List Coloring Halin Graphs

Wei-Fan Wang
Department of Mathematics
Zhejiang Normal University
Jinhua 321004, P. R. China

Ko-Wei Lih*
Institute of Mathematics
Academia Sinica
Nankang, Taipei 115, Taiwan

Abstract

A Halin graph is a plane graph $H = T \cup C$, where T is a tree with no vertex of degree two and at least one vertex of degree three or more, and C is a cycle connecting the pendant vertices of T in the cyclic order determined by the drawing of T. In this paper we determine the list chromatic number, the list chromatic index, and the list total chromatic number (except when $\Delta = 3$) of all Halin graphs, where Δ denotes the maximum degree of H.

Key words. list coloring, Halin graph

1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges unless otherwise stated. A plane graph is a particular drawing in the Euclidean plane of a certain planar graph. For a graph G, we denote its vertex set, edge set, order, maximum degree, and minimum degree by $V(G)$, $E(G)$, $|G|$, $\Delta(G)$, and $\delta(G)$, respectively. Let $d_G(v)$ denote the degree of v in G.

A proper *k-coloring* of a graph G is a mapping ϕ from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for every edge xy of G. We say that G is *k-colorable* if it has a proper k-coloring. The chromatic number $\chi(G)$ is the smallest integer k such that G is k-colorable. We make the convention that adjacent or incident elements receive different colors for all coloring notions discussed in this paper. A mapping L is said to be an *assignment* for the graph G if it assigns a list $L(v)$ of possible colors to each vertex v of G. If G has a coloring ϕ such that $\phi(v) \in L(v)$ for all vertices v, we say that G is *L-colorable* or ϕ is an *L-coloring* of

*Corresponding author. E-mail: makwlih@sinica.edu.tw

G. A graph G is *k-choosable* if it is L-colorable for every assignment L satisfying $|L(v)| = k$ for all vertices v. The *list chromatic number* $\chi_\ell(G)$, also known as the *choice number*, of G is the smallest k such that G is k-choosable. We can define analogous notions such as edge k-colorability, edge k-choosability, the chromatic index $\chi'(G)$, and the list chromatic index $\chi'_\ell(G)$ when colorings of $E(G)$ are taken into consideration. If we consider colorings of $V(G) \cup E(G)$, we can define further analogous notions such as totally k-colorable, totally k-choosable, the total chromatic number $\chi''(G)$, and the list total chromatic number $\chi''_\ell(G)$.

Conjecture 1 *If G is a multigraph, then* $\chi'_\ell(G) = \chi'(G)$.

This is the well-known List-Edge-Coloring Conjecture and was proposed independently by Vizing, by Gupta, by Albertson and Collins, and by Bollobás and Harris (see [6, 9]). It has been proved for a few special cases, such as bipartite multigraphs [5], complete graphs of odd order [7], multicircuits [17], line-perfect multigraphs [15], and planar graphs G with $\Delta(G) \geq 12$ [2].

Conjecture 2 *If G is a multigraph, then* $\chi''_\ell(G) = \chi''(G)$.

Borodin, Kostochka and Woodall [2] proposed this conjecture which is also known as the List-Total-Coloring Conjecture. They proved the following results: (i) $\chi''_\ell(G) \leq \lfloor \frac{3}{2}\Delta(G) \rfloor + 2$ for any multigraph G. (ii) $\chi''_\ell(G) \leq \Delta(G) + 2$ for a bipartite multigraph G. (iii) $\chi''_\ell(G) = \Delta(G) + 1$ if G is a planar graph with $\Delta(G) \geq 12$. Juvan, Mohar and Škrekovki [10] also independently proposed Conjecture 2 and confirmed it for multigraphs with maximum degree 2. Kostochka and Woodall proved the conjecture for multicircuits of orders 3, 4, 5, and a wide class of even orders in [11] . They finally established the conjecture for all multicircuits in [12, 13]. (A *multicircuit* is a multigraph whose underlying simple graph is a circuit.)

The purpose of this paper is to study three choosability notions for Halin graphs. A *Halin* graph is a plane graph $H = T \cup C$, where T is a tree with no vertex of degree two and at least one vertex of degree three or more, and C is a cycle connecting the pendant vertices of T in the cyclic order determined by the drawing of T. By convention, we draw the tree T inside the cycle C. Vertices (or edges) of C are called *outer* vertices (or edges) of H and vertices of $H - C$ are called *inner* vertices of H. A Halin graph H is called a *wheel* if H contains only one inner vertex. If we delete an outer edge from a wheel, the remaining graph is called a *fan*. The reader is referred to [14] and [18] for results on colorings of Halin graphs.

Halin graphs possess some fairly interesting properties. It is easy to construct 3-regular Halin graphs with no non-trivial automorphisms. Halin graphs are 3-connected, but none of their proper subgraphs are [8].

In this paper, we will characterize the list chromatic number and the list chromatic index of Halin graphs, and determine the list total chromatic number of a Halin graph H when $\Delta(H) \neq 3$. In particular, Conjecture 1 holds for a Halin graph H, so does Conjecture 2 when $\Delta(H) \neq 3$.

2 Preliminaries

Let $H = T \cup C$ be a Halin graph. Then every vertex of $V(C)$ is adjacent to exactly one vertex in $V(H) \setminus V(C)$, and every edge of $E(C)$ is adjacent to exactly two edges in $E(H) \setminus E(C)$. Since H is a minimally 3-connected plane graph, we have $\Delta(H) \geq \delta(H) = 3$. A graph is *$k$-degenerate* if each of its induced subgraphs contains a vertex of degree at most k. It is easy to see by induction that a k-degenerate graph is $(k+1)$-choosable. An inner vertex u of a Halin graph H is called *special* if it is a neighbor of a unique inner vertex. Let $v_1, v_2, \ldots, v_k$ denote the neighbors of u on C. If H is not a wheel, then $\{u, v_1, v_2, \ldots, v_k\}$ induces a fan. Proofs of the following Lemmas 1 to 7 either are straightforward or appeared elsewhere.

Lemma 1 *Every Halin graph is 3-degenerate.*

Lemma 2 *If a Halin graph H is not a wheel, then it contains at least two special inner vertices.*

Lemma 3 ([1]) *Every Halin graph H contains cycles of all lengths k, $3 \leq k \leq |H|$, except for one possible even value of k. In particular, H is a Hamiltonian graph.*

Lemma 4 ([4, 10, 16]) *Let C_n be a cycle of length n. Then*

(a) $\chi_\ell(C_n) = \chi'_\ell(C_n) = \begin{cases} 2, & \text{if } n \equiv 0 \pmod 2; \\ 3, & \text{if } n \not\equiv 0 \pmod 2; \end{cases}$

(b) $\chi''_\ell(C_n) = \chi''(C_n) = \begin{cases} 3, & \text{if } n \equiv 0 \pmod 3; \\ 4, & \text{if } n \not\equiv 0 \pmod 3. \end{cases}$

Lemma 5 *Let C be a cycle of length n. Let L be an assignment that satisfies $|L(u)| = 2$ (or $|L(e)| = 2$) for each vertex u (or edge e) and $L(u') \neq L(u'')$ (or $L(e') \neq L(e'')$) for some pair of consecutive vertices u' and u'' (or edges e' and e''.) Then C is L-colorable (or edge L-colorable.)*

Lemma 6 *Let T be a tree with $\Delta(T) \geq 2$. Then*

(a) $\chi_\ell(T) = \chi(T) = 2$;

(b) $\chi'_\ell(T) = \chi'(T) = \Delta(T)$;

(c) $\chi''_\ell(T) = \chi''(T) = \Delta(T) + 1$.

Lemma 7 ([3]) *Every d-regular edge d-colorable plane graph is edge d-choosable.*

The next lemma is a technical one whose straightforward proof needs detailed case analyses. This lemma is essential in establishing Theorem 11.

Lemma 8 *Let K denote a graph obtained by adding an edge uv to a 4-cycle $uxvyu$. Let L be an assignment for $V(K) \cup E(K)$ that satisfies $|L(x)| = |L(y)| = |L(u)| = |L(ux)| = |L(uy)| = 3$, $|L(vx)| = |L(uv)| = |L(vy)| = 4$, and $|L(v)| = 5$. If $L(u) \neq L(ux)$ and $L(u) \neq L(uy)$, then K is totally L-colorable.*

Proof. We are going to construct a total L-coloring of K in each possible case (up to symmetry.) We use nonnegative integers to denote colors in L. When no ambiguities arise, we use $t \Rightarrow \alpha$ to indicate that elements of K appearing in the expression t are colored with the color α. Let $L'(z)$ denote the shortened list for z when all colors that have already been used on elements adjacent or incident to z are deleted from $L(z)$. An *m-set* means a set with m elements.

Case 1. $L(x) \cap L(uv) \neq \emptyset$ or $L(y) \cap L(uv) \neq \emptyset$: We may assume that $L(x) \cap L(uv) \neq \emptyset$ without loss of generality.

We may choose colors so that $(x, uv) \Rightarrow 1$, $ux \Rightarrow 2$, and (since $L(u) \neq L(uy)$) $u \Rightarrow 3$, $uy \Rightarrow 4$, and $y \Rightarrow c_y$. If v, vx and vy cannot be properly colored from their shortened lists L', then $L'(vy) \subseteq L'(v) = L'(vx)$ and $|L'(v)| = 2$. In this case we may assume that $L(v) = \{1, 3, 5, 6, c_y\}$, $L(vx) = \{1, 2, 5, 6\}$, $L(vy) \subseteq \{1, 4, 5, 6, c_y\}$, and $c_y \notin \{1, 3, 4, 5, 6\}$, so that $L'(vy) \subseteq L'(v) = L'(vx) = \{5, 6\}$. Without loss of generality, we may suppose $5 \in L(vy)$. If $L(y) \neq \{3, 4, c_y\}$, then we recolor y with some color in $L(y) \setminus \{3, 4, c_y\}$ to make $L'(v) \neq L'(vx)$, and the required L-coloring can then be completed. Similarly, if $L(u) \not\subseteq \{1, 2, 3, 4, c_y\}$ then we can recolor u to make $L'(v) \neq L'(vx)$. If there exists $\alpha \in L(uy) \setminus \{1, 2, 3, 4\}$, we let $uy \Rightarrow \alpha$ and $y \Rightarrow 4$ to make $L'(v) \neq L'(vx)$. So we may assume

$$L(y) = \{3, 4, c_y\}, \quad L(u) \subseteq \{1, 2, 3, 4, c_y\}, \text{ and } L(uy) \subseteq \{1, 2, 3, 4\}. \qquad (*)$$

First let $vy \Rightarrow 5$. There remain three subcases to consider.

Subcase 1.1. $2 \in L(uy)$ and $4 \in L(ux)$.

We let $(uy, vx) \Rightarrow 2$, $(ux, y) \Rightarrow 4$, and $v \Rightarrow 6$.

Subcase 1.2. $4 \notin L(ux)$.

If $L(ux) \neq \{1, 2, 3\}$, we can recolor ux with some color in $L(ux) \setminus \{1, 2, 3, 4\}$ and set $vx \Rightarrow 2$ and $v \Rightarrow 6$. So we may assume $L(ux) = \{1, 2, 3\}$. If 2 or $c_y \in L(u)$, then (regardless of whether or not $c_y = 2$) let $u \Rightarrow 2$ or c_y, $vx \Rightarrow 2$, $(ux, y) \Rightarrow 3$, and $v \Rightarrow 6$. So we may assume $L(u) = \{1, 3, 4\}$ by $(*)$. If $3 \in L(uy)$, let $(v, uy) \Rightarrow 3$, $u \Rightarrow 4$, and $vx \Rightarrow 6$. So we furthermore assume $L(uy) = \{1,2,4\}$ by (*). Now let $(uy, vx) \Rightarrow 2$, $(ux, y) \Rightarrow 3$, $u \Rightarrow 4$, and $v \Rightarrow 6$.

Subcase 1.3. $4 \in L(ux)$ and $2 \notin L(uy)$.

Then $L(uy) = \{1,3,4\}$ by (*). If $4 \in L(u)$, let $(v, uy) \Rightarrow 3$, $u \Rightarrow 4$, and $vx \Rightarrow 6$. If $c_y \in L(u)\setminus\{1,2,3\}$, let $(v, uy) \Rightarrow 3$, $u \Rightarrow c_y$, $y \Rightarrow 4$, and $vx \Rightarrow 6$. So we assume $L(u) = \{1,2,3\}$ by (*). Now let $(u, vx) \Rightarrow 2$, $(uy, v) \Rightarrow 3$, and $(ux, y) \Rightarrow 4$.

Case 2. $L(x) \cap L(y) \neq \emptyset$ and $(L(x) \cup L(y)) \cap L(uv) = \emptyset$.

We may start with the following assignments: $(x, y) \Rightarrow 1$, $ux \Rightarrow i$, $u \Rightarrow j$, and $uy \Rightarrow k$. Let $L(y) = \{1,2,3\}$ and $L(uv) = \{4,5,6,7\}$. In view of symmetry, we only consider the following cases.

Subcase 2.1. $\{i,j,k\} \subset L(uv)$.

We remove the color 1 from x and y, then color uv, vx, vy, v, x, and y in succession.

Subcase 2.2. $\{i,j,k\} \cap L(uv) = \emptyset$.

We color vx, vy, v, and uv in succession.

Subcase 2.3. $|\{i,j,k\} \cap L(uv)| = 1$.

Subcase 2.3.1. $j = 4$ and $i, k \notin L(uv)$.

The shortened lists will satisfy $|L'(v)| \geq 3$, $|L'(uv)| = 3$, $|L'(vx)| \geq 2$, and $|L'(vy)| \geq 2$. If $L'(v) \neq L'(uv)$, we color vx, vy, uv, and v in succession. If $L'(v) = L'(uv)$, it is easy to see $L(v) = \{1,4,5,6,7\}$. Since $4,5,6,7 \notin L(x) \cup L(y)$, we may recolor x and y, respectively, with some color different from 1 to make $L'(v) = \{1,5,6,7\}$. Finally $L'(v) \neq L'(uv)$.

Subcase 2.3.2. $i = 4$ and $j, k \notin L(uv)$.

If v, vx, vy, and uv cannot be properly colored from their shortened lists L', then $L'(vx), L'(vy) \subseteq L'(v) = L'(uv) = \{5,6,7\}$. It follows that $L(v) = \{1, j, 5, 6, 7\}$. If $1 \in L(vx)$, we color vx with 1 and recolor x with some color $c_x \in L(x) \setminus \{1, j\}$. Since $c_x \notin L(v)$, we may color vy, uv, and v in succession. So suppose $1 \notin L(vx)$. If $|L(y) \cap \{j,k\}| \leq 1$, we recolor x with some color in $L(x) \setminus \{1, j\}$ and y with some color in $L(y) \setminus \{1, j, k\}$, afterward let $v \Rightarrow 1$, and finally color vy, vx, and uv in succession. Now assume $|L(y) \cap \{j,k\}| = 2$. Without loss of generality, suppose $j = 2$, $k = 3$, and hence $L(v) = \{1,2,5,6,7\}$. If $L(vx) \neq L(uv)$, we color vx with some color in $L(vx) \setminus L(uv)$, then color vy, v, and uv in succession. Therefore let $L(vx) = \{4,5,6,7\}$, and similarly $L(vy) \subset \{1,3,5,6,7\}$. First erase the color 1 from x. If $1 \in L(uy)$, then let $(v, uy) \Rightarrow 1$, $y \Rightarrow 3$, and color vy, vx, uv, and x in succession. So suppose $1 \notin L(uy)$. If there exists $\beta \in L(uy) \setminus \{2,3,4\}$, let $uy \Rightarrow \beta$, $y \Rightarrow 3$, $v \Rightarrow 1$, and color vy, uv, vx, and x in succession. Hence let $L(uy) = \{2,3,4\}$. If $3 \in L(ux)$, then let $ux \Rightarrow 3$, $(vx, uy) \Rightarrow 4$, $x \Rightarrow 1$, and color vy, v, and uv in succession. Therefore let $3 \notin L(ux)$. If there exists $\gamma \in L(ux) \setminus \{1,2,4\}$, we need to handle two cases. When $\gamma \in L(v)$, let $(v, ux) \Rightarrow \gamma$, $x \Rightarrow 1$, and color vy, vx, and uv in succession. When $\gamma \notin L(v)$, implying $\gamma \notin L(uv)$, let $ux \Rightarrow \gamma$, $x \Rightarrow 1$, and color vy, vx, v, and uv in succession. Finally, if $L(ux) = \{1,2,4\}$, let $ux \Rightarrow 1$, $vx \Rightarrow 4$, and color vy, v, uv, and x in succession.

Subcase 2.4. $|\{i,j,k\} \cap L(uv)| = 2$.

Subcase 2.4.1. $j = 4$, $k = 5$, and $i \neq 6,7$.

Remove the color 1 from x and y. If $i \notin L(x)$, we color vx, uv, vy, v, x, and y in succession. If $i \in L(x)$, we color vx with some color in $L(vx) \setminus L(x)$, then color uv, vy, v, x, and y in succession.

Subcase 2.4.2. $i = 4$, $k = 5$, and $j \neq 6,7$.

If 6 or 7 $\in L(u)$, we recolor u with 6 or 7, then the problem can be reduced to Subcase 2.1. Thus assume $6,7 \notin L(u)$. Let either 4 or 5 belong to $L(u)$, say $4 \in L(u)$. If $L(ux) \neq \{1,4,5\}$, we let $u \Rightarrow 4$ and $ux \Rightarrow \alpha \in L(ux) \setminus \{1,4,5\}$. Afterward, the problem is reduced to Subcase 2.1 if $\alpha \in \{6,7\}$ and to Subcase 2.4.1 otherwise. So assume $L(ux) = \{1,4,5\}$. If $L(uy) \neq \{1,4,5\}$, we color u with 4, ux with 5, and uy with a color in $L(uy) \setminus \{1,4,5\}$. Again, the problem is reduced to Subcase 2.1 or Subcase 2.4.1. So suppose $L(uy) = \{1,4,5\}$. If $4 \in L(vx)$, we let $(uy, vx) \Rightarrow 4$ and $ux \Rightarrow 5$. The shortened lists satisfy $|L'(uv)| = 2$, $|L'(v)| \geq 2$, and $|L'(vy)| \geq 2$. If $L'(v) = L'(vy) = L'(uv) = \{6,7\}$, we let $uv \Rightarrow 5$, $v \Rightarrow 6$, $vy \Rightarrow 7$, $ux \Rightarrow 1$, and recolor x with some color in $L(x) \setminus \{1,j\}$. Otherwise, the required coloring can be established. Therefore assume $4,5 \notin L(vx) \cup L(vy)$. Now let $uv \Rightarrow 6$. It suffices to consider the case when $L'(v) = L'(vx) = L'(vy) = \{a,b\}$, implying $L(v) = \{1,6,j,a,b\}$, $L(vx) = L(vy) = \{1,6,a,b\}$, and $j \neq a,b$. If $7 \notin \{a,b\}$, we let $uv \Rightarrow 7$, $v \Rightarrow 6$, $vx \Rightarrow a$, and $vy \Rightarrow b$. If $7 \in \{a,b\}$, let $v \Rightarrow 7$, $vy \Rightarrow 1$, then color y with a color in $\{2,3\} \setminus \{j\}$ and vx with a color in $\{a,b\} \setminus \{7\}$. Hence we assume $4,5 \notin L(u)$.

Let us erase the color j from u. Note that the partly shortened lists satisfy $|L'(vx)| \geq 2$, $|L'(vy)| \geq 2$, and $L'(uv) = \{6,7\}$. If at most one of $L'(vx)$ and $L'(vy)$ is identical to $L'(uv)$, we color vx, vy, uv, v, and u in succession. So assume $L'(vx) = L'(vy) = L'(uv) = \{6,7\}$. It follows that $L(vx) = \{1,4,6,7\}$ and $L(vy) = \{1,5,6,7\}$. If there exists $\alpha \in L(u) \setminus L(y)$, we let $u \Rightarrow \alpha$, $uv \Rightarrow 6$, $vx \Rightarrow 7$, $vy \Rightarrow 1$, $v \Rightarrow \beta \in L(v) \setminus \{1,6,7,\alpha\}$, then color y with a color in $L(y) \setminus \{1,\beta\}$. So we may assume $L(x) = L(u) = L(y) = \{1,2,3\}$. In this case, let $(u, vy) \Rightarrow 1$, $(x,y) \Rightarrow 2$, $vx \Rightarrow 6$, $uv \Rightarrow 7$, and color v with a color in $L(v) \setminus \{1,2,6,7\}$.

Case 3. $L(x) \cap L(y) = L(x) \cap L(uv) = L(y) \cap L(uv) = \emptyset$.

Let $L(x) = \{0,1,2\}$, $L(uv) = \{3,4,5,6\}$, and $L(y) = \{7,8,9\}$. We first color ux, u, and uy with i, j, and k, respectively. In fact, we only need to consider the following cases.

Subcase 3.1. $\{i,j,k\} \subset L(uv)$.

We color uv, vx, vy, v, x, and y in succession.

Subcase 3.2. $\{i,j,k\} \cap L(uv) = \emptyset$.

Assume $j \notin L(x)$ (otherwise, $j \notin L(y)$.) Color x with $\alpha \in L(x) \setminus \{i\}$ and y with $\beta \in L(y) \setminus \{j,k\}$. Then $|L'(uv)| = 4$. If all lists $L'(v)$, $L'(vx)$, and $L'(vy)$ are identical to a 2-set, we recolor x with some color in $L(x) \setminus \{i, \alpha\}$ to make them not entirely identical. Thus a proper coloring can be constructed.

Subcase 3.3. $|L(uv) \cap \{i,j,k\}| = 2$.

First assume that $j = 3$, $k = 4$, and $i \neq 5, 6$. It suffices to color vx (using a color in $L(vx) \setminus L(x)$ when $i \in L(x)$), uv, vy, v, x, and y in succession. Next assume that $i = 3$, $k = 4$, and $j \neq 5, 6$. If $j \notin L(x) \cup L(y)$, we color vx, uv, vy, v, x, and y in succession. So we may assume $j = 0$. If $0 \in L(vx)$, we let $vx \Rightarrow 0$, then color vy, uv, v, x, and y in succession. If $0 \notin L(vx)$, we color vx with some color in $L(vx) \setminus \{1,2,3\}$, then color uv, vy, v, x, and y in succession.

Subcase 3.4. $|L(uv) \cap \{i,j,k\}| = 1$.

Subcase 3.4.1. $j = 3$ and $i, k \notin L(uv)$.

If $i \notin L(x)$ and $k \notin L(y)$, we color vx, vy, uv, v, x, and y in succession. If $i \in L(x)$ and $k \notin L(y)$, we first color vx with some color in $L(vx) \setminus L(x)$, then color vy, uv, v, x, and y in succession. Finally suppose $i \in L(x)$ and $k \in L(y)$. If there are $\alpha \in L(vx) \setminus L(x)$ and $\beta \in L(vy) \setminus L(y)$ such that $\alpha \neq \beta$, we color vx with α and vy with β. Then we color uv, v, x, and y in succession. Otherwise, we should have $L(vx) = \{0,1,2,\beta\}$ and $L(vy) = \{7,8,9,\beta\}$. Let $x \Rightarrow a \in \{0,1,2\} \setminus \{i\}$, $vx \Rightarrow b \in \{0,1,2\} \setminus \{i,a\}$, $y \Rightarrow c \in \{7,8,9\} \setminus \{k\}$, and $vy \Rightarrow d \in \{7,8,9\} \setminus \{k,c\}$. If $L(v) \neq \{3,a,b,c,d\}$, we can further color v and uv in succession. If $L(v) = \{3,a,b,c,d\}$, we color v with b, vx with β, and uv with some color in $\{4,5,6\} \setminus \{\beta\}$.

Subcase 3.4.2. $i = 3$ and $j, k \notin L(uv)$.

At first, we assume $j \notin L(x)$. Let $y \Rightarrow a \in L(y) \setminus \{j,k\}$ and $vy \Rightarrow b \in L(vy) \setminus \{a,k\}$. If v, vx, uv, and x cannot be properly colored from their shortened lists L', then we suppose, without loss of generality, $L'(v) = L'(uv) = L'(vx) = \{5,6\}$ and $|L'(x)| = 3$. This implies that $b = 4$, $L(v) = \{4,5,6,a,j\}$, and $L(vx) = \{3,4,5,6\}$. If at most one of j and k belongs to $L(y)$, we recolor y with some color in $L(y) \setminus \{a,j,k\}$ so that $L'(v), L'(uv)$, and $L'(vx)$ are not entirely identical. So suppose $j = 7, k = 8$, and $a = 9$. If there is $\alpha \in L(ux) \setminus \{3,7,8\}$, we let $ux \Rightarrow \alpha$, $uv \Rightarrow 3$, and $x \Rightarrow 0$. When $\alpha \in \{4,9\}$, further let $vx \Rightarrow 5$ and $v \Rightarrow 6$. When $\alpha \notin \{4,9\}$, let $vx \Rightarrow \beta \in \{5,6\} \setminus \{\alpha\}$ and color v with some color in $\{5,6\} \setminus \{\beta\}$. Thus suppose $L(ux) = \{3,7,8\}$. Let $ux \Rightarrow 7$. If there is $c \in L(u) \cap L(uv)$, we may recolor u with c so that the problem is reduced to Subcase 3.4.1. Otherwise, we color u with some color in $L(u) \setminus \{7,8\}$ so that the problem is reduced to Subcase 3.2.

Now assume $j \in L(x)$. In view of the previous argument, we may suppose $L(u) = L(x) = \{0,1,2\}$. We let $v \Rightarrow a \in L(v) \setminus L(u)$, $y \Rightarrow b \in L(y) \setminus \{k,a\}$, $vy \Rightarrow c \in L(vy) \setminus \{a,b,k\}$, and $vx \Rightarrow d \in L(vx) \setminus \{3,a,c\}$. If there exists $\alpha \in \{4,5,6\} \setminus \{a,c,d\}$, let $uv \Rightarrow \alpha$, then color properly u and x. So suppose $\{a,c,d\} = \{4,5,6\}$. Now let $uv \Rightarrow c$ and $vy \Rightarrow e \in L(vy) \setminus \{4,5,6\}$. If $e = b$, we further recolor y with some color in $L(y) \setminus \{e,k\}$. If $e = k$, we need to recolor uy with some color in $L(uy) \setminus \{b,e\}$. Let k' denote the color of uy after possible recoloring. We color u with $\beta \in L(u) \setminus \{k'\}$, and x with some color in $L(x) \setminus \{\beta, d\}$. □

3 Main Results

Theorem 9 *For a Halin graph H, we have*

$$\chi_\ell(H) = \begin{cases} 4, & \text{if } H \text{ is a wheel of even order,} \\ 3, & \text{otherwise.} \end{cases}$$

Proof. Let $H = T \cup C$ where C is the cycle $u_1u_2\cdots u_nu_1$. Since H contains at least one 3-face, $\chi_\ell(H) \geq \chi(H) \geq 3$. On the other hand, $\chi_\ell(H) \leq 4$ since H is 3-degenerate by Lemma 1.

If H is a wheel of even order, then $\chi_\ell(H) \geq \chi(H) = 4$. Thus $\chi_\ell(H) = 4$.

Now suppose that H is not a wheel of even order. Let L be an arbitrary assignment for H such that $|L(u)| = 3$ for each $u \in V(H)$. By Lemma 6, $T - V(C)$ has an L-coloring ϕ. Define an assignment L' of C by $L'(v) = L(v) \setminus \{\phi(w)\}$ for every $v \in V(C)$, where w is the inner vertex adjacent to v. Thus $|L'(v)| \geq 2$ for all $v \in V(C)$. Suppose that C is not L'-colorable. By Lemmas 4 and 5, $|C|$ is odd and all new lists are equal to the same 2-set. By Lemma 2, H contains a special inner vertex u. Without loss of generality, let $y, u_1, u_2, \ldots, u_k$ be the neighbors of u, where y is an inner vertex. Note that u_{k+1} is not a neighbor of u. We recolor u with some color in $L(u) \setminus \{\phi(u), \phi(y)\}$. After this change, $|L'(u_k)| \geq 2$ and $L'(u_k) \neq L'(u_{k+1})$. So C can be properly colored by Lemma 5 and H is L-colorable. □

Theorem 10 *If H is a Halin graph, then $\chi'_\ell(H) = \chi'(H) = \Delta(H)$.*

Proof. Let the given Halin graph be expressed as $H = T \cup C$, where C is the cycle $u_1u_2\cdots u_nu_1$. It suffices to prove $\chi'_\ell(H) \leq \Delta(H)$. Let L be an assignment satisfying $|L(e)| = \Delta(H)$ for each edge e of H. We are going to show that H is edge L-colorable.

If $\Delta(H) = 3$, then H is a 3-regular 3-connected plane graph. By Lemma 3, H is a Hamiltonian graph. It follows that H is edge 3-colorable. By Lemma 7, H is edge L-colorable.

Suppose $\Delta(H) \geq 4$. By Lemma 6, T has an edge L-coloring ϕ. For every $e \in E(C)$, define the new list $L'(e) = L(e) \setminus \{\phi(e_1), \phi(e_2)\}$, where e_1 and e_2 are edges of T that are adjacent to e. Thus $|L'(e)| \geq \Delta(H) - 2 \geq$

2. Suppose that C is not edge L'-colorable. By Lemmas 4 and 5, $|C|$ is odd and $L'(e) = \{a, b\}$ for every edge e of C. In this case we must have $\Delta(H) = 4$. By Lemma 2, H contains a special inner vertex u. Without loss of generality, let $y, u_1, u_2, \ldots, u_k$ be the neighbors of u, where y is an inner vertex. Thus uu_1u_2 forms a 3-face of H. Suppose $\phi(uu_1) = c_1$ and $\phi(uu_2) = c_2$. Thus $L(u_1u_2) = \{a, b, c_1, c_2\}$, $c_1 \in L(u_nu_1)$, and $c_2 \in L(u_2u_3)$. Erase c_1 from uu_1 and c_2 from uu_2. Now there are at least two colors available for each of uu_1 and uu_2. We color uu_1 with a new color α different from c_1, then color uu_2 properly. We then modify L' accordingly. Hence $|L'(u_nu_1)| \geq 2$ and $L'(u_nu_1) \neq L'(u_{n-1}u_n)$. In fact, if $|L'(u_nu_1)| = 2$, then $c_1 \in L'(u_nu_1) \setminus L'(u_{n-1}u_n)$. By Lemma 5, C is edge L'-colorable. Consequently, H is edge L-colorable. □

Theorem 11 *If H is a Halin graph with $\Delta(H) \geq 4$, then $\chi''_\ell(H) = \chi''(H) = \Delta(H) + 1$.*

Proof. Let the given Halin graph be expressed as $H = T \cup C$, where C is the cycle $u_1u_2 \cdots u_nu_1$. It suffices to prove $\chi''_\ell(H) \leq \Delta(H) + 1$. Let L be an assignment for H satisfying $|L(x)| = \Delta(H) + 1$ for every $x \in V(H) \cup E(H)$. We are going to show that H is totally L-colorable.

Suppose $\Delta(H) \geq 5$. By Lemma 6, T has a total L-coloring. Let ϕ be a coloring obtained from a total L-coloring of T by uncoloring the pendant vertices. Since every element $x \in V(C) \cup E(C)$ is adjacent or incident to two colored elements of T (either two edges or one edge and one vertex), x has at least four available colors to choose from. It follows from Lemma 4 that ϕ can be extended to C.

Now suppose $\Delta(H) = 4$. If H is a wheel of order 5, then (by using Lemma 8 as in the last paragraph of this proof) it is easy to prove that $\chi''_\ell(H) = 5$. Otherwise, H contains a special inner vertex u by Lemma 2. Without loss of generality, let $y, u_1, u_2, \ldots, u_k$, $2 \leq k \leq 3$, be the neighbors of u, where y is an inner vertex.

If $k = 2$, then $d_H(u) = 3$ and uu_1u_2 forms a 3-face of H. Let ϕ be a coloring obtained from a total L-coloring of T (which exists by Lemma 6) by uncoloring u, uu_1, uu_2, and all pendant vertices. Each $t \in V(C) \cup E(C)$ has at least three available colors to choose from. Since every path is totally 3-list colorable, we extend ϕ to $(V(C) \cup E(C)) \setminus \{u_1, u_2, u_1u_2\}$. Now every vertex or edge on the 3-cycle uu_1u_2u has at least three available colors to choose from. By Lemma 4, it can be properly colored.

If $k = 3$, then $\{u, u_1, u_2, u_3\}$ induces a fan K of order 4. Let ϕ be a coloring obtained from a total L-coloring of T (which exists by Lemma 6) by uncoloring u, uu_1, uu_2, uu_3, and all pendant vertices. Then we extend ϕ to $(V(C) \cup E(C)) \setminus \{u_1, u_2, u_3, u_1u_2, u_2u_3\}$. Let x be a vertex or an edge of K. From the list $L(x)$, we remove all $\phi(z)$, where $z \in V(H) \cup E(H)$

is adjacent or incident to x. Denote this modified assignment of K by L'. Then $|L'(u_2)| = 5$, $|L'(t)| \geq 4$ for $t \in \{uu_2, u_1u_2, u_2u_3\}$, and $|L'(s)| \geq 3$ for $s \in \{u, u_1, u_3, uu_1, uu_3\}$. Without loss of generality, we may assume that $|L'(t)| = 4$ for all t considered and $|L'(s)| = 3$ for all s considered (otherwise we can select their subsets having such property.) If $L'(u) = L'(uu_3)$, we change $\phi(u_3u_4)$ to some color which is different from its three adjacent or incident colors. When this final modification is done, $|L'(uu_3)| = 3$ and $L'(uu_3) \neq L'(u)$. If $L'(u) = L'(uu_1)$, we have a similar recoloring. Thus we may suppose $L'(uu_1) \neq L'(u)$. By Lemma 8, K is totally L'-colorable. Consequently, H is totally L-colorable. □

Remarks. Zhang, Liu, Wang, and Li [18] proved that $\chi''(H) = \Delta(H) + 1$ for every Halin graph H with $\Delta(H) \geq 4$. Theorem 11 establishes the stronger result for the list total chromatic number.

For a Halin graph H with $\Delta(H) = 3$, we actually have $4 \leq \chi''_\ell(H) \leq 5$. This follows from a result of Juvan, Mohar, and Škrekovski [10] which states that every graph G is totally 5-list colorable if $\Delta(G) \leq 3$. Note that the complete graph K_4 and the complement H of a 6-cycle are 3-regular Halin graphs having $\chi''_\ell(K_4) = 5$ and $\chi''(H) = 4$. If Conjecture 2 is true, we have $\chi''_\ell(H) = 4$.

Finally, we note that our proofs of Theorems 9 to 11 actually provide polynomial-time algorithms for finding those list colorings.

Acknowledgment. This work was done while the first author was visiting the Institute of Mathematics, Academia Sinica, Taipei. The financial support provided by the Institute is greatly appreciated. The authors are very grateful to the referee for giving them detailed comments on the revision of this paper.

References

[1] J. A. Bondy and L. Lovász, Length of cycles in Halin graphs, J. Graph Theory 8 (1985) 397-410.

[2] O. V. Borodin, A. V. Kostochka, and D. R. Woodall, List edge and list total coloring of multigraphs, J. Combin. Theory Ser. B 71 (1997) 184-204.

[3] M. N. Ellingham and L. Goddyn, List edge colorings of some 1-factorable multigraphs, Combinatorica 16 (1996) 343-352.

[4] P. Erdős, A. L. Rubin, and H. Taylor, Choosability in graphs, Congr. Numer. 26 (1979) 125-157.

[5] F. Galvin, The list chromatic index of a bipartite multigraph, J. Combin. Theory Ser. B 63 (1995) 153-158.

[6] R. Häggkvist and A. Chetwynd, Some upper bounds on the total and list chromatic numbers of multigraphs, J. Graph Theory 16 (1992) 503-516.

[7] R. Häggkvist and J. Janssen, New bounds on the list-chromatic index of the complete graph and other simple graphs, Combin. Probab. Comput. 6 (1997) 295-313.

[8] R. Halin, Studies on minimally n-connected graphs, in Combinatorial Mathematics and its Applications (Proc. Conf. Oxford, 1969), (Academic Press, London, 1971) 129-136.

[9] T. R. Jensen and B. Toft, Graph Coloring Problems, Wiley-Interscience, New York, 1995.

[10] M. Juvan, B. Mohar, and R. Škrekovski, List total colorings of graphs, Combin. Probab. Comput. 7 (1998) 181-188.

[11] A. V. Kostochka and D. R. Woodall, Choosability conjectures and multicircuits, Discrete Math. 240 (2001) 123-143.

[12] A. V. Kostochka and D. R. Woodall, Total choosability of multicircuits I, J. Graph Theory 40 (2002) 26-43.

[13] A. V. Kostochka and D. R. Woodall, Total choosability of multicircuits II, J. Graph Theory 40 (2002) 44-67.

[14] P. C. B. Lam and Z. Zhang, The vertex-face total chromatic number of Halin graphs, Networks 30 (1997) 167-170.

[15] D. Peterson and D. R. Woodall, Edge-choosability in line-perfect multigraphs, Discrete Math. 202 (1999) 191-199.

[16] Wang Weifan and Ko-Wei Lih, Choosability, edge choosability, and total choosability of outerplane graphs, Europ. J. Combin. 22 (2001) 71-78.

[17] D. R. Woodall, Edge-choosability of multicircuits, Discrete Math. 202 (1999) 271-277.

[18] Z. Zhang, L. Liu, J. Wang, and H. Li, A note on the total chromatic number of Halin graphs with maximum degree 4, Appl. Math. Lett. 11(5) (1998) 23-27.