A Family of Group Divisible Designs of Block Size Four and Three Groups With λ_1=2 and λ_2=1 Using MOLS

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Abstract

We give a construction for a new family of Group Divisible Designs $(6s+2, 3, 4; 2, 1)$ using Mutually Orthogonal Latin Squares for all positive integers s. Consequently, we have proved that the necessary conditions are sufficient for the existence of GDD's of block size four with three groups, λ_1=2 and λ_2=1.

1 Introduction

A group divisible design GDD(n, m, k; λ_1, λ_2) is a set V of mn elements, together with a partition of V into m sets of size n , called groups, and a collection of k-subsets of V called blocks, such that every pair of elements occurring in a group together appears in precisely λ_1 blocks, while every pair of elements not occurring together in a group appears in exactly λ_2 blocks. A restricted version of this original definition with $\lambda_1 = 0$ is more commonly used as the definition of GDD. Little is known about group divisible designs with block size four when the number of groups is less than four. In fact, Clatworthy's table only lists eleven designs with block size four, three groups, and replication numbers ≤ 10 [1]. For easy reference these designs are listed below, Column 1 refers to the name in [1]. The reader is referred to [2,3] for the terms used in this note.

Table 1:

name	v	r	k	b	m	n	λ_1	λ_2
S1	6	2	4	3	3	2	2	1
S2	6	4	4	6	3	2	4	2
S3	6	6	4	9	3	2	6	3
S4	6	8	4	12	3	2	8	4
S5	6	10	4	15	3	2	10	5
R96	6	8	4	12	3	2	4	5
R104	9	4	4	9	3	3	3	1
R105	9	8	4	18	3	3	6	2
R111	12	10	4	30	3	4	2	3
R117	15	8	4	30	3	5	1	2
R127	24	10	4	60	3	8	2	1

The present note is motivated by the two nontrivial designs, R111 and R127, in Table 1. The first design is very easy to construct using RBIBDs. In fact we have,

Theorem 1 *If a RBIBD(4m,4,λ) exists then a GDD(4m, r'=r-1, m=m, n=4; λ_1=λ-1, λ_2= λ) exists by taking out a parallel class and converting the blocks of the parallel class to groups.*

2 Main Result

This section deals with the structure of the second design R127. For GDDs with block size 4 and 3 groups, the replication number r and the number of blocks b are r = ((n-1)λ_1+ 2nλ_2)/3 and b = (n(n-1)λ_1+ $2n^2\lambda_2$)/4 = 3nr/4. Hence when λ_1 = 2 and λ_2 = 1, we obtain the necessary condition on n, n≡2(mod6). We now give a construction of a GDD(6s+2,3,4;2,1) using Mutually Orthogonal Latin Squares (MOLS) and BIBDs, for all positive integers s.
It is known that MOLS exist for all orders except n=2,6. Consider two MOLS of order 3s+1, L_1 and L_2. For each i∈{1, 2, 3}, partition the group G_i in two sets $G_{i,1}$, and $G_{i,2}$ of size 3s+1, label the rows of L_1 and L_2 by the elements of $G_{i,1}$ and the columns by the elements of $G_{i,2}$. Replace the entries of L_1 by the elements in $G_{i+1,1}$ to get L_1^* and the entries of L_2 by the elements in $G_{i+1,2}$ in any order to get L_2^*, where (i+1) is taken modulo 3. Now we construct the blocks. Each block consists of the four elements: the row label, column label, and the corresponding entries from

L_1^* and L_2^*. The elements of $G_{1,1}$ and $G_{1,2}$ occur together once as well as the elements of $G_{2,1}$ and $G_{2,2}$ are forced to come with each other only once because of the property of MOLS. This process is done three times: with G_1 and G_2, with G_2 and G_3, and lastly with G_3 and G_1. The index is increased by one between the elements of $G_{i,1}$ and $G_{i,2}$ each time a group G_i appears. They appear twice (once with each of the other two groups), so $\lambda_1 = 2$ between the elements of $G_{i,1}$ and $G_{i,2}$. Note that the pairs of distinct points from each $G_{i,1}$ and each $G_{i,2}$ have not occurred amongst themselves in any blocks yet. Therefore along with the blocks made from the MOLS we include the blocks of BIBD(3s+1,4,2) on each of $G_{i,1}$, $G_{i,2}$, i=1,2,3. This gives the required $\lambda_1 = 2$.
We now demonstrate the above procedure to construct a GDD(8,3,4;2,1):

As n=8, let $G_{1,1}$={1,2,3,4}, $G_{1,2}$={5,6,7,8}, $G_{2,1}$={9,10,11,12}, $G_{2,2}$= {13,14,15,16}, $G_{3,1}$={17,18,19,20}, $G_{3,2}$={21,22,23,24}. The squares L_1^* and L_2^* are

L_1^*	5	6	7	8
1	9	10	11	12
2	12	11	10	9
3	10	9	12	11
4	11	12	9	10

L_2^*	5	6	7	8
1	13	14	15	16
2	15	16	13	14
3	16	15	14	13
4	14	13	16	15

The blocks constructed using G_1 and G_2 are constructed as follows:

1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
5	6	7	8	5	6	7	8	5	6	7	8	5	6	7	8
9	10	11	12	12	11	10	9	10	9	12	11	11	12	9	10
13	14	15	16	15	16	13	14	16	15	14	13	14	13	16	15

To complete the rest of the blocks, repeat this process for G_2 and G_3 and then for G_3 and G_1. Also needed are the blocks of BIBD(4,4,2) on the point sets $G_{i,j}$ where i = 1,2,3 and j = 1,2.
In conclusion, we have the following results.

Theorem 2 *The necessary conditions are sufficient for the existence of GDD(n,3,4;2,1)*

References

1. W.H. Clatworthy, Tables of Two-Associate-Class Partially Balanced Designs, National Bureau of Standards (U.S.) Applied Mathematics Series No. 63, 1973.
2. C.J. Colbourn and J.H. Dinitz, editors, The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, Fl, 1996.
3. C.J. Colbourn and A. Rosa, Triple Systems, Oxford Science Publications, 1999.