Embedding a restricted class of partial K_4-designs

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Abstract

Given a partial K_4-design (X, P), if $x \in X$ is a vertex which occurs in exactly one block of P, then call x a *free vertex*. In this paper, a technique is described for obtaining a cubic embedding of any partial K_4-design with the property that every block in the partial design contains at least two free vertices.

1 Introduction

Let G be a simple graph. A *G-design of order n* is a pair (S, B) where S is the vertex set of K_n, and B is a collection of edge-disjoint subgraphs of K_n, all isomorphic to G, whose edge sets partition the edges of K_n. A *partial G-design of order n* is a pair (X, P), where X is the vertex set of K_n, and P is a collection of edge-disjoint subgraphs of K_n, all isomorphic to G, such that every vertex of K_n occurs in at least one copy of G. Thus, the copies of G in a partial G-design of order n must use all of the vertices of K_n, but not necessarily all of the edges.

A partial G-design of order n with the property that the unused edges of K_n can be partitioned into copies of G is said to be *completable*. It is clear that for almost any simple graph G, not all partial G-designs are completable. Thus, the problem of embedding partial G-designs arises. The partial G-design (X, P) is said to be *embedded* in the G-design (S, B) if $X \subseteq S$ and $P \subseteq B$. Wilson [8] has shown that for any simple graph G, a partial G-design can always be embedded in some G-design. Unfortunately, the order of the containing G-design in Wilson's method is exponentially large with respect to the order of the partial G-design. Hence we have the problem of finding a "small" embedding for an arbitrary partial G-design.

The problem of finding small embeddings for partial G-designs has received much attention. Much progress has been made for the case where G is C_m, the cycle of length m. For example, Lindner [7] has shown that any partial C_4-design of order n can be embedded in a C_4-design of order at most $2n+15$. In fact, linear embeddings are known for partial C_m-designs for any given m. (See [3], [6].) In contrast to this, the search for even a polynomial embedding for partial K_m-designs, for any given $m \geq 4$, has been unsuccessful. The closest result of interest to this problem is Hoffman and Lindner's $8n+16\sqrt{n}+82$ embedding for partial $(K_4 \setminus K_2)$-designs [2]. While the graph $K_4 \setminus K_2$ differs from K_4 by only one edge, Hoffman and Lindner's embedding relies heavily on the fact that $K_4 \setminus K_2$ is tripartite. Thus a small embedding for partial K_4-designs appears to be beyond the reach of the current methods.

The aim of this paper is to describe a technique for embedding a very restricted class of partial K_4-designs. In what follows, where convenient, we shall refer to a copy of K_4 as a *block of size* 4, or simply a *block*. Given a partial K_4-design (X, P), if $x \in X$ is a vertex which occurs in exactly one block of P, then call x a *free vertex*. We shall present a cubic embedding of any partial K_4-design with the property that every block contains at least two free vertices. Note that such a partial K_4-design can be formed by taking any simple graph G, and replacing each edge by a copy of K_4, as shown in Figure 1.

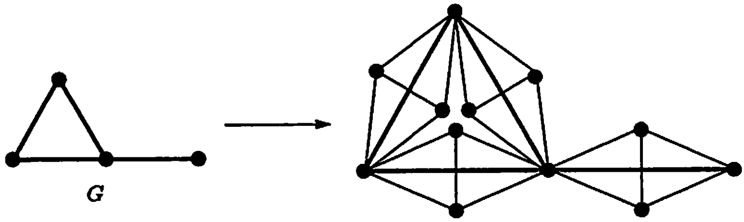

Figure 1: Construction of a partial K_4-design from the graph G.

Specifically, it will be shown that any partial K_4-design of order n with the property that every block contains at least two free vertices can be embedded in a K_4-design of order at most $(n+24)[(n+28)(n+22)+24]/48$.

2 Preliminary Results

A *group divisible design of order v and block size* 4 (or 4-GDD) is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ where $\mathcal{V}$ is a set of size v, $\mathcal{G}$ is a partition of $\mathcal{V}$ into parts (*groups*) of possibly different sizes, and $\mathcal{B}$ is a collection of blocks of size 4 such that every pair of distinct elements of $\mathcal{V}$ in different groups occurs together in exactly one block. If $v = a_1g_1 + a_2g_2 + \ldots + a_sg_s$, and if there are a_i groups of size g_i, $i = 1, 2, \ldots, s$, then the 4-GDD is said to be of type $g_1^{a_1}g_2^{a_2}\cdots g_s^{a_s}$.

Example 2.1 The blocks of a 4-GDD of type 3^4 with groups $\{0,1,2\}$, $\{3,4,5\}$, $\{6,7,8\}$, and $\{9,10,11\}$:

$$\begin{array}{l}\{0,3,6,9\},\ \{0,4,7,10\},\ \{0,5,8,11\},\\ \{1,3,8,10\},\ \{1,4,6,11\},\ \{1,5,7,9\},\\ \{2,3,7,11\},\ \{2,4,8,9\},\ \{2,5,6,10\}.\end{array}$$

□

The following results guarantee the existence of 4-GDDs of certain types.

Lemma 2.2 ([4]) *Let t and u be positive integers. If $t \equiv 0 \pmod 6$, $u \geq 3$, and $(t,u) \neq (6,3)$ then there exists a 4-GDD of type $t^u(t(u-1)/2)^1$.*

Corollary 2.3 *There exists a 4-GDD of type $(12)^{2m+1}(12m)^1$ for all positive integers m.*

Lemma 2.4 ([4]) *Let t and u be positive integers. If $t \equiv 0 \pmod 6$, $u \geq 4$, and $(t,u) \neq (6,4)$ then there exists a 4-GDD of type t^u.*

The existence of K_4-designs of certain orders is also needed for our main result. The following well-known lemma gives the spectrum of K_4-designs.

Lemma 2.5 ([1]) *Let n be a positive integer. There exists a K_4-design of order n if and only if $n \equiv 1$ or $4 \pmod{12}$.*

Let $\Psi(n)$ denote the partial K_4-design (X, P) of order $n + n(n-1)$ obtained as follows. Let $X = \mathbb{Z}_n \cup (2^{\mathbb{Z}_n} \times \{1,2\})$ where $2^{\mathbb{Z}_n}$ denotes the set of 2-element subsets of $\mathbb{Z}_n$. For each pair a, b of distinct elements in $\mathbb{Z}_n$, place in P the block $\{a, b, (\{a,b\},1), (\{a,b\},2)\}$.

Example 2.6 Let $X = \mathbb{Z}_4 \cup (2^{\mathbb{Z}_4} \times \{1,2\})$, and let P contain the following blocks.

$$\{0,1,(\{0,1\},1),(\{0,1\},2)\},$$
$$\{0,2,(\{0,2\},1),(\{0,2\},2)\},$$
$$\{0,3,(\{0,3\},1),(\{0,3\},2)\},$$
$$\{1,2,(\{1,2\},1),(\{1,2\},2)\},$$
$$\{1,3,(\{1,3\},1),(\{1,3\},2)\},$$
$$\{2,3,(\{2,3\},1),(\{2,3\},2)\}.$$

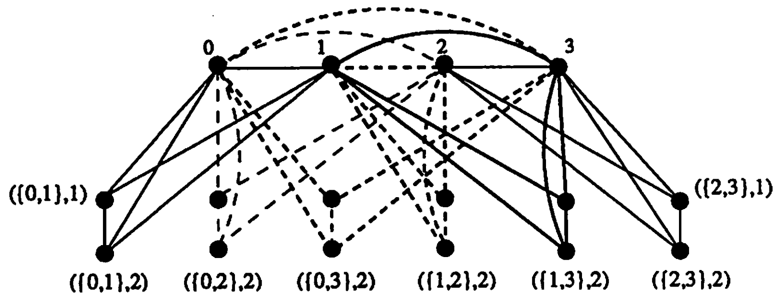

Figure 2: $\Psi(4)$.

Clearly, $(X,P)=\Psi(4)$. This partial K_4-design is shown diagrammatically in Figure 2. Of particular importance to the main result of this paper is the fact that $\Psi(4)$ is completable to a full K_4-design of order 16. For let C contain the following blocks.

$$\{0,(\{1,2\},1),(\{2,3\},1),(\{1,3\},1)\},$$
$$\{0,(\{1,2\},2),(\{2,3\},2),(\{1,3\},2)\},$$
$$\{1,(\{0,3\},1),(\{2,3\},1),(\{0,2\},1)\},$$
$$\{1,(\{0,3\},2),(\{2,3\},2),(\{0,2\},2)\},$$
$$\{2,(\{0,3\},1),(\{0,1\},1),(\{1,3\},2)\},$$
$$\{2,(\{0,3\},2),(\{0,1\},2),(\{1,3\},1)\},$$
$$\{3,(\{0,1\},1),(\{1,2\},1),(\{0,2\},2)\},$$
$$\{3,(\{0,1\},2),(\{1,2\},2),(\{0,2\},1)\},$$
$$\{(\{0,3\},1),(\{0,1\},2),(\{1,2\},1),(\{2,3\},2)\},$$
$$\{(\{0,3\},1),(\{1,2\},2),(\{0,2\},2),(\{1,3\},1)\},$$
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$$\{(\{0,1\},1),(\{2,3\},2),(\{0,2\},1),(\{1,3\},1)\},$$
$$\{(\{0,1\},2),(\{2,3\},1),(\{0,2\},2),(\{1,3\},2)\}.$$

It is easy to verify that $(X,P\cup C)$ is a K_4-design of order 16. □

Lemma 2.7 *A partial K_4-design of order n with the property that every block contains at least two free vertices can be embedded in a partial K_4-design isomorphic to $\Psi(z)$ for any integer $z \geq n/2$.*

Proof. Let (X, P) be a partial K_4-design of order n with the stated property, and let z be an integer with $z \geq n/2$. Define a partition $\{U, V\}$ of X as follows.

(i) For each block $B \in P$, place two free vertices of B in V.

(ii) Let $U = X \setminus V$.

Figure 3 shows an example of a partial K_4-design with the vertices partitioned in this way.

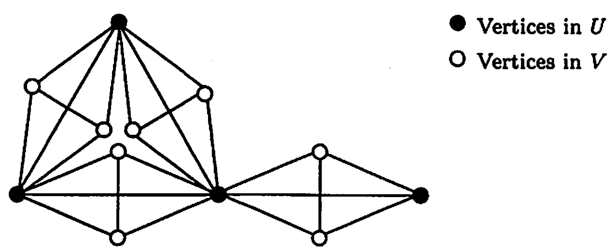

Figure 3: A partial K_4-design (X, P) with the vertex set partitioned into sets U and V.

It is clear that any partial K_4-design of order n contains at least $n/4$ blocks. Consequently, $|V| \geq n/2$, and so $|U| \leq n/2 \leq z$. Construct a partial K_4-design (S, B) as follows.

(i) Let $X \subseteq S$ and $P \subseteq B$.

(ii) If $|U| < z$, let U' be a set containing $z - |U|$ new vertices, and let $U' \subseteq S$.

(iii) For each pair a, b of distinct elements of $U \cup U'$, if the edge $\{a, b\}$ does not occur in any block in P, place two new vertices, x and y, in S, and place the block $\{a, b, x, y\}$ in B. (See Figure 4.)

It is easy to see that (S, B) is a partial K_4-design isomorphic to $\Psi(z)$, which contains the partial K_4-design (X, P). □

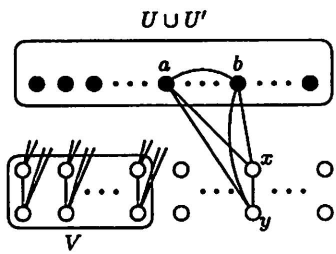

Figure 4: Construction of the partial K_4-design (S, B).

3 The main embedding result

Theorem 3.1 *A partial K_4-design of order n with the property that every block contains at least two free vertices can be embedded in a K_4-design of order at most*

$$\frac{1}{48}(n+24)[(n+28)(n+22)+24].$$

Proof. Let (X, P) be a partial K_4-design of order n with the stated property, and let m be the smallest positive integer such that $12m+4 \geq n/2$. By Lemma 2.7, (X, P) can be embedded in a partial K_4-design isomorphic to $\Psi(12m+4)$. Thus it is sufficient to show that for any positive integer m, it is possible to construct a "small" K_4-design which contains a partial K_4-design isomorphic to $\Psi(12m+4)$.

Let Y be a set of size $12m+4$, and let $(Y, \{H_0, \ldots, H_{t-1}\})$ be a K_4-design of order $12m+4$, where $t = (3m+1)(4m+1)$. We shall construct a K_4-design (S, B) of order $12(2m+1)(3m+1)(4m+1)+12m+4$ where $S = (\mathbb{Z}_t \times \mathbb{Z}_{2m+1} \times \mathbb{Z}_{12}) \cup Y$ and B contains the blocks from the following K_4-decompositions.

(1) For each $i \in \mathbb{Z}_t$, if $H_i = \{a, b, c, d\}$ place a K_4-design of order 16 on $H_i \cup \{(i, 0, k) \mid k \in \mathbb{Z}_{12}\}$ which includes the following blocks.

 (i) $\{a, b, (i,0,0), (i,0,1)\}$,

 (ii) $\{a, c, (i,0,2), (i,0,3)\}$,

 (iii) $\{a, d, (i,0,4), (i,0,5)\}$,

 (iv) $\{b, c, (i,0,6), (i,0,7)\}$,

 (v) $\{b, d, (i,0,8), (i,0,9)\}$, and

 (vi) $\{c, d, (i,0,10), (i,0,11)\}$.

Notice that this collection of blocks is isomorphic to $\Psi(4)$, which was shown to be completable in Example 2.6. Thus, the desired K_4-design certainly exists. Furthermore, it is clear that combining blocks (i) to (vi) for all $i \in \mathbb{Z}_t$ results in a partial K_4-design isomorphic to $\Psi(12m+4)$, as required.

(2) For each $i \in \mathbb{Z}_t, j \in \mathbb{Z}_{2m+1} \setminus \{0\}$, place a K_4-decomposition of $K_{16} \setminus K_4$ on $H_i \cup \{(i,j,k) \mid k \in \mathbb{Z}_{12}\}$ with the four vertices of H_i in the hole.

(3) For each $i \in \mathbb{Z}_t$, place a 4-GDD of type $12^{2m+1}(12m)^1$ on $(Y \setminus H_i) \cup \{(i,j,k) \mid j \in \mathbb{Z}_{2m+1}, k \in \mathbb{Z}_{12}\}$ with groups $\{(i,j,k) \mid k \in \mathbb{Z}_{12}\}$ for each $j \in \mathbb{Z}_{2m+1}$, and one group $Y \setminus H_i$. (See Corollary 2.3.)

(4) Place a 4-GDD of type $(12(2m+1))^t$ on $S \setminus Y$ with groups $\{(i,j,k) \mid j \in \mathbb{Z}_{2m+1}, k \in \mathbb{Z}_{12}\}$ for each $i \in \mathbb{Z}_t$. (See Lemma 2.4.)

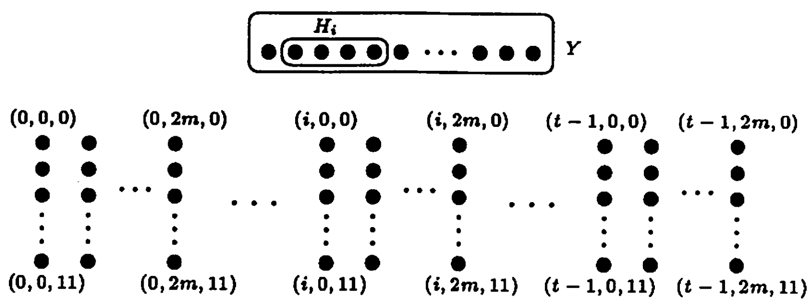

Figure 5: The vertex set S.

It is natural to think of the vertices of S being arranged as shown in Figure 5, where each block $H_i, i \in \mathbb{Z}_t$, is associated with a "cluster" of $2m+1$ columns of 12 vertices. With this arrangement, the following observations can be made.

(i) For each $i \in \mathbb{Z}_t$, the type (1) and (2) decompositions use the six edges of H_i, all edges between the vertices of H_i and the vertices of the associated cluster, and all edges between vertices of the same column in each column of the cluster.

(ii) For each $i \in \mathbb{Z}_t$, the type (3) decompositions use all edges between $Y \setminus H_i$ and the vertices of the associated cluster, and all edges between vertices in distinct columns of the cluster.

(iii) The type (4) decomposition uses all edges between vertices in distinct clusters.

It now follows that (S, B) is indeed a K_4-design of order

$$12(2m+1)(3m+1)(4m+1)+12m+4.$$

Finally, since m was as small as possible subject to $12m+4 \geq n/2$, we have $12m+4 \leq n/2+12$, and so $m \leq n/24+2/3$. It follows that the order of (S, B) is at most

$$\frac{1}{48}(n+24)[(n+28)(n+22)+24].$$

This completes the proof. □

4 Concluding remarks

The class of partial K_4-designs for which this embedding is applicable is indeed very restricted. A closely related problem which remains open is to find a small embedding for the class of partial K_4-designs in which every block contains at least *one* free vertex. It can be shown that a solution to this problem implies the existence of a small embedding of an arbitrary partial Steiner triple system into a *resolvable* Steiner triple system. (See [5].)

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