Magic Squares, Finite Planes and Simple Quasilattices

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1. *Deriving quasilattices from magic squares and finite planes*

Noncommutative generalizations of lattices have been studied for over fifty years. One of the first to study such objects was the physicist Pascual Jordan who in the 1950s and early 1960s wrote numerous papers on the subject. Many types of noncommutative lattices are quasilattices. By definition, a ***quasilattice*** is a set Q together with associative binary operations $\vee$ and $\wedge$ that are also idempotent ($x \vee x = x = x \wedge x$) and dualize each other in that

$$x \vee y \vee x = x \text{ if and only if } y \wedge x \vee \wedge y = y.$$

Given that $\vee$ and $\wedge$ are associative and idempotent, this is equivalent to the following absorption identities being satisfied:

$$x \wedge (y \vee x \vee y) \wedge x = x = x \vee (y \wedge x \wedge y) \vee x.$$

Ordinary lattices are quasilattices and so are ***antilattices*** - sets with associative, idempotent binary operations, $\vee$ and $\wedge$, satisfying

$$x \vee y \vee z = x \vee z \text{ and } x \wedge y \wedge z = x \wedge z.$$

Finite antilattices can be described using rectangular arrays. Suppose we have a finite set A along with two (usually different) ways of storing its elements in a rectangular array with each array corresponding to one of the operations. The join $x \vee y$ and the meet $x \wedge y$ of $x, y \in$ A is the element z in the row of x and the column of y of the assigned array. For example, let $\{1, 2, 3, \ldots, 9\}$ be stored in the following arrays:

$$(\vee) \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad (\wedge) \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}.$$

Then $1 \vee 6 = 3$ while $1 \wedge 6 = 6$. Similarly, $3 \vee 2 = 2$ while $3 \wedge 2 = 7$. Allowing for infinite arrays, every antilattice can be described in this way.

In this paper our interest is with simple quasilattices. While simplicity is defined in the next section, suffice to say here that a simple quasilattice is either a simple lattice or a simple antilattice. We are especially interested in uncovering (families of) simple antilattices, and seeing how arrays such as

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magic squares and representations of finite planes can give rise to simple antilattices.

Recall that a ***magic square*** is a square array of distinct numbers where all rows, columns and the two diagonals have a common ***magic sum***. A classic instance is the *Lo-Shu* with a magic sum of 15:

$$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}.$$

Given a magic square, its ***derived antilattice*** arises by letting the given square be the $\wedge$-array and letting the $\vee$-array be the square array storing the same numbers entered in their natural ordering. The above antilattice example illustrates this in the case of the *Lo-Shu*.

Square arrays also come from ***finite planes***, that is, vector spaces of dimension 2 over finite fields. For instance, given the field $\mathbf{Z}_5$, the plane $\mathbf{Z}_5 \times \mathbf{Z}_5$ can be represented as the 5×5 array of ordered pairs on the left below, but with parentheses deleted. Alternatively, one could view these pairs as base 5 representations of integers in base 10. Thus 3,2 represents $32_5 = 17_{10}$. The planar array could thus be encoded using the numbers in the right array.

$$\begin{bmatrix} 0,0 & 1,0 & 2,0 & 3,0 & 4,0 \\ 0,1 & 1,1 & 2,1 & 3,1 & 4,1 \\ 0,2 & 1,2 & 2,2 & 3,2 & 4,2 \\ 0,3 & 1,3 & 2,3 & 3,3 & 4,3 \\ 0,4 & 1,4 & 2,4 & 3,4 & 4,4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 & 24 \end{bmatrix}$$

In either case, this plane has 25 points and 30 lines, the latter arranged in six classes of five parallel lines each. The rows of the array consist of all lines of slope 0, the columns consist of all lines of undefined slope, the main diagonal plus the four broken descending diagonals yield all five lines of slope 1, and the counter-diagonal plus all four broken ascending counter-diagonals yield all five lines of slope 4. In all, between the rows, columns, diagonals and counter-diagonals, 20 out of 30 lines are accounted for in this way, with only lines of slopes 2 or 3 left out.

Alternatively, $\mathbf{Z}_5 \times \mathbf{Z}_5$ can be represented by storing the five lines of slope 1 in the five rows and the five lines of slope 4 in the five columns in the left array below.

$$\begin{bmatrix} 0,0 & 1,1 & 2,2 & 3,3 & 4,4 \\ 2,3 & 3,4 & 4,0 & 0,1 & 1,2 \\ 4,1 & 0,2 & 1,3 & 2,4 & 3,0 \\ 1,4 & 2,0 & 3,1 & 4,2 & 0,3 \\ 3,2 & 4,3 & 0,4 & 1,0 & 2,1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 6 & 12 & 18 & 24 \\ 13 & 19 & 20 & 1 & 7 \\ 21 & 2 & 8 & 14 & 15 \\ 9 & 10 & 16 & 22 & 3 \\ 17 & 23 & 4 & 5 & 11 \end{bmatrix}.$$

In the right array, not only do all rows, columns, the main diagonal and the counter-diagonal sum to 60, but so do all broken diagonals and counter-diagonals. This makes the right array a ***pandiagonal*** magic square. Returning to the left array, the (broken) diagonals and counter-diagonals are precisely the lines of slope 3 and 2 respectively. Indeed, the line arrangement of the left array forces the right array to be pandiagonal. That finite planes can yield pandiagonal squares is well known. Together, the two representations of this plane in integer format describe an antilattice induced from a magic square. As is shown below, both derived antilattices encountered thus far are simple.

For further background on quasilattices and their congruences, see Laslo and Leech [3]. For more information on noncommutative lattices in general, see the introductory remarks in Leech [4].

2. *Congruences and simple algebras*

Quasilattices, and antilattices in particular, are algebras. Recall that a ***congruence*** on an algebra A is any equivalence θ on its underlying set A that is consistent with the operations. For a quasilattice Q, this means that for all $a, b, c \in Q$, $a\,\theta\,b$ implies $a\vee c\;\theta\;b\vee c$, $c\vee a\;\theta\;c\vee b$, $a\wedge c\;\theta\;b\wedge c$ and $c\wedge a\;\theta\;c\wedge b$. The set of congruences on Q forms a complete lattice **Con**(Q). Given $\theta_1, \theta_2 \in$ **Con**(Q), $\theta_1 \vee \theta_2$ is the equivalence generated from $\theta_1 \cup \theta_2$, while $\theta_1 \wedge \theta_2$ is just $\theta_1 \cap \theta_2$.

On any quasilattice Q a canonical congruence $\mathcal{D}$ is defined by $a\,\mathcal{D}\,b$ if both $a\vee b\vee a = a$ and $b\vee a\vee b = b$ in Q; equivalently, $a\,\mathcal{D}\,b$ if $a\wedge b\wedge a = a$ and $b\wedge a\wedge b = b$. The ***Clifford-McLean Theorem for quasilattices*** states: (1) $\mathcal{D}$ is the least lattice congruence on Q *and (hence)* $Q/\mathcal{D}$ *is the greatest lattice image of* Q. (2) *The* $\mathcal{D}$*-equivalence classes of* Q *are just its maximal sub-antilattices.* Put briefly: *every quasilattice is a lattice of antilattices.* There is more.

By results in [3], **Con($\mathcal{Q}$)** is isomorphic to a complete sublattice of the direct product **Con($\mathcal{Q}/\mathcal{D}$)** $\times \Pi_{\delta\in\Delta}$**Con($D_\delta$)** where $\{D_\delta \mid \delta \in \Delta\}$ is an indexed set of all maximal sub-antilattices of $\mathcal{Q}$. Since $\mathcal{Q}/\mathcal{D}$ a lattice, **Con($\mathcal{Q}/\mathcal{D}$)** is distributive. Thus, **Con($\mathcal{Q}$)** must be distributive [or modular] when all of the **Con(D_δ)** are thus. Even when the **Con(D_δ)** are not that well-behaved, they play a significant role in our understanding of congruence lattices of a quasilattice.

Of special interest in the study of any class of algebras is the determination of those algebras **A** that are ***simple*** in that **Con(A)** = $\{\Delta, \nabla\}$ where $\Delta = \{(a, a) \mid a \in A\}$ is the ***trivial congruence***, and $\nabla = A\times A$ is the ***universal congruence***, where A is the set underlying **A**. In general, *a simple quasi-lattice is either a simple lattice or the simple antilattice*. Thus, uncovering simple antilattices is a worthy ongoing project in the study of quasilattices. In particular, in this paper we are interested in simple antilattices induced from magic squares or finite planes. In the case of an antilattice induced from a magic square, *its congruence lattice and hence its potential simplicity is unaffected by any dihedral variation of the magic square.*

To begin, given any pair $a, b \in A$, recall that the ***principal congruence*** $\theta_{(a,b)}$ is the smallest congruence on **A** relating a and b. Clearly: $\theta_{(a,b)} = \cap\{\theta \in$ **Con(A)** $\mid a\,\theta\, b\}$. In particular, $\theta_{(a,b)} = \Delta$ precisely when $a = b$. Clearly:

Proposition 1. *An algebra* **A** *is simple if* $\theta_{(a,b)} = \nabla$ *all for* $a \neq b$ *in its underlying set*. □

For antilattices, this obvious criterion can be simplified. Consider an antilattice **A** deter-mined by a pair of rectangular arrays. Let R_0 and C_0 represent a row and a column of, say, the $\vee$-array of **A**. (*Which* array is unimportant. But both R_0 and C_0 must come from the same array.)

Theorem 2. (*Simplicity Criterion for Antilattices*) *Given an antilattice* $\mathcal{A}$ *determined by a pair of rectangular arrays, let* R_0 *and* C_0 *denote respectively a row and a column of the* $\vee$*-array. Then* $\mathcal{A}$ *is a simple algebra iff* $\theta_{(a,b)} = \nabla$ *for all* $a \neq b$ *in* R_0 *and all* $a \neq b$ *in* C_0. *In particular, any given* $\theta_{(a,b)}$ *must equal* ∇ *if* $\mathcal{A}$ *is generated from* $\{a, b\}$ *using both* $\vee$ *and* $\wedge$.

Proof. The condition is clearly necessary. To see sufficiency, suppose that the condition holds for row R_0 and column C_0 intersecting at element c in the $\vee$-array. Given $a \neq b$ in A, both $c\vee a$ and $c\vee b$ lie in R_0, while $a\vee c$ and $b\vee c$ lie in C_0. Since $a \neq b$, either $c\vee a \neq c\vee b$ in R_0 or $a\vee c \neq b\vee c$ in C_0. Say $c\vee a \neq c\vee b$, so that $\theta_{(c\vee a, c\vee b)} = \nabla$. But since $c\vee a\ \theta_{(a,b)}\ c\vee b$, $\theta_{(c\vee a,c\vee b)}$ refines $\theta_{(a,b)}$ so that $\theta_{(a,b)} = \nabla$ also.

Thus $\theta_{(a,b)} = \nabla$ for all $a \neq b \in A$ and $\mathcal{A}$ is simple. Since $\vee$ and $\wedge$ are idempotent, the sub-algebra $\langle a, b\rangle$ generated from $\{a, b\}$ lies in the $\theta_{(a,b)}$-class of a. The final statement follows. □

Comment. In the case of a square antilattice determined from a pair of $n{\times}n$ arrays, this theorem says that the number of principal congruences needing to be checked can be reduced from $(n^4 - n^2)/2$ to just $n^2 - n$. Although the check to see that $\theta_{(a,b)} = \nabla$ for $a \neq b$ in either $R_0{\times}R_0$ or $C_0{\times}C_0$ can be initially tedious, as the check continues some random recursion enters the process. Thus, if say $\theta_{(a,b)}$ has been shown to equal ∇ and $a\ \theta_{(c,d)}\ b$ is encountered in the check of $\theta_{(c,d)}$, then one can immediately conclude that $\theta_{(c,d)} = \nabla$ also holds.

Theorem 2 can be used to establish the following result showing that simple antilattices of all composite orders greater than 4 exist. Its proof is given in [3]. (See Theorem 14.)

Theorem 3. *Given positive integers m and n with $n \geq m \geq 2$ and $n \geq 3$, the antilattice determined by the following pair m×n arrays with distinct entries is simple.* □

$$(\vee)\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (\wedge)\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{21} \\ a_{22} & a_{23} & \cdots & a_{2,n} & a_{31} \\ a_{32} & a_{33} & \cdots & a_{3,n} & a_{41} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m2} & a_{m3} & \cdots & a_{mn} & a_{1n} \end{bmatrix}$$

Note that the $(\wedge)$-array is the result of first removing a_{1n} from the $(\vee)$-array, moving all remaining elements forward so that a_{21} takes the vacated place, a_{22} takes the place of a_{21} and so forth, and finally placing the removed a_{1n} in the now-vacated lower right corner.

Returning to connections with magic squares, we have:

Example. The antilattice derived from the *Lo-Shu* is simple.

$$(\vee)\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad (\wedge)\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}.$$

Take $\{1, 2\}$. From the $\wedge$-array, it is clear that $6, 9 \in \langle 1, 2\rangle$. But $\{1, 2, 6, 9\}$ clearly generates the $\vee$-array and thus the algebra. Hence $\theta_{(1,2)} = \nabla$. Similar remarks hold for any other pair $a \neq b$ in any row or column of *either* array. □

These remarks deserve a more precise analysis. Given distinct elements a and b in a common row (column) of a 3×3 array, the elements c and d lying in neither the row (column) or the two columns (rows) of a and b is called the ***dual pair***. The relationship is symmetrical. Thus $\{1, 2\}$ and $\{6, 9\}$ form dual pairs in the $\vee$-array above, but not in the $\wedge$-array. *Any pair of dual pairs in a* 3×3 *array generates the entire array under the ambient idempotent operation.* Given two distinct elements in a common row or column of one of the above arrays, this pair immediately generates its dual pair in the opposite array. In this sense, *these two arrays are* ***complementary*** 3×3 *arrays, so that any pair of elements lying in a common row or column in either array generates the entire antilattice which thus is simple.*

What can be said in general about congruences on an antilattice?

Given a rectangular array A, a ***cartesian partition*** of A is a partition $\mathcal{P}$ that is induced in cartesian fashion from a partition of the rows and a partition of the columns of A. For example, a cartesian partition of $\begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \end{bmatrix}$ is

given by: $\left[\begin{bmatrix} a & b & c \\ f & g & h \end{bmatrix} \begin{bmatrix} d & e \\ i & j \end{bmatrix}\right.$ $\left.\begin{bmatrix} k & l & m \end{bmatrix} \begin{bmatrix} n & o \end{bmatrix}\right]$.

Given such a partition $\mathcal{P}$, an equivalence $\mathcal{P}^{\#}$ on A is given by $a\ \mathcal{P}^{\#}\ b$ iff a and b lie in the same $\mathcal{P}$-class. Such an equivalence is called a ***cartesian equivalence*** on A. From semigroup theory, and the study of rectangular bands in particular, *the congruences on an array that are consistent with a single operation (using just one of $\vee$ or $\wedge$) are precisely its cartesian equivalences.* Thus:

Proposition 4. *Given an antilattice $\mathcal{A}$, its congruences arise from pairs of cartesian partitions of its two arrays sharing the same equivalence classes.* □

Example. $(\vee)\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad (\wedge)\begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}$

The magic square on the right appears in Albrecht Dürer's *Melancholia*. This antilattice is rich in congruences. Its congruences besides Δ and ∇, as classified by their corresponding partitions, are:

$\alpha_1 \leftrightarrow [1,2,3,4,13,14,15,16 \mid 5,6,7,8,9,10,11,12]$

$\alpha_2 \leftrightarrow [1,4,5,8,9,12,13,16 \mid 2,3,6,7,10,11,14,15]$

$\alpha_{12} \leftrightarrow [1,4,13,16 \mid 2,3,14,15 \mid 5,8,9,12 \mid 6,7,10,11]$

$\beta_1 \leftrightarrow [1,2,13,14 \mid 3,4,15,16 \mid 5,6,9,10 \mid 7,8,11,12]$

$\beta_2 \leftrightarrow [1,3,13,15 \mid 2,4,14,16 \mid 5,7,9,11 \mid 6,8,10,12]$

$\gamma_1 \leftrightarrow [1,4,5,8 \mid 2,3,6,7 \mid 9,12,13,16 \mid 10,11,14,15]$

$\gamma_2 \leftrightarrow [1,4,9,12 \mid 2,3,10,11 \mid 5,8,13,16 \mid 6,7,14,15]$

$\delta_1 \leftrightarrow [1,13 \mid 2,14 \mid 3,15 \mid 4,16 \mid 5,9 \mid 6,10 \mid 7,11 \mid 8,12]$

$\delta_2 \leftrightarrow [1,4 \mid 2,3 \mid 5,8 \mid 6,7 \mid 9,12 \mid 10,11 \mid 13,16 \mid 14,15]$

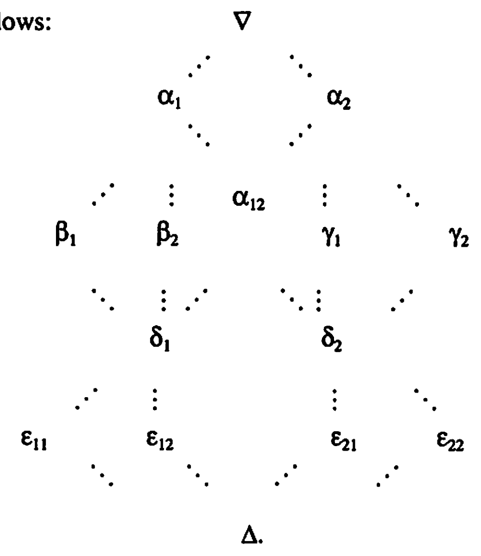
$\varepsilon_{11} \leftrightarrow [1,13 \mid 2,14 \mid 3,15 \mid 4,16 \mid 5 \mid 9 \mid 6 \mid 10 \mid 7 \mid 11 \mid 8 \mid 12]$

$\varepsilon_{12} \leftrightarrow [1 \mid 13 \mid 2 \mid 14 \mid 3 \mid 15 \mid 4 \mid 16 \mid 5,9 \mid 6,10 \mid 7,11 \mid 8,12]$

$\varepsilon_{21} \leftrightarrow [1,4 \mid 2 \mid 3 \mid 5,8 \mid 6 \mid 7 \mid 9,12 \mid 10 \mid 11 \mid 13,16 \mid 14 \mid 15]$

$\varepsilon_{22} \leftrightarrow [1 \mid 4 \mid 2,3 \mid 5 \mid 8 \mid 6,7 \mid 9 \mid 12 \mid 10,11 \mid 13 \mid 16 \mid 14,15]$

Con($\mathcal{A}$) is as follows:



Since a copy of $\mathcal{N}_5$ arises as a sublattice (say $\{\alpha_{12}, \delta_1, \varepsilon_{11}, \delta_2, \Delta\}$), the full congruence lattice is not modular, much less distributive. Its order 15, however, should be contrasted to 225, the order of the lattice of all distinct cartesian partitions of a 4×4 array. □

3. *Antilattices induced from* 3×3 *and* 4×4 *magic squares.*

The *Lo-Shu* is one of infinitely many possible 3×3 magic squares that can arise if we agree to store integers *besides* 1 - 9. Others include the following two squares:

$$\begin{bmatrix} 71 & 89 & 17 \\ 5 & 59 & 113 \\ 101 & 29 & 47 \end{bmatrix} \qquad \begin{bmatrix} 252 & 171 & 363 \\ 373 & 262 & 151 \\ 161 & 353 & 272 \end{bmatrix}.$$

The magic square on the left consists entirely of primes, with a magic sum of 177, the least possible such sum for any magic square of primes. (Magic squares of primes is a significant topic among magic square enthusiasts.) The magic square on the right consists of 3-digit palindromes. It turns out that all three magic squares induce simple antilattices. Is this true for all 3×3 magic squares? To answer this, we begin with the following result on complementary pairs of arrays.

Proposition 5. *Given* 3×3 *arrays,* A *and* A′, *each storing the same* 9 *distinct elements, the following assertions are equivalent*:

1. A *and* A′ *form a complimentary pair of* 3×3 *arrays.*
2. *If two distinct elements are either row-related or column-related in either array, they are unrelated in either sense in the other array.*
3. *The rows [columns] in* A *either all become (extended) diagonals in* A′ *or all become extended counterdiagonals in* A′; *similar remarks hold in passing from* A′ *to* A.

Proof. Clearly (1) implies (2). For the converse, observe that the status of (2) is unchanged if either array undergoes row or columns interchanged! Thus, we assume (2) in the case where elements *a* and *b* lie in a common row of A, as in

$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Assertion (2) implies that A′ has either form $\begin{bmatrix} a & f & x \\ i & b & y \\ u & v & w \end{bmatrix}$ or $\begin{bmatrix} a & i & x \\ f & b & y \\ u & v & w \end{bmatrix}$. Applying (2) further, A′ must be either $\begin{bmatrix} a & f & h \\ i & b & d \\ e & g & c \end{bmatrix}$ or its transpose $\begin{bmatrix} a & i & e \\ f & b & g \\ h & d & c \end{bmatrix}$. In either case we have a complementary pair of arrays. Similarly, assuming a and b lie in the same column of A, (2) forces A′ to be a complementary array. Likewise, if a and b are row-[column-] related in A′, then (2) forces A to be a complementary to A′. Thus, (1) and (2) are equivalent. Clearly (3) implies (1) and (2). Given the latter, every row/column in either array must be an (extended) [counter]diagonal in the other array. But this can only happen if (3) holds. □

We are ready to state our main results about antilattices induced from 3×3 magic squares.

Theorem 6. *Given a* 3×3 *array* A = $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ *consisting of nine distinct positive integers in their natural (increasing) order and a* 3 × 3 *magic square* A′ *storing the same integers, then either* A *and* A′ *are complementary or else* A′ *is a dihedral variation of* $\begin{bmatrix} b & i & c \\ f & e & d \\ g & a & h \end{bmatrix}$. ($\begin{bmatrix} 3 & 11 & 4 \\ 7 & 6 & 5 \\ 8 & 1 & 9 \end{bmatrix}$ is a minimal instance of the latter.)

Proof. Using a dihedral replacement of A′ if need be, distinct $\beta > \gamma > 0$ and $\alpha > \beta + \gamma$ exist such that:

$$A' = \alpha\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \beta\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \gamma\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha - \beta & \alpha + \beta + \gamma & \alpha - \gamma \\ \alpha + \beta - \gamma & \alpha & \alpha - \beta + \gamma \\ \alpha + \gamma & \alpha - \beta - \gamma & \alpha + \beta \end{bmatrix}.$$

Clearly $a = \alpha - \beta - \gamma$ and $b = \alpha - \beta$. What is the next smallest element? If it is $\alpha - \gamma$, then the ascending sequence

$$\alpha-\beta-\gamma < \alpha-\beta < \alpha-\gamma < \alpha-\beta+\gamma < \alpha < \alpha+\beta-\gamma < \alpha+\gamma < \alpha+\beta < \alpha+\beta+\gamma$$

must occur. In this case we have the displayed array. Otherwise, we must have:

$$\alpha-\beta-\gamma < \alpha-\beta < \alpha-\beta+\gamma < \alpha-\gamma < \alpha < \alpha+\gamma < \alpha+\beta-\gamma < \alpha+\beta < \alpha+\beta+\gamma$$

yielding an array complementary to A. □

This theorem has the following consequences:

. **Corollary 7.** *Given the arrays* A *and* A′ *of the prior theorem, the induced antilattice* A *is simple if and only if* A *and* A′ *are complementary. Otherwise,* **Con**(A) *is a* 3*-element chain.*

Proof. In the complementary case, any pair of distinct elements generates A, which thus is simple. Otherwise, a single nontrivial, proper congruence is given by the partition $\{a, b, c, g, h, i \mid d, e, f\}$. □

Corollary 8. *All antilattices induced from* 3×3 *magic squares are congruence distributive.* □

We next consider antilattices induced from 4×4 magic squares that store 1 - 16. While just one 3×3 magic square stores 1 - 9 (with eight dihedral variations), 880 essentially distinct magic squares store 1 to 16. A list of all 880 squares was given by Bernard Frénicle de Bessy in a posthumous 1693 publication. A mathematical analysis was given in the 1983 paper of Dame Kathleen Ollerenshaw and Sir Hermann Bondi [5]. Thanks to the following observation, these 880 cases decompose into 220 classes of 4.

Lemma 9. *Given a* 4×4 *magic square* A, *let squares* B, C *and* D *be induced from* A *by simultaneous row and column permutations determined by* (2 3), (1 2)(3 4) *and* (1 3 4 2) *respectively. Then* A – D *are all magic squares, but none are dihedrally equivalent. Moreover all four squares induce the same antilattice.* □

Thus one can get by checking the leading array in each row of four squares in the Ollerenshaw-Bondi list. Among these, the nonsimple cases are easily spotted, thanks to a theorem about semimagic squares (all rows and columns add up to 34). In its statement, the ***index*** of a congruence μ counts its number of congruence classes.

Theorem 10. *If a semi-magic square* A *storing* 1 – 16 *induces a nonsimple antilattice* A, *then* A *has a maximal congruence* μ *of index* 2 *whose corresponding congruence class partition is either*

$$\pi_R = \{1-4, 13-16 \mid 5-12\}$$

or

$$\pi_C = \{1, 4, 5, 8, 9, 12, 13, 16 \mid 2, 3, 6, 7, 10, 11, 14, 15\}$$

where π_R *and* π_C *are outer/inner partitions splitting rows[columns]* 1 & 4 *against rows[columns]* 2 & 3 *in the standard array.*

Example. Consider the following magic squares in the Ollerenshaw-Bondi listing:

$$(1)\begin{bmatrix} 1 & 7 & 12 & 14 \\ 10 & 16 & 3 & 5 \\ 15 & 9 & 6 & 4 \\ 8 & 2 & 13 & 11 \end{bmatrix} \quad (9)\begin{bmatrix} 1 & 4 & 15 & 14 \\ 13 & 16 & 3 & 2 \\ 12 & 9 & 6 & 7 \\ 8 & 5 & 10 & 11 \end{bmatrix} \quad (25)\begin{bmatrix} 1 & 16 & 6 & 11 \\ 13 & 4 & 10 & 7 \\ 12 & 5 & 15 & 2 \\ 8 & 9 & 3 & 14 \end{bmatrix}.$$

Square (1) induces a simple antilattice because 1 - 4 lie in distinct rows and columns (denying π_R) and 1, 5, 9, 13 lie in distinct rows and columns (denying π_C). By contrast both π_R and π_C work for (9), while π_C works, but not π_R, for square (25). Thus both (9) and (25) are nonsimple. □

In any case, a quick survey of the 220 leading squares in the Ollerenshaw-Bondi list yields:

Theorem 11. *Of the* 880 *magic squares storing* 1 – 16, 416 *cases yield simple antilattices and* 464 *yield nonsimple algebras, giving a breakdown of* 47.27% to 52.73%.

Caveat. In the Ollerenshaw-Bondi list, the arrays actually store 0 – 15, instead of 1 – 16, and do so in base 4 notation.

Proof of Theorem 10. (All arrays in this proof are identified to within row and column permutations.)

All possible cartesian partitions of a 4×4 square with distinct elements can only have indices among the following: 1, 2, 3, 4. 6, 8, 9, 12, 16. Thus if A is nonsimple, the index $|\mu|$ of its maximal proper congruence μ must lie among 2, 3, 4, 6, 8, 9, 12.

If $|\mu| = 2$, then any cartesian partition of the standard array is one of four cases: one row and three rows, or one column and three columns, or two rows and two rows, or two columns and two columns. The first two cases are impossible when A is included, as no row or column in the standard array has the magic sum of 34. In the final cases, the sum of each pair of rows or columns must be $2 \times 34 = 68$. This occurs only for $\{\text{row } 1 \cup \text{row } 4 \mid \text{row } 2 \cup \text{row } 3\}$ or $\{\text{column } 1 \cup \text{column } 4 \mid \text{column } 2 \cup \text{column } 3\}$, just as stated.

$|\mu| = 3$ is impossible in the antilattice context since that would mean a row or column in the standard array would sum to 34 (because it would appear as a row or column in A), which is impossible.

$|\mu| = 4$ is possible. But in this case, the quotient algebra A/μ would have order 4 and thus be nonsimple by [3] Proposition 13. Hence μ was not really maximal after all.

$|\mu| = 6$ is also possible with the cartesian partition of the standard array having either template $\begin{bmatrix} [1\times 2] & [1\times 2] \\ [1\times 2] & [1\times 2] \\ [2\times 2] & [2\times 2] \end{bmatrix}$ or template $\begin{bmatrix} [1\times 1] & [1\times 3] \\ [1\times 1] & [1\times 3] \\ [2\times 1] & [2\times 3] \end{bmatrix}$ or a transpose. In any of these cases, the two bottom cells would still be adjacent in the cartesian partition of the magic square, thus inducing a congruence of index two.

Similarly for the remaining indices of 8, 9 and 12, congruences with these indices are always refined by a properly larger congruence of index 2, thus returning us to the $|\mu| = 2$ case. □

Example. Pandiagonal magic squares first appear in the 4×4 case. In considering pandiagonal magic squares in general, two such squares of the same dimension are ***equivalent*** if either is obtained from the other using a combination of dihedral operations, along with cyclic permutations of the rows and/or the columns. Any given pandiagonal square storing 1 through 16 is thus one of $8 \times 4 \times 4 = 128$ equivalent pandiagonal magic squares. In the 4×4 case, 48 dihedrally distinct pandiagonal squares exist, all being equivalent to exactly one of the following three pandiagonal magic squares:

$$\begin{bmatrix} 1 & 8 & 10 & 15 \\ 12 & 13 & 3 & 6 \\ 7 & 2 & 16 & 9 \\ 14 & 11 & 5 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 11 & 14 \\ 12 & 13 & 2 & 7 \\ 6 & 3 & 16 & 9 \\ 15 & 10 & 5 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 14 & 4 & 15 \\ 8 & 11 & 5 & 10 \\ 13 & 2 & 16 & 3 \\ 12 & 7 & 9 & 6 \end{bmatrix}.$$

The first two pandiagonal magic arrays induce simple antilattices; but the antilattice induced by the third array does not. In fact, its congruence structure is precisely that of the Dürer example since the Dürer square and the third pandiagonal square are equivalent under row and column interchanges. Indeed, the third square is column equivalent to $\begin{bmatrix} 4 & 15 & 14 & 1 \\ 5 & 10 & 11 & 8 \\ 16 & 3 & 2 & 13 \\ 9 & 6 & 7 & 12 \end{bmatrix}$ which is row equivalent to Dürer's square. □

4. *Simple antilattices over finite planes*

Finite affine planes are fruitful sources of both simple antilattices and magic squares. We begin with a finite plane construction of simple antilattices that generalizes an earlier result given by the author and Gratiela Laslo. (See [3] Theorem 15.).

Construction. Let $\mathbf{F}$ be a finite field of order p^n for p a prime and pick $\mu \neq \nu$ in $\mathbf{F}^* = \mathbf{F}\backslash\{0\}$. On the affine plane $\mathbf{P} = \mathbf{F} \times \mathbf{F}$ set $(a, b) \vee (c, d) = (c, b)$, the point on the line of slope 0 through (a, b) and the line of undefined slope through (c, d). Let $(a, b) \wedge (c, d)$ be the point on the line of slope μ through (a, b) and the line of slope ν through (c, d), that is:

$$(a,b)\wedge(c,d) = \left(\frac{\mu a - b - \nu c + d}{\mu - \nu}, \frac{\mu\nu a - \nu b - \mu\nu c + \mu d}{\mu - \nu}\right).$$

Such an algebra is called the ***affine antilattice*** on $\mathbf{P}$ with parameters μ and ν. It is indeed an anti-lattice with **L**- and **R**-classes for $\vee$ and $\wedge$ consisting of lines of slopes $0, \infty, \mu$ and ν.

Theorem 12. *Given a finite field* $\mathbf{F}$ *with* $\mu \neq \nu$ *in* $\mathbf{F}^*$, *the affine antilattice on* $\mathbf{P}$ *with parameters* μ *and* ν *is simple if either* $\mu/(\mu - \nu)$ *or* $-\nu/(\mu - \nu)$ *is a multiplicative generator of* $\mathbf{F}^*$.

Proof. That both operations form rectangular band operations is easily checked. We show that if $\delta = \mu/(\mu - \nu)$ generates $\mathbf{F}^*$, then $(\mathbf{P}, \vee, \wedge)$ is generated by any pair of distinct points, and thus is simple. To see this, observe that scalar multiplication and vector addition distribute over $\vee$ and $\wedge$ in that both

$$k(a, b) \vee k(c, d) = k[(a, b) \vee (c, d)]$$

and

$$[(a, b) + (e, f)] \vee [(c, d) + (e, f)] = [(a, b) \vee (c, d)] + (e, f)$$

with similar identities holding for $\wedge$. Thus in showing that all of $\mathbf{P}$ is generated from any two distinct points, we may assume that one of the points is (0, 0). If (a, b) is the other point, then from $(a, 0) = (0, 0)\vee(a, b)$ and $(0, b) = (a, b)\vee(0, 0)$ we may assume the given nonzero point lies on either the x-axis, consisting of pairs $(x, 0)$, or on the y-axis consisting of pairs $(0, y)$. From $(y/(\mu - \nu), 0) = (0, 0)\vee[(0, 0)\wedge(0, y)]$, we may suppose further that the given pair of points is (0, 0) and $(a, 0)$ on the x-axis. But $\{(0, 0), (a, 0)\}$ is the scalar a times $\{(0, 0), (1, 0)\}$. Hence $\langle(0, 0), (a, 0)\rangle$, the subset of $\mathbf{P}$ generated from $\{(0, 0), (a, 0)\}$ is the scalar a times the subset $\langle(0, 0), (1, 0)\rangle$. We show that the latter is $\mathbf{P}$ from which it follows that $\langle(0, 0), (a, 0)\rangle = \mathbf{P}$ also.

First observe that $(0, 0) \vee [(1, 0) \wedge (0, 0)] = (\delta, 0)$. Repeatedly applying $(0, 0) \vee [\_\_ \wedge (0, 0)]$ yields all points of the form $(\delta^n, 0)$. By our assumption on δ, the entire x-axis lies in $\langle(0, 0), (1, 0)\rangle$. From $[(a, 0) \wedge (0, 0)] \vee (0, 0) = (0, \mu\nu a/(\mu - \nu))$, it follows that the y-axis also lies in $\langle(0,0), (1,0)\rangle$. Hence all $(a, b) = (0, b) \vee (a, 0)$ lie in $\langle(0, 0), (1, 0)\rangle$ and the latter must be $\mathbf{P}$.

The case when $-\nu/(\mu - \nu)$ generates $\mathbf{F}^*$ is shown similarly, but by using the identity:

$$(0,0) \vee [(0,0) \wedge (c, 0)] = (-\nu c/(\mu - \nu), 0). \quad \square$$

Corollary 13. *Simple affine antilattices exist for all prime power orders of the form p^{2n} except for 2^2.*

Proof. This follows from the previous theorem and the fact that the group $\mathbf{F}^*$ is cyclic. □

Not until $\mathbf{Z}_7$ can parameters be found such that neither $\dfrac{\mu}{\mu - \nu}$ nor $\dfrac{-\nu}{\mu - \nu}$ generates $\mathbf{F}^*$. But even here the antilattice must be simple.

Theorem 14. *For p an odd prime and all $\mu \neq \nu \in \mathbf{Z}_p^*$, the affine antilattice on $\mathbf{P} = \mathbf{Z}_p \times \mathbf{Z}_p$ with parameters μ and ν is simple.*

Proof. Again, we need only show that (0, 0) and (1, 0) generate $\mathbf{P}$. Consider what can be generated by repeated application of $f(x, y) = (0, 0) \vee (x \wedge y)$. We immediately obtain $(\mu/(\mu + \pi), 0)$ and $(\pi/(\mu + \pi), 0)$, where π denotes $-\nu$. In general, all x-coordinates of the form $p(\mu, \pi)/(\mu + \pi)^k$, where $p(\mu, \pi)$ is any subpolynomial of $(\mu + \pi)^k$, arise from repeated application of $f(x, y)$. In particular, for $k = p - 1$, $(\mu + \pi)^{p-1} = 1$ and $0, 1\mu^{p-2}\pi, 2\mu^{p-2}\pi, \ldots, (p-1)\mu^{p-2}\pi$ are x-coordinates of f-generated vectors. But these are precisely the p distinct multiples of $\mu^{p-2}\pi$ and hence are all of $\mathbf{Z}_p$. Thus $(0, 0), (1, 0), \ldots, (p-1, 0)$ is in the generated set. But the latter occupy distinct rows and columns in the $\wedge$-array and so collectively generate all of **P** *via* $\vee$ and $\wedge$. □

Not until $\mathbf{F}_9$ can a nonsimple affine antilattice be defined on **P** (by using $\mu = 2$ and $\nu = 1$). In the case of Theorem 14, where $\mathbf{F} = \mathbf{Z}_p$, the $\wedge$-arrays of affine antilattices over **P** yield examples of pandiagonal matrices, provided $p \geq 5$, thus allowing ≥ 4 slopes in $\mathbf{Z}^*_p$. To do so, however, one may need to interchange rows or columns (leaving the antilattice structure unchanged). As is, the $\wedge$-array is at least semimagical if we let its coordinate pairs denote integers, base p from 0 through $p^2 - 1$, with rows and columns all summing to $p(p^2 - 1)/2$.

Example. Consider the $\wedge$-arrays below for $\mathbf{P} = \mathbf{Z}_5 \times \mathbf{Z}_5$ in the case where $\mu = 1$ and $\nu = 4$. Considered as integers base 5, all rows and columns have a sum of 220, base 5, or 60, base 10.

(*a*)

00	11	22	33	44
14	20	31	42	03
23	34	40	01	12
32	43	04	10	21
41	02	13	24	30

$h = 1 = k \quad \alpha = 0, \beta = \infty$

(*b*)

00	11	22	33	44
23	34	40	01	12
41	02	13	24	30
14	20	31	42	03
32	43	04	10	21

$h = 1, k = 2 \quad \alpha = 3, \beta = 2$

(*c*)

00	11	22	33	44
32	43	04	10	21
14	20	31	42	03
41	02	13	24	30
23	34	40	01	12

$h = 1, k = 3 \quad \alpha = 2, \beta = 3$

(*d*)

00	22	44	11	33
14	31	03	20	03
23	40	12	34	01
32	04	21	43	10
41	13	30	02	24

$h = 2, k = 1 \quad \alpha = 2, \beta = 3$

In (*a*), the descending diagonals consist of lines of slope 0, while all ascending diagonals consist of lines of undefined slope. Hence the corresponding numerical diagonals do not share the same sum of 220_5. As one goes across the rows from left to right, the entries are all incremented by $h = +1$ in the x-coordinate; and as one goes down the columns from top to bottom, the entries are also incremented by $k = +1$ in the x-coordinate.

What happens if we change this scheme by choosing h and k values between 1 and $p - 1$ such that $h \neq \pm k$? This would require only row switches and column switches. The result would create diagonals also sharing the same magic sum. In terms of the coordinates, the descending diagonals would consist of lines of slope $\alpha = (h - k)/(h + k)$ and the ascending diagonals would consist of all lines of inverse slope $\beta = (h + k)/(h - k)$ and the corresponding numerical array would be a pandiagonal magic square. Several such cases are given in (*b*) – (*d*). □

Two further comments: Any simple antilattice constructed from an affine plane $\mathbf{Z}_p \times \mathbf{Z}_p$ using slopes $\mu, \nu \in \{1, 2, 3, \dots, p - 1\}$ is also induced from any of the pandiagonal magic squares themselves induced from this affine situation. Secondly, and once again, constructing pandiagonal magic squares from finite planes is a well-known process. In this regard, the paper makes no claim of novelty.

5. *Simple antilattices from some classic magic squares*

In 1693, Simon de la Loubère gave the following rule for constructing magic squares for any odd order n:

De la Loubère's Rule. *Place* 00 *in the middle of the first row. In ascending (broken) diagonal fashion place in order the remaining* 01 *through* $0,n-1$. *Beneath* $0,n-1$ *place* 10 *and again in ascending diagonal fashion place* 11 *through* $1,n-1$. *Beneath* $1,n-1$ *place* 20, *and repeat the process until an entire* $n \times n$ *array is filled. The resulting array, is a magic square of odd order* n *storing* 0 *through* n^2-1 *in base* n.

The array to the right is the $n = 5$ case in base 5 notation storing 0 – 24.

(∨)

00	01	02	03	04
10	11	12	13	14
20	21	22	23	24
30	31	32	33	34
40	41	42	43	44

(∧)

31	43	00	12	24
42	04	11	23	30
03	10	22	34	41
14	21	33	40	02
20	32	44	01	13

Theorem 15. *The magic squares of odd order given by De la Loubère's rule together with their corresponding standard array yield simple antilattices precisely when the order is prime.*

Proof. Suppose that $x \,\theta\, y$ where $x \neq y$. By Theorem 2, we may assume that x and y either lie in a common row or in a common column of the $\vee$-array. If x and y are in the same column, then their meets yield θ-related elements u and v in distinct columns of the $\vee$-array. From $\{u, v, u \vee v, v \vee u\}$ we gain a pair of distinct θ-related elements lying in a common row of the $\vee$-array.

Thus at the outset we may assume that $x \,\theta\, y$ with x and y distinct elements in a common row of the $\vee$-array. If the order of the magic square is $p = 2n + 1$ (and the order of the algebra is p^2), then $n0 \vee x$ and $n0 \vee y$ must be distinct θ-related elements in the middle row of the $\vee$-array, say ni and nj. But ni and nj are also lie the main ascending diagonal of the $\wedge$-array and from them we can generate *via* $f(X, Y) = ni \vee (X \wedge Y)$, all n, $i{\pm}mk$ where $k = j - i$. If p is prime, the main ascending diagonal in the $\wedge$-array must lie in a common θ-class. Since this diagonal generates the entire algebra, $\theta = \nabla$.

If p is composite, say $p = ab$ with $1 < a, b < p$, then define an equivalence α by $ij \,\alpha\, kl$ if both $i \equiv k \pmod a$ and $j \equiv l \pmod a$. That α is a $\vee$-congruence is clear. In the case of $\wedge$, observe that in the $\wedge$-array any horizontal or vertical displacement of a positions from any starting position yields an α-related element. Conversely, any pair of α-related elements are connected by a sequence of such displacements. Thus given $x \,\alpha\, y$ and $u \,\alpha\, v$, the $\wedge$-columns of $x \wedge u$ and $y \wedge v$, being the $\wedge$-columns of u and v, differ in their position by a multiple of a. Likewise the $\wedge$-rows of $x \wedge u$ and $y \wedge v$, being the $\wedge$-rows of x and y, differ in their position by a multiple of a. It follows that $x \wedge u \,\alpha\, y \wedge v$ so that α is a $\wedge$–congruence also. Clearly α is neither Δ or ∇. □

A variation of de la Loubère's rule had been given previously by Claude Gaspar Bachet de Méziriac, the same individual who 1621 published the edition of Diophantus' *Arithmetica* of which Fermat owned a copy.

Bachet de Méziriac's Rule. *Place* 00 *directly above the middle position of an* $n{\times}n$ array. *In ascending (broken) diagonal fashion place. in order, the remaining* 01 *through* $0,n{-}1$. *Next, place* 10 two rows directly above $0,n{-}1$. *In ascending (broken) diagonal fashion place, in order,* 11 *through* $1,n{-}1$. *Next, place* 20 *two rows directly above* $1,n{-}1$. *Repeat the process until an entire* $n{\times}n$ *array is filled. The resulting array, is a magic square storing* 0 *through* $n^2{-}1$ *in base n.*

Theorem 16. *The magic squares of odd order given by Bachet de Méziriac's rule induce simple antilattices precisely when the order is prime.*

00	01	02	03	04	05	06
10	11	12	13	14	15	16
20	21	22	23	24	25	26
30	31	32	33	34	35	36
40	41	42	43	44	45	46
50	51	52	53	54	55	56
60	61	62	63	64	65	66

(Standard ∨-array for 0 – 48, base 7)

63	20	54	11	45	02	36
26	53	10	44	01	35	62
52	16	43	00	34	61	25
15	42	06	33	60	24	51
41	05	32	66	23	50	14
04	31	65	22	56	13	40
30	64	21	55	12	46	03

(de Méziriac array for 0 – 48. base 7)

Proof. Suppose first that p is prime with $p = 2n + 1$ and let θ be a congruence with $x\ \theta\ y$ where $x \neq y$. As with the previous theorem, things may be reduced to the case where $x = ni$ and $y = nj$ in the middle row of the ∨-array and the ascending diagonal of the ∧-array. If $F(X, Y) = n0 \vee (X \wedge Y)$, then $F(ni, nj) = n\,(i + j)/2$ in the same θ-class as $ni = n0 \vee ni$. (Here $(i + j)/2$ is calculated in $\mathbf{Z}_p$.) Since $n0, n1, \dots, n\,p{-}1$ generates the algebra, simplicity follows if we can show that from ni and nj one can F-generate the entire n^{th} ∨-row. This is equivalent to showing that from any two $i \neq j$ in $\mathbf{Z}_p$, all of $\mathbf{Z}_p$ is generated via the function $f(x, y) = (x + y)/2$. Let S be the set of all numbers in $\mathbf{Z}_p$ thus generated. If $0 \in S$, then S must be closed under addition and thus is a nontrivial subgroup of $\mathbf{Z}_p$. Since $\mathbf{Z}_p$ is simple as a group, this forces $S = \mathbf{Z}_p$. Indeed $f(0, (x + y)/2) = (x + y)/4$, $f(0, (x + y)/4) = (x + y)/8$, etc. Hence all $(x + y)/2^n$ lie in S. Since some power of 2 equals 1 in $\mathbf{Z}_p$, we get $x + y \in S$ so that S is as claimed. Otherwise, suppose $0 \notin S$. From $f(x + k, y + k) = f(x, y) + k$, the general case can be shifted to the 0-case, so that no matter what pair i, j is given, the f-generated set is all of $\mathbf{Z}_p$.

If p is composite, say $p = ab$ with $1 < a, b < p$, then define an equivalence α by $ij\ \alpha\ kl$ if both $i \equiv k \pmod{a}$ and $j \equiv l \pmod{a}$. The argument that α is a congruence is identical to that in the case involving de la Loubère's rule. □

6. *A gallery of select examples.*

The minimum consecutive prime pandiagonal magic square of order 6 displayed below stores the 36 consecutive primes from 67 through 251 and also induces a simple antilattice.

67	193	71	251	109	239
139	233	113	181	157	107
241	97	191	89	163	149
73	167	131	229	151	179
199	103	227	101	127	173
211	137	197	79	223	83

Next, consider the 7×7 pandiagonal magic square of prime numbers:

11	3851	9257	1747	6481	881	5399
6397	827	5501	71	3779	9221	1831
3881	9281	1759	6361	911	5417	17
839	5381	101	3797	9227	1861	6421
9311	1777	6367	941	5441	29	3761
5387	131	3821	9239	1741	6451	857
1801	6379	821	5471	47	3767	9341

Replacing each prime number in the array by the integer from 1 through 49 representing its position among the other entries under the natural ordering, we obtain the pandiagonal magic square displayed to the left below. Subtracting 1 from each entry, we obtain a pandiagonal magic square with entries from 0 to 48. Writing each number in base 7 notation, we obtain the square to the right. Interpreting these entries as point-pairs on the finite plane, $\mathbf{Z}_7 \times \mathbf{Z}_7$, the rows of this square are the lines of slope 4; the columns are the lines of slope 2; the ascending diagonals are the lines of slope 6; and the descending diagonals are the lines of slope 1. By Theorem 14, all of the isomorphic antilattices induced from these pandiagonal arrays must be simple.

1	27	46	16	42	12	31
39	9	35	5	24	43	20
28	47	17	36	13	32	2
10	29	6	25	44	21	40
48	18	37	14	33	3	22
30	7	26	45	15	41	11
19	38	8	34	4	23	49

00	35	63	21	56	14	42
53	11	46	04	32	60	25
36	64	22	50	15	43	01
12	40	05	33	61	26	54
65	23	51	16	44	02	30
41	06	34	62	20	55	13
24	52	10	45	03	31	66

Consider next the 9 × 9 ***nested*** magic square of Frénicle de Bessy. Its nested magic squares have magic sums of 369, 287, 205 and 123 forming a descending arithmetical sequence with the main square inducing a simple antilattice.

16	81	79	78	77	13	12	11	2
76	28	65	62	61	26	27	18	6
75	23	36	53	51	35	30	59	7
74	24	50	40	45	38	32	58	8
9	25	33	39	41	43	49	57	73
10	60	34	44	37	42	48	22	72
14	63	52	29	31	47	46	19	68
15	64	17	20	21	56	55	54	67
80	1	3	4	5	69	70	71	66

The antilattice induced from the following 13 × 13 nested magic square of primes is also simple. The status of all five antilattices induced from its five properly nested magic subarrays is unchecked.

1153	8923	1093	9127	1327	9277	1063	9133	9611	1693	991	8887	8353
9967	8161	3253	2857	6823	2143	4447	8821	8713	8317	3001	3271	907
1831	8167	4093	7561	3631	3457	7573	3907	7411	3967	7333	2707	9043
9907	7687	7237	6367	4597	4723	6577	4513	4831	6451	3637	3187	967
1723	7753	2347	4603	5527	4993	5641	6073	4951	6271	8527	3121	9151
9421	2293	6763	4663	4657	9007	1861	5443	6217	6211	4111	8581	1453
2011	2683	6871	6547	5227	1873	5437	9001	5647	4327	4003	8191	8863
9403	8761	3877	4783	5851	5431	9013	1867	5023	6091	6997	2113	1471
1531	2137	7177	6673	5923	5881	5233	4801	5347	4201	3697	8737	9343
9643	2251	7027	4423	6277	6151	4297	6361	6943	4507	3847	8623	1231
1783	2311	3451	3313	7243	7417	3301	6067	3463	6907	6781	8563	9091
9787	7603	7621	8017	4051	8731	6427	2053	2161	2557	7873	2713	1087
2421	1961	9781	1747	9547	1597	9811	1741	1213	9181	9883	1987	9721

This example and all others in this section are taken from Clifford Pickover's recent book [6]. With the exception of the second example, the simplicity in each case is checked using the tedious, but accessible method of Theorem 2. In doing so one can often strategize to some extent, depending on the precise layout of the square being considered.

As we have seen, not every magic square induces a simple antilattice. Nearly all of Benjamin Franklin's magic squares fail to induce simple antilattices. These squares are typically of the form $4n \times 4n$ and one can often spot a nontrivial congruence almost right away. Consider the Franklin square on the left below that has been partitioned in half. The left cell contains the contents of rows 1,2, 7 and 8 of the standard array to the right storing 1 – 64 in their natural order. The right cell of the Franklin square contains the contents of the four middle rows of the standard array.

52	61	4	13	20	29	36	45
14	32	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
7	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

The same simple pattern can be detected in Franklin's 16×16 "most magical of any magical square ever made." (See [6] page 151. This square is also easily found in an internet search.) One scans the right half of the square to see that 1 – 64 and 193 – 256 never occur in this region. These squares provide further instances of outer/inner partitions leading to congruences.

We conclude this section, and the paper, with several open problems.

Our discussion of the 4×4 case considered only ***normal*** magic squares storing 1 – 16. Are there criteria for inducing simple antilattices from abnormal 4×4 magic squares as in the 3×3 case?

Using a computer, in 1973 Richard Schroeppel showed there exist 275,305,224 essentially distinct 5×5 magic squares storing 1 - 25. While the growth of the normal magic square count as the dimensions increase is staggering, this need not prohibit relatively simple tests for [non-]simplicity from arising in the normal 5×5 case or even in higher dimensions. Thus, computer-assisted determinations of precisely what happens in the 5×5 and higher cases may be possible.

In Section 5 we recalled two classic magic square constructions and determined which cases of each induced simple antilattices. Many other constructions have been found. For each construction one can seek necessary and sufficient conditions for the constructed magic square to induce a simple antilattice.

One can also ask similar questions about when induced antilattices have distributive (or modular) congruence lattices. We saw that congruence lattices are distributive for all 3×3 magic squares.

Theorem 12 gives a sufficient, but not necessary, condition that affine antilattices over finite fields be simple. A necessary and sufficient condition would be nice.

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