An addition structure on incidence matrices of a BIB design

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Abstract. For a balanced incomplete block (BIB) design, the following problem is considered: Find s different incidence matrices of the BIB design such that (i) for $1 \leq t \leq s-1$, sums of any t different incidence matrices yield BIB designs and (ii) the sum of all s different incidence matrices becomes a matrix all of whose elements are one. In this paper, we show general results and present four series of such BIB designs with examples of other three BIB designs.

Keywords: balanced incomplete block (BIB) design; incidence matrix.

1. Introduction

A balanced incomplete block (BIB) design is a system with v points and b blocks each containing k different points, each point appearing in r different blocks and any two different points appearing in exactly λ blocks (see Colbourn and Dinitz [1]). This is denoted by BIBD(v, b, r, k, λ). Let $\boldsymbol{N} = (n_{ij})$ be the $v \times b$ incidence matrix of the BIB design, where $n_{ij} = 1$ or 0, for all i, j, according as the ith point occurs in the jth block or otherwise. Hence the incidence matrix $\boldsymbol{N}$ satisfies the following condition:

1. $n_{ij} = 0$ or 1 for all i $(= 1, 2, ..., v)$ and j $(= 1, 2, ..., b)$.

2. $\sum_{j=1}^{b} n_{ij} = r$ for all $i = 1, 2, ..., v$.
3. $\sum_{i=1}^{v} n_{ij} = k$ for all $j = 1, 2, ..., b$.
4. $\sum_{j=1}^{b} n_{ij} n_{i'j} = \lambda$ for all i, i' $(i \neq i') = 1, 2, ..., v$.

Now the present problem can be stated as follows. Does a BIBD(v, b, r, k, λ) have s different incidence matrices $\boldsymbol{N}_1, \boldsymbol{N}_2, ..., \boldsymbol{N}_s$ such that

(1) for $1 \leq t \leq s-1$, $\boldsymbol{N}_{i_1} + \boldsymbol{N}_{i_2} + \cdots + \boldsymbol{N}_{i_t}$ is the incidence matrix of a BIB design for any distinct $i_1, i_2, ..., i_t \in \{1, 2, ..., s\}$, and

(2) $\sum_{i=1}^{s} \boldsymbol{N}_i = \boldsymbol{J}$, where $\boldsymbol{J}$ is a $v \times b$ matrix whose elements are all 1?

Because of (2) we necessarily have $s = v/k = b/r$. If the condition (1) becomes free, then this includes a problem of decomposing the matrix $\boldsymbol{J}$ into a sum of different incidence matrices each of which yields a BIB design with the same parameters.

In this paper we provide general results, four series for some s and three examples of BIB designs for $s = 4, 5$.

2. Statements

When $s = 2$, any self-complementary BIB design (i.e., $v = 2k$) gives a complete answer to the present problem. In this case the conditions (1) and (2) in Section 1 coincide.

When $s = 3$, the conditions (1) and (2) are equivalent, because of a relation of the complementation of designs in the both conditions. When $s \geq 4$ the conditions (1) and (2) make sense independently.

Now we present three series of BIB designs that solve the present problem. Let a Galois field $GF(p^n) = \{0, 1, x, x^2, ..., x^{p^n-2}\}$, where x is a primitive element of $GF(p^n)$, p is a prime and n is a positive integer. Consider the following array:

$$
\begin{array}{llllll}
 & x & x^2 & x^3 & \cdots & x^{p^n-1} \\
 & x^2 & x^3 & x^4 & \cdots & x \\
 & \vdots & \vdots & \vdots & & \vdots \\
(2.1) & x^{\frac{p^n-1}{2}} & x^{\frac{p^n-1}{2}+1} & x^{\frac{p^n-1}{2}+2} & \cdots & x^{\frac{p^n-1}{2}-1} \\
 & \vdots & \vdots & \vdots & & \vdots \\
 & x^{p^n-1} & x & x^2 & \cdots & x^{p^n-2}
\end{array}
$$

Then we can obtain $p^n - 1$ initial blocks, by taking any p $(= k$, say) columns exclusively in the array (2.1) with $p^n - 1$ columns, for example,

$$
\begin{array}{l}
\{x, x^2, ..., x^p\}, \\
\{x^2, x^3, ..., x^{p+1}\}, \\
\quad\vdots \\
\{x^{p^n-1}, x, ..., x^{p^n+p-2}\},
\end{array}
$$

which, after development, can be shown to yield a BIBD($v = p^n, b = p^n(p^n - 1), r = p(p^n - 1), k = p, \lambda = p(p-1)$), because among differences arising from elements in the initial blocks each of non-zero elements of GF(p^n) occurs $p(p-1)$ times and other parameters are obvious. The iteration of this procedure of taking other p columns shows the existence of $p^{n-1} - 1$ different incidence matrices, $\boldsymbol{N}_1, \boldsymbol{N}_2, ..., \boldsymbol{N}_{s-1}$, where $s = p^{n-1}$. Furthermore, a design with the last incidence matrix $\boldsymbol{N}_s$ has $p^n - 1$ initial blocks consisting of elements in the remaining $p-1$ columns of the array and an additional element 0. Thus the procedure of taking disjoint choices of p columns (and lastly $p-1$ columns) in (2.1) to form each $\boldsymbol{N}_i$ shows that any sum of $\boldsymbol{N}_i$'s yields a BIB design. Hence it can be shown that the s constructed incidence matrices $\boldsymbol{N}_1, \boldsymbol{N}_2, ..., \boldsymbol{N}_s$ satisfy the conditions (1) and (2) as in Section 1.

Next, when p is an odd prime the procedure mentioned above can be improved in the sense of having less numbers of blocks. That is, by considering the first $(p^n - 1)/2$ rows only of (2.1) and taking the same procedure as before, we can get a BIBD($v = p^n$,

$b = p^n(p^n - 1)/2, r = p(p^n - 1)/2, k = p, \lambda = p(p-1)/2)$, whose $s\ (= p^{n-1})$ different incidence matrices are shown to satisfy the conditions (1) and (2).

Thus we have obtained the following two series that give an answer to the present problem.

Series 1: For a prime p and a positive integer $n \geq 2$, a BIBD($v = p^n, b = p^n(p^n - 1), r = p(p^n - 1), k = p, \lambda = p(p-1)$) solves the problem.

Series 2: For an odd prime p and a positive integer $n \geq 2$, a BIBD($v = p^n, b = p^n(p^n - 1)/2, r = p(p^n - 1)/2,\ k = p,\ \lambda = p(p-1)/2$) solves the problem.

For example, when $p = 3$, Series 2 gives a BIBD($v = 3^n$, $b = 3^n(3^n - 1)/2, r = 3(3^n - 1)/2, k = 3, \lambda = 3$), in which $s\ (= 3^{n-1})$ incidence matrices satisfying the conditions (1) and (2) are given by

$$\begin{aligned}
&\boldsymbol{N}_1 : \text{initial blocks } \{x, x^2, x^3\}, \{x^2, x^3, x^4\}, \ldots, \{x^{\frac{3^n-1}{2}}, x^{\frac{3^n-1}{2}+1}, x^{\frac{3^n-1}{2}+2}\},\\
&\boldsymbol{N}_2 : \text{initial blocks } \{x^4, x^5, x^6\}, \{x^5, x^6, x^7\}, \ldots, \{x^{\frac{3^n-1}{2}+3}, x^{\frac{3^n-1}{2}+4}, x^{\frac{3^n-1}{2}+5}\},\\
&\quad\vdots\\
&\boldsymbol{N}_s : \text{initial blocks } \{x^{3^n-2}, x^{3^n-1}, 0\}, \{x^{3^n-1}, x, 0\}, \ldots, \{x^{\frac{3^n-1}{2}-2}, x^{\frac{3^n-1}{2}-1}, 0\}.
\end{aligned}$$

In the array (2.1), similarly to the construction of Series 2, by taking any p^{n-1} columns exclusively, we can get a BIBD($v = p^n$, $b = p^n(p^n - 1)/2, r = p^{n-1}(p^n - 1)/2,\ k = p^{n-1}, \lambda = p^{n-1}(p^{n-1} - 1)/2$), whose p initial blocks are given, for example, by

$$\{x, x^2, \ldots, x^{p^{n-1}}\}, \{x^{p^{n-1}+1}, x^{p^{n-1}+2}, \ldots, x^{2p^{n-1}}\}, \ldots, \{x^{(p-1)p^{n-1}+1}, \ldots, x^{p^n-1}, 0\}.$$

By the iteration of this procedure, we can form p incidence matrices satisfying the conditions (1) and (2). Hence, as the third series, we can present the following.

Series 3: For an odd prime p and a positive integer $n \geq 2$, a BIBD($v = p^n, b = p^n(p^n - 1)/2$, $r = p^{n-1}(p^n - 1)/2, k = p^{n-1}$, $\lambda = p^{n-1}(p^{n-1} - 1)/2$) solves the problem.

A BIBD(v, b, r, k, λ) is said to be resolvable if its b blocks can be grouped into r resolution sets of v/k (= s, say) blocks each such that every treatment appears in each resolution set precisely once.

For the present problem, we will show a general result with $s = 3$ for a class of resolvable BIB designs.

Theorem 2.1. Any resolvable BIBD($v = 3k, b, r, k, \lambda$) has three incidence matrices satisfying the conditions (1) and (2) in Section 1.

Proof. Suppose that a resolvable BIBD($v = 3k, b = 3r, r, k, \lambda$) with the incidence matrix $\boldsymbol{N}_1$ (say) has r resolution sets of three blocks each, i.e., they are denoted by B_{ij}, $i = 1, 2, ..., r$ and $j = 1, 2, 3$, being the jth block in the ith resolution set $\mathcal{B}_i^{(1)} = \{B_{i1}, B_{i2}, B_{i3}\}$. Now the other two incidence matrices $\boldsymbol{N}_2$ and $\boldsymbol{N}_3$ can be formed as follows. The incidence matrix $\boldsymbol{N}_2$ corresponds to a resolvable BIBD($v = 3k, b, r, k, \lambda$) with r resolution sets of three blocks each, whose ith resolution set is given by $\mathcal{B}_i^{(2)} = \{B_{i2}, B_{i3}, B_{i1}\}$ for $i = 1, 2, ..., r$, i.e., a collection $\{\mathcal{B}_i^{(2)}, i = 1, 2, ..., r\}$. In this case it follows that $\boldsymbol{N}_1 + \boldsymbol{N}_2$ is the incidence matrix of a BIBD($v = 3k$, $b, 2r, 2k, 2\lambda(2k - 1)/(k - 1)$). In fact, it is clear that the complement of this design yields a resolvable BIBD($v = 3k, b, r, k, \lambda$) with the incidence matrix $\boldsymbol{N}_3$ $(= \boldsymbol{J} - (\boldsymbol{N}_1 + \boldsymbol{N}_2))$ which, in fact, has r resolution sets $\mathcal{B}_i^{(3)} = \{B_{i3}, B_{i1}, B_{i2}\}$. Since the numbering of blocks within the resolution sets discussed above is arbitrary, the sum of any two incidence matrices yields a BIB design. Hence the proof is complete. □

Many resolvable BIB designs are available in literature (cf. Colbourn and Dinitz [1]). We can find individual examples of resolvable BIB designs with $v = 3k$, for example, a resolvable BIBD($v = 9, b = 12, r = 4, k = 3, \lambda = 1$) whose incidence matrices are given, taking the procedure described in the proof of Theorem 2.1, by

$$N_1 = \left[\begin{array}{ccc|ccc|ccc|ccc} 1&0&0&1&0&0&1&0&0&1&0&0\\ 1&0&0&0&1&0&0&1&0&0&1&0\\ 1&0&0&0&0&1&0&0&1&0&0&1\\ 0&1&0&1&0&0&0&1&0&0&0&1\\ 0&1&0&0&1&0&0&0&1&1&0&0\\ 0&1&0&0&0&1&1&0&0&0&1&0\\ 0&0&1&1&0&0&0&0&1&0&1&0\\ 0&0&1&0&1&0&1&0&0&0&0&1\\ 0&0&1&0&0&1&0&1&0&1&0&0 \end{array}\right],$$

$$N_2 = \left[\begin{array}{ccc|ccc|ccc|ccc} 0&0&1&0&0&1&0&0&1&0&0&1\\ 0&0&1&1&0&0&1&0&0&1&0&0\\ 0&0&1&0&1&0&0&1&0&0&1&0\\ 1&0&0&0&0&1&1&0&0&0&1&0\\ 1&0&0&1&0&0&0&1&0&0&0&1\\ 1&0&0&0&1&0&0&0&1&1&0&0\\ 0&1&0&0&0&1&0&1&0&1&0&0\\ 0&1&0&1&0&0&0&0&1&0&1&0\\ 0&1&0&0&1&0&1&0&0&0&0&1 \end{array}\right],$$

$$N_3 = \left[\begin{array}{ccc|ccc|ccc|ccc} 0&1&0&0&1&0&0&1&0&0&1&0\\ 0&1&0&0&0&1&0&0&1&0&0&1\\ 0&1&0&1&0&0&1&0&0&1&0&0\\ 0&0&1&0&1&0&0&0&1&1&0&0\\ 0&0&1&0&0&1&1&0&0&0&1&0\\ 0&0&1&1&0&0&0&1&0&0&0&1\\ 1&0&0&0&1&0&1&0&0&0&0&1\\ 1&0&0&0&0&1&0&1&0&1&0&0\\ 1&0&0&1&0&0&0&0&1&0&1&0 \end{array}\right].$$

One more series of BIB designs that solve the present problem is given. By taking $(n-1)$-flats (as blocks) in an n-dimensional affine geometry AG(n,q), where q is a prime or a prime power (cf. Raghavarao [2; page 78]), there exists a resolvable BIBD($v = q^n, b = q(q^n - 1)/(q-1)$, $r = (q^n - 1)/(q-1)$, $k = q^{n-1}$, $\lambda = (q^{n-1} - 1)/(q-1)$), which can be shown to satisfy the conditon (2) in Section 1, by taking an idea used in the proof of Theorem 2.1. In particular, when $q = 3$, by Theorem 2.1, the resolvable BIB design can satisfy the conditions (1) and (2). That is,

Series 4: A resolvable BIBD($v = 3^n, b = 3(3^n - 1)/2$, $r = (3^n - 1)/2, k = 3^{n-1}$, $\lambda = (3^{n-1} - 1)/2$) solves the problem.

A recursive method of construction of a resolvable BIBD($v = 3k, b, r, k, \lambda$) is presented as Theorem 2.2 below. Through this method, we can get more series of BIB designs satisfying the conditions (1) and (2).

Theorem 2.2. The existence of a resolvable BIBD($v = 3k, b, r, k, \lambda$) implies the existence of a resolvable BIBD($9k, 3(4r-3\lambda), 4r - 3\lambda, 3k, r$).

Proof. By Theorem 2.1, we can let a resolvable BIBD($3k, b, r, k, \lambda$) have three incidence matrices $\boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3$ satisfying the conditions (1) and (2) as in Section 1. Then the following incidence matrix can yield the required design:

$$\left[\begin{array}{ccc|ccc} \boldsymbol{N}_{i_1} & \boldsymbol{N}_{i_2} & \boldsymbol{N}_{i_1} & \boldsymbol{J} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{N}_{i_1} & \boldsymbol{N}_{i_1} & \boldsymbol{N}_{i_2} & \boldsymbol{O} & \boldsymbol{J} & \boldsymbol{O} \\ \boldsymbol{N}_{i_2} & \boldsymbol{N}_{i_1} & \boldsymbol{N}_{i_1} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{J} \end{array}\right]$$

for any distinct $i_1, i_2 \in \{1, 2, 3\}$, where $\boldsymbol{J}$ is a $3k \times (r - 3\lambda)$ matrix whose elements are all 1. □

3. Examples

In this section, we provide three examples for $s = 4$ and 5 that do not belong to the series and theorem given in Section 2. One of them satisfies the conditions (1) and (2) in Section 1 completely. As to the notation PC(n), for example, in $\{1, 3\}, \{2, 6\}$PC(4), mod 8, PC(4) means a short cycle of order

4, i.e., a cyclic development of the initial block $\{2,6\}$ four times and reducing modulo 8 when necessary. The other block $\{1,3\}$ is developed modulo 8. In fact, this case means that

$\{\mathit{1},\mathit{3}\},\{2,4\},\{3,5\},\{4,6\},\{5,7\},\{6,0\},\{7,1\},\{0,2\},$
$\{\mathit{2},\mathit{6}\},\{3,7\},\{4,0\},\{5,1\}.$

Example 3.1. Consider a BIBD($v=8,b=28,r=7,k=2$, $\lambda=1$). Then there are four (i.e., $s=4$) incidence matrices of this BIB design as follows:

(i) $\boldsymbol{N}_1$ is generated by initial blocks $\{0,1\},\{0,3\},\{0,2\}$, $\{0,4\}$PC(4), mod 8.
(ii) $\boldsymbol{N}_2$ is generated by initial blocks $\{4,5\},\{2,5\},\{1,3\}$, $\{2,6\}$PC(4), mod 8.
(iii) $\boldsymbol{N}_3$ is generated by initial blocks $\{3,6\},\{1,7\},\{6,7\}$, $\{1,5\}$PC(4), mod 8.
(iv) $\boldsymbol{N}_4$ is generated by initial blocks $\{2,7\},\{4,6\},\{4,5\}$, $\{3,7\}$PC(4), mod 8.

Note that the order of initial blocks above is essential to solve the present problem. Now it follows that

(1) $\boldsymbol{N}_i$ is a BIBD($v=8,b=28,r=7,k=2$, $\lambda=1$) for all $i=1,2,3,4$;
(2) $\boldsymbol{N}_i+\boldsymbol{N}_j$ is a BIBD($v=8,b=28,r=14,k=4$, $\lambda=6$) for all i,j $(i\neq j)=1,2,3,4$;
(3) $\boldsymbol{N}_{i_1}+\boldsymbol{N}_{i_2}+\boldsymbol{N}_{i_3}$ is a BIBD($v=8,b=28,r=21,k=6$, $\lambda=15$) for all distinct $i_1,i_2,i_3\in\{1,2,3,4\}$;
(4) $\boldsymbol{N}_1+\boldsymbol{N}_2+\boldsymbol{N}_3+\boldsymbol{N}_4=\boldsymbol{J}$.

□

Example 3.2. Consider a BIBD($v=10,b=90,r=18,k=2$, $\lambda=2$). Then there are five (i.e., $s=5$) incidence matrices of this BIB design as follows:

(i) $\boldsymbol{N}_1$ is generated by initial blocks $\{1,2\},\{0,4\},\{0,1\},\{1,5\}$, $\{1,3\},\{0,7\}$, $\{4,6\},\{1,4\},\{0,5\}$, mod 10.

(ii) N_2 is generated by initial blocks $\{0,4\}, \{1,2\}, \{2,5\}, \{0,2\}, \{0,7\}, \{1,3\}, \{0, 1\}, \{0,6\}, \{2,7\}$, mod 10.

(iii) N_3 is generated by initial blocks $\{3,6\}, \{3,6\}, \{6,8\}, \{6,8\}, \{2,8\}, \{2,8\}, \{7, 8\}, \{7,8\}, \{4,9\}$, mod 10.

(iv) N_4 is generated by initial blocks $\{7,8\}, \{5,9\}, \{7,9\}, \{4,7\}, \{4,5\}, \{5,9\}, \{3, 5\}, \{2,9\}, \{1,6\}$, mod 10.

(v) N_5 is generated by initial blocks $\{5,9\}, \{7,8\}, \{3,4\}, \{3,9\}, \{6,9\}, \{4,6\}, \{2, 9\}, \{3,5\}, \{3,8\}$, mod 10.

Now it follows that

(1) N_i is a BIBD($v = 10, b = 90, r = 18, k = 2, \lambda = 2$) for all $i = 1,2,3,4,5$;

(2) N_1+N_2 or N_4+N_5 is a BIBD($v = 10, b = 90, r = 36, k = 4, \lambda = 12$);

(3) $N_1+N_2+N_3$ or $N_3+N_4+N_5$ is a BIBD($v = 10, b = 90, r = 54, k = 6, \lambda = 30$);

(4) $N_{i_1}+N_{i_2}+N_{i_3}+N_{i_4}$ is a BIBD($v = 10, b = 90, r = 72, k = 8, \lambda = 56$) for any distinct $i_1, i_2, i_3, i_4 \in \{1,2,3,4,5\}$;

(5) $N_1 + N_2 + N_3 + N_4 + N_5 = J$.

The above properties (2) and (3) are only valid for the present combinations. □

Example 3.3. Consider a BIBD($v = 12, b = 44, r = 11, k = 3, \lambda = 2$). Then there are four (i.e., $s = 4$) incidence matrices of this BIB design as follows:

(i) N_1 is generated by initial blocks $\{0,1,3\}, \{0,1,6\}, \{0,2,5\}, \{0,4, 8\}$PC(4), $\{0,4,8\}$PC(4), mod 12.

(ii) N_2 is generated by initial blocks $\{5,10,11\}, \{7,8,10\}, \{3,6,8\}, \{1,5,9\}$PC(4), $\{1,5,9\}$PC(4), mod 12.

(iii) N_3 is generated by initial blocks $\{6,8,9\}, \{2,3,9\}, \{1,4,11\}, \{2,6,10\}$PC(4), $\{2,6,10\}$PC(4), mod 12.

(iv) N_4 is generated by initial blocks $\{2,4,7\}, \{4,5,11\}, \{7,9,10\}, \{3,7,11\}$PC(4), $\{3,7,11\}$PC(4), mod 12.

Now it follows that

(1) $\boldsymbol{N}_i$ is a BIBD($v = 12, b = 44, r = 11, k = 3, \ \lambda = 2$) for all $i = 1, 2, 3, 4$;

(2) $\boldsymbol{N}_1 + \boldsymbol{N}_2$ or $\boldsymbol{N}_3 + \boldsymbol{N}_4$ is a BIBD($v = 12, b = 44, r = 22, k = 6, \ \lambda = 10$);

(3) $\boldsymbol{N}_{i_1} + \boldsymbol{N}_{i_2} + \boldsymbol{N}_{i_3}$ is a BIBD($v = 12, b = 44, r = 33, k = 9, \lambda = 24$) for any distinct $i_1, i_2, i_3 \in \{1, 2, 3, 4\}$;

(4) $\boldsymbol{N}_1 + \boldsymbol{N}_2 + \boldsymbol{N}_3 + \boldsymbol{N}_4 = \boldsymbol{J}$.

The above property (2) is only valid for the present combinations. □

The condition (1) given in Section 1 is strong. Note that Examples 3.2 and 3.3 do not satisfy all cases of the conditon (1). However, the present stratum structure on incidence matrices are quite interesting from a combinatorial point of view.

References

[1] Colbourn, C. J. and Dinitz, J. H. (ed.) (1996). *The CRC Handbook of Combinatorial Designs.* CRC Press, Boca Raton, USA.

[2] Raghavarao, D. (1988). *Constructions and Combinatorial Problems in Design of Experiments.* Dover.