x^n}{n!},$$

that is, the final sequence is $B_n(\delta)$.

Now since the continued fraction representation for the ogf of the Bernoulli numbers is (ref. [1] Eq. A6, A12)

$$a(x) = \sum_{n=0}^{\infty} B_n x^{n+1} = \frac{x}{1+}\frac{x/2}{1}-\frac{x/6}{1}+\cdots+\frac{\frac{n^2x}{2(2n-1)}}{1}-\frac{\frac{n^2x}{2(2n+1)}}{1}+\cdots. \tag{22}$$

From Proposition 3

$$\begin{aligned}\bar{a}(x) &= \sum_{n=0}^{\infty} B_n(\delta)x^{n+1} = a\left(\frac{x}{1-\delta x}\right)\\ &= \frac{x}{1-\delta x+}\frac{x/2}{1}-\frac{x/6}{1-\delta x+}\cdots+\frac{\frac{n^2x}{2(2n-1)}}{1}-\frac{\frac{n^2x}{2(2n+1)}}{1-\delta x+}\cdots.\end{aligned} \tag{23}$$

Hence we get the continued fraction representation for the ogf of the Bernoulli polynomials.

5 Analogies with Euler polynomials

Let the initial sequence be the Euler numbers E_n in a generalized Seidel matrix with $\alpha = 2\delta - 1$ and $\beta = 1$. Then the egf of the final sequence becomes

$$\bar{A}(x) = e^{(2\delta-1)x}A(x) = e^{(2\delta-1)x}\,\mathrm{sech}\,x = \frac{2e^{2\delta x}}{e^{2x}+1} = \sum_{n=0}^{\infty} \frac{E_n(\delta)2^n x^n}{n!}, \tag{24}$$

that is, the final sequence is $2^n E_n(\delta)$.

Now since the continued fraction representation for the ogf of the Euler numbers is (ref. Corollary 3.1 of [2])

$$a(x) = \sum_{n=0}^{\infty} E_n x^{n+1} = \frac{x}{1+}\frac{x^2}{1+}\frac{2^2x^2}{1+}\cdots\frac{n^2x^2}{1+}\cdots. \tag{25}$$

From Proposition 3

$$\begin{aligned}\bar{a}(x) &= \sum_{n=0}^{\infty} 2^n E_n(\delta) x^{n+1} = a\left(\frac{x}{1-(2\delta-1)x}\right) \\ &= \frac{x}{1-(2\delta-1)x+}\frac{x^2}{1-(2\delta-1)x+}\cdots\frac{n^2x^2}{1-(2\delta-1)x+}\cdots.\end{aligned}$$

Hence we get the continued fraction representation for the ogf of the Euler polynomials:

$$\sum_{n=0}^{\infty} E_n(\delta) t^{n+1} = \frac{2t}{2-(2\delta-1)t+}\frac{t^2}{2-(2\delta-1)t+}\cdots\frac{n^2t^2}{2-(2\delta-1)t+}\cdots. \tag{26}$$

Remark 2. Using the same trick as above we can get the continued fraction representations for the ogf of some of the other classical sequences that appeared in Section 1. In case A, we can use Eq. 25 to get the continued fraction representation for the ogf of the sequence of tangent numbers. In case B, we can use Eq. 14 to get the continued fraction representations for the ogf of $(n+1)E_n$. In case E, F, and G, we can use Eq. 25 or the continued fraction representation for the ogf of the sequence of tangent numbers to get the continued fraction representations for the ogf of the sequence of Springer numbers.

Remark 3. The continued fraction representations of the ogf of B_n, G_n, and E_n are not unique (see e.g. Eq. 14, 15, 16). When we change the representations in the above statements, the corresponding continued fraction representations of the ogf of $B_n(x)$, $G_n(x)$, and $E_n(x)$ are changed, simultaneously. For example, the corresponding continued fraction representation of the ogf of $G_n(x)$, with Eq. 15, is

$$\begin{aligned}&\sum_{n=0}^{\infty} G_n(x) t^{n+1} \\ &= \frac{t^2}{1+(1-2x)t+x(x-1)t^2+}\frac{t^2}{1+}\frac{t^2}{1+(1-2x)t+x(x-1)t^2+}\cdots \\ &\quad \frac{(nt)^2}{1+}\frac{(nt)^2}{1+(1-2x)t+x(x-1)t^2+}\cdots. \end{aligned} \tag{27}$$

6 Congruences for Euler polynomials

Let p be an odd prime and δ an integer. Let $f(t) = \sum_n a_n t^n$ and $g(t) = \sum_n b_n t^n$ $(n \geq 0)$ be two formal power series with integral coefficients. For a non-negative integer m we write

$$f(t) \equiv g(t) \pmod{m} \quad \text{iff} \quad a_n \equiv b_n \pmod{m} \quad \text{for all } n \geq 0. \tag{28}$$

It is straightforward that

$$(1 + 4z^2 h(z))^{-1} \equiv 1 \pmod{4}$$

for any $h(z)$ with integral coefficients. Thus by Eq. 25

$$\begin{aligned} \sum_{n=0}^{\infty} E_{2n} x^{n+1} &\equiv \frac{x}{1+x^2} \\ &= \sum_{n=0}^{\infty} (-1)^n x^{2n+1} \qquad \pmod{4} \end{aligned}$$

This implies (ref. [5])

$$E_{2n} \equiv (-1)^n \qquad \pmod{4}.$$

Since $(2, p) = 1$ it is clear that

$$\begin{aligned} &\sum_{n=0}^{\infty} E_n(\delta) t^{n+1} \\ &\equiv \frac{2t}{2 - (2\delta - 1)t +} \frac{t^2}{2 - (2\delta - 1)t +} \cdots \frac{(p-1)^2 t^2}{+ 2 - (2\delta - 1)t} \pmod{p}. \end{aligned}$$

This enables us to obtain congruences for Euler polynomials. In the sequel we give two examples with $p = 3$ and $p = 5$.

Proposition 4. *If* $(\delta, 3) = 1$, *then for* $n \geq 1$

$$\left\{ \begin{array}{rcl} E_0(\delta) & \equiv & 1 \\ E_{2n}(\delta) & \equiv & 1 - \delta \\ E_{2n-1}(\delta) & \equiv & 1 + \delta \end{array} \right. \pmod{3}. \tag{29}$$

If $\delta \equiv 0 \pmod{3}$, *then for* $n \geq 1$

$$\left\{ \begin{array}{rcl} E_0(\delta) & \equiv & 1 \\ E_{2n}(\delta) & \equiv & 0 \\ E_{2n-1}(\delta) & \equiv & 1 \end{array} \right. \pmod{3}. \tag{30}$$

Proof. Since $(2,3)=1$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_n(\delta)x^{n+1} \\
&\equiv \frac{2x}{2-(2\delta-1)x+\cfrac{x^2}{2-(2\delta-1)x+\cfrac{2^2x^2}{2-(2\delta-1)x}}} \pmod 3 \\
&\equiv \frac{x+(1+\delta)x^2+(\delta^2-\delta+2)x^3}{1+2x^2+(\delta-\delta^3)x^3} \pmod 3.
\end{aligned}$$

If $(\delta,3)=1$, then

$$\begin{aligned}
\sum_{n=0}^{\infty} E_n(\delta)x^{n+1} &\equiv \frac{x+(1+\delta)x^2-\delta x^3}{1-x^2} \\
&\equiv (x+(1+\delta)x^2-\delta x^3)\sum_{n=0}^{\infty} x^{2n} \\
&\equiv x+(1+\delta)\sum_{n=1}^{\infty} x^{2n}+(1-\delta)\sum_{n=1}^{\infty} x^{2n+1}.
\end{aligned}$$

If $\delta\equiv 0 \pmod 3$, then

$$\begin{aligned}
\sum_{n=0}^{\infty} E_n(\delta)x^{n+1} &\equiv \frac{x+x^2+2x^3}{1-x^2} \\
&\equiv (x+x^2+2x^3)\sum_{n=0}^{\infty} x^{2n} \\
&\equiv x+\sum_{n=1}^{\infty} x^{2n}.
\end{aligned}$$

Now we compare the coefficients of x^n, then we complete the proof. □

Using a similar trick, we can easily get the following proposition.

Proposition 5. *If $(\delta,5)=1$, then for $n\geq 0$*

$$\begin{cases}
E_0(\delta) \equiv 1 \\
E_{4n+1}(\delta) \equiv \delta+2 \\
E_{4n+2}(\delta) \equiv \delta^2-\delta \\
E_{4n+3}(\delta) \equiv \delta^3+\delta^2-1 \\
E_{4n+4}(\delta) \equiv 3\delta^3+\delta+1
\end{cases} \pmod 5. \tag{31}$$

If $\delta \equiv 0 \pmod 5$, *then for* $n \geq 0$

$$\left\{\begin{array}{rcl} E_0(\delta) & \equiv & 1 \\ E_{4n+1}(\delta) & \equiv & 2 \\ E_{4n+3}(\delta) & \equiv & -1 \\ E_{2n+2}(\delta) & \equiv & 0 \end{array}\right. \pmod 5. \tag{32}$$

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