Hamilton Paths in Certain Arithmetic Graphs

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Abstract

For each integer $m \geq 1$, consider the graph G_m whose vertex set is the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers and whose edges are the pairs xy with $y = x+m$ or $y = x-m$ or $y = mx$ or $y = x/m$. Our aim in this note is to show that, for each m, the graph G_m contains a Hamilton path. This answers a question of Lichiardopol.

For each integer $m \geq 1$, consider the graph G_m whose vertex set is the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers and whose edges are the pairs xy with $y = x + m$ or $y = x - m$ or $y = mx$ or $y = x/m$. We show that, for each m, the graph G_m contains a Hamilton path. Here, by 'Hamilton path' we mean a 'one-way infinite Hamilton path', i.e. a sequence x_0, x_1, x_2, ... of vertices of G_m such that each vertex appears precisely once and, for all i, the vertices x_i and x_{i+1} are adjacent. We shall use this to answer a question of Lichiardopol [1] about two-way infinite Hamilton paths in graphs defined similarly but with vertex set the set $\mathbb{Z}$ of integers.

The case $m = 1$ is trivial so we begin at $m = 2$. The construction of the Hamilton path in the graph G_2 is similar in spirit to those used later, but this case is much easier.

Proposition 1. *The graph G_2 contains a Hamilton path.*

Proof. Our approach is to define inductively a strictly increasing sequence x_0, x_1, x_2, ... of natural numbers with $x_0 = 0$, and show that, for each $i = 0, 1, 2, \dots$, there is a Hamilton path in $G_2[x_i, x_{i+1}]$ from x_i to x_{i+1}; putting these paths end-to-end gives the required Hamilton path in G_2.

Now, take

- $x_0 = 0$;

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- $x_1 = 3$;
- $x_i = 2x_{i-1} + 5$ $(i \geq 2)$.

Our path in $G_2[x_0, x_1]$ is simply 0,2,1,3. To show that there is such a path in $G_2[x_i, x_{i+1}]$ for $i \geq 1$, it suffices to exhibit a Hamilton path in $G_2[x, 2x+5]$ for odd $x > 0$; such a path is given by

$$x, 2x, 2x-2, 2x-4, \ldots, x+1, 2x+2, 2x+4, x+2, x+4, x+6, \ldots, 2x+5.$$

□

We next consider the case of even $m > 2$. The approach is similar to that used for the graph G_2, but instead of splitting $\mathbb{N}$ up into intervals we need to use slightly more complicated sets.

Proposition 2. *For all even $m > 2$, the graph G_m contains a Hamilton path.*

Proof. Define inductively a strictly increasing sequence x_0, x_1, x_2, ... of natural numbers by

- $x_0 = 0$;
- $x_i = m(x_{i-1} + 2)$ $(i \geq 1)$.

Note that each x_i is divisible by m.

For $i = 0, 1, 2, \ldots$, let $G_m^{(i)}$ be the graph

$$G_m^{(i)} = G_m\left[([x_i, x_{i+1}] - m\mathbb{N}) \cup ([mx_i, mx_{i+1}] \cap m\mathbb{N})\right].$$

Note that, for all i, the sets $V(G_m^{(i)})$ and $V(G_m^{(i+1)})$ intersect only at mx_{i+1}; and for all i and j with $|i-j| > 1$, the sets $V(G_m^{(i)})$ and $V(G_m^{(j)})$ are disjoint. Moreover, the union of the sets $V(G_m^{(i)})$ (i = 0, 1, 2, ...) is the whole of $\mathbb{N}$. Hence it is enough to construct, for each i, a Hamilton path in $G_m^{(i)}$ from mx_i to mx_{i+1}; putting these paths end-to-end again gives a Hamilton path in G_m as required.

So, fix i. Observe that, for each j = 1, 2, ..., $m-1$, there is a path P_j in $G_m^{(i)}$ from $m(x_i + j)$ to $m(x_{i+1} - m + j)$ whose internal vertices are precisely those vertices of $G_m^{(i)}$ which are congruent to j (mod m), namely the path

$$m(x_i + j), x_i + j, x_i + m + j, x_i + 2m + j, \ldots, x_{i+1} - m + j, m(x_{i+1} - m + j).$$

Note that the $V(P_j)$ $(1 \leq j \leq m-1)$ partition $V(G_m^{(i)})$ except for the vertices mx_i, $m(x_i + m)$, $m(x_i + m + 1)$, $m(x_i + m + 2)$, ..., $m(x_{i+1} - m)$,

mx_{i+1} which are missed. Moreover, the first (last) vertex of P_j is adjacent to the first (last) vertex of P_{j+1} $(1 \le j \le m-2)$. Hence it is possible to join these paths together to make the required Hamilton path in $G_m^{(i)}$, namely

$$mx_i, P_1, m(x_{i+1}-m), m(x_{i+1}-m-1), \ldots, m(x_i+m),$$
$$P_{m-1}, P_{m-2}^{-1}, P_{m-3}, \ldots, P_2^{-1}, mx_{i+1}$$

(where P^{-1} denotes the path obtained by traversing the path P in reverse). □

This only leaves us to deal with odd m. The construction used in Proposition 2 will not work here as, since m is odd, we would have to finish by traversing the path P_2 *forwards*, and so we would be unable to reach the point mx_{i+1} at the end of each intermediate path. However, it turns out that it is possible to adapt the construction by modifying the definition of our sequence $x_0, x_1, x_2, \ldots$ and changing the points where the intermediate paths end. This is sufficient to get around the obstruction.

Proposition 3. *For all odd m, the graph G_m has a Hamilton path.*

Proof. For convenience, we shall assume initially that $m \ge 5$. This time we inductively define our strictly increasing sequence x_0, x_1, x_2, ... by

- $x_0 = 0$;
- $x_1 = 2m$;
- $x_2 = m(m+3)$;
- $x_i = m(x_{i-2}+1)$ $(i \ge 3)$.

Note that each x_i is divisible by m.

For each $i = 0, 1, 2, \ldots$, let $G_m^{(i)}$ be the graph

$$G_m^{(i)} = G_m\left[([x_i, x_{i+1}] - m\mathbb{N}) \cup ([mx_i, mx_{i+1}-m] \cap m\mathbb{N})\right].$$

Note that the sets $V(G_m^{(i)})$ $(i = 0, 1, 2, \ldots)$ form a partition of $\mathbb{N}$.

We shall construct a Hamilton path in $G_m^{(i)}$ which for $i = 0$ goes from 0 to $m(m+2)$, and for $i > 0$ goes from $m(x_{i+1}-m)$ to $mx_i = x_{i+2}-m$; note that these are genuinely distinct vertices of $G_m^{(i)}$ as $x_{i+1} > x_i + m$ for all i. Moreover, the last vertex of the path we shall define in $G_m^{(i)}$ will be adjacent to the first vertex of the path in $G_m^{(i+1)}$ so it will indeed be possible to join them together to make a Hamilton path in G_m.

Consider first the case $i = 0$. For each $j = 1, 2, \ldots, m-1$, consider the path Q_j given by jm, j, $m+j$, $m(m+j)$. The Q_j are vertex-disjoint paths in $G_m^{(0)}$, and, for each $j = 1, 2, \ldots, m-2$, the first (last) vertex of the path Q_j is adjacent to the first (last) vertex of the path Q_{j+1}. It is then easy to see that we may take as our Hamilton path in $G_m^{(0)}$ the path

$$0, Q_1, m^2, Q_{m-1}, Q_{m-2}^{-1}, Q_{m-3}, \ldots, Q_4, Q_3^{-1}, Q_2.$$

Now fix $i \geq 1$. Similarly to the case of even m, for each j $(1 \leq j \leq m-1)$ we have a path P_j in $G_m^{(i)}$ from $m(x_i + j)$ to $m(x_{i+1} - m + j)$ whose internal vertices are precisely those vertices of $G_m^{(i)}$ which are congruent to j (mod m). Here, the vertex sets $V(P_j)$ $(1 \leq j \leq m-1)$ partition $V(G_m^{(i)})$ except for the vertices mx_i, $m(x_i+m)$, $m(x_i+m+1)$, $m(x_i+m+2)$, $\ldots$, $m(x_{i+1}-m)$.

Again, the first (last) vertex of P_j is adjacent to the first (last) vertex of P_{j+1} $(1 \leq j \leq m-2)$ and so again it is possible to join these paths together to make the required Hamilton path in $G_m^{(i)}$, namely

$$m(x_{i+1}-m), m(x_{i+1}-m-1), \ldots, m(x_i+m),$$
$$P_{m-1}, P_{m-2}^{-1}, P_{m-3}, \ldots, P_1^{-1}, mx_i.$$

This only leaves us to consider the case $m = 3$. The above construction fails only because $x_3 = x_2 + 3$. So if we can construct a Hamilton path from 0 to $3x_2$ in the graph $G_3\left[([0, x_3] - 3\mathbb{N}) \cup ([0, 3x_3 - 3] \cap 3\mathbb{N})\right]$ then we can put this path together with the paths constructed above for $i \geq 3$ to make our Hamilton path in G_3. But what we need is simply a Hamilton path from 0 to 54 in $G_3\left[[0,21] \cup \{24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57, 60\}\right]$, for which we may take

$$0, 3, 1, 4, 7, 10, 13, 16, 19, 57, 60, 20, 17, 14, 11, 8, 5, 2, 6, 9, 12, 15, 18, 21,$$
$$24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54.$$

□

So we have now constructed a Hamilton path in G_m for each positive integer m.

Lichiardopol [1] asked if the graph $G_m(\mathbb{Z})$, defined similarly but with vertex set the set $\mathbb{Z}$ of integers, contained a Hamilton path.

We note first that it is clear that $G_m(\mathbb{Z})$ cannot contain a one-way infinite Hamilton path. For suppose that there were such a path, say $x_0, x_1, x_2, \ldots$. Then the set

$$A = \{x_i : i \in \mathbb{N}, x_i > 0, x_{i+1} \leq 0\}$$

would be infinite; but it would also have to be a subset of the finite set $\{1, 2, \ldots, m\}$, a contradiction.

However, turning to the more interesting question of whether $G_m(\mathbb{Z})$ contains a two-way infinite Hamilton path, we observe that our construction answers this question positively. Indeed, since our one-way infinite path in G_m starts at 0, we may put together two copies of it to form a two-way infinite path in $G_m(\mathbb{Z})$.

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References

[1] N. Lichiardopol, Problem 7 in *Problems from the Nineteenth British Combinatorial Conference* (2003) (edited by P.J. Cameron).