Composite Path Algebras for Solving Path Problems in Graphs

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Abstract. Path problems in graphs can generally be formulated and solved by using an algebraic structure whose instances are called path algebras. Each type of path problem is characterized by a different instance of the structure. This paper proposes a method for combining already known path algebras into new ones. The obtained composite algebras can be applied to solve relatively complex path problems, such as explicit identification of optimal paths or multi-criteria optimization. The paper presents proofs showing that the proposed construction is correct. Also, prospective applications of composite algebras are illustrated by examples. Finally, the paper explores possibilities of making the construction more general.

Keywords: directed graphs, path problems, algebraic approach, path algebras, optimization, identification of paths, multi-criteria optimization.

1. Introduction

Path problems are a family of optimization and enumeration problems, which reduce to generation or comparison of paths in directed or undirected graphs. Some examples are: checking path existence, finding shortest or most reliable paths, listing all paths.

Each particular type of path problem can be treated separately, and solved by dedicated algorithms [3]. However, a more economic approach is to establish a general framework for the whole family of problems, and to use general algorithms. The latter can be achieved by introducing a suitable abstract algebraic structure.

Many variants of the algebraic approach to path problems have been proposed [1, 4, 8, 9, 10, 12]. Our favorite variant from [1] uses a structure whose instances are called "path algebras". The approach from [1] relies heavily on matrices and on analogies with ordinary linear algebra. Each type of path problem is formulated by using a different algebra. Solving a concrete problem reduces to computing with matrices over the corresponding algebra.

The aim of this paper is to propose a method for building more complex path algebras from simpler ones. The obtained composite algebras can be applied to formulate and solve relatively complex but still meaningful path problems.

The paper is organized as follows. Section 2 gives preliminaries about path algebras, graphs and the adopted algebraic approach. Section 3 presents our construction of a composite path algebra and proves that the construction is indeed correct and useful. Section 4 illustrates by examples how composite algebras can be applied to solve more complicated tasks, such as optimization with explicit identification of optimal paths or multi-criteria optimization. Section 5 explains why our construction of a composite path algebra cannot be made even more general. Finally, Section 6 gives a conclusion.

2. Path algebras and graphs

We start with the definition of our algebraic structure. A *path algebra* is a set P equipped with two binary operations, $\vee$ (*join*) and $\circ$ (*multiplication*), which have the following properties.

- The operation $\vee$ is idempotent, commutative and associative. Or in other words, for all $x, y, z \in P$: $x \vee x = x$, $x \vee y = y \vee x$, $(x \vee y) \vee z = x \vee (y \vee z)$.
- The operation $\circ$ is associative, left-distributive and right-distributive over $\vee$. Thus for all $x, y, z \in P$: $(x \circ y) \circ z = x \circ (y \circ z)$, $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$, $(y \vee z) \circ x = (y \circ x) \vee (z \circ x)$.
- There exist a zero element $\phi \in P$ and a unit element $\epsilon \in P$, such that for any $x \in P$: $\phi \vee x = x$, $\phi \circ x = \phi = x \circ \phi$, $\epsilon \circ x = x = x \circ \epsilon$.

When evaluating algebraic expressions over P, we will always assume that $\circ$ takes precedence over $\vee$, unless otherwise regulated by parentheses. Concrete instances of our algebraic structure can be found in [1, 7, 8, 9, 11], and some of them are also repeated in Table 1.

notation	P	$x \vee y$	$x \circ y$	ϕ	ϵ
P_2	$\mathbf{R} \cup \{\infty\}$	$\min\{x, y\}$	$x + y$	∞	0
P_3	$\mathbf{R} \cup \{-\infty\}$	$\max\{x, y\}$	$x + y$	$-\infty$	0
P_4	$\{r \in \mathbf{R} \mid 0 \leq r \leq 1\}$	$\max\{x, y\}$	$x \cdot y$	0	1
P_5	$\{r \in \mathbf{R} \mid r \geq 0\} \cup \{\infty\}$	$\max\{x, y\}$	$\min\{x, y\}$	0	∞
P_6	$\mathcal{P}(\Sigma^*)$	$x \cup y$	$\{w_x * w_y \mid w_x \in x, w_y \in y\}$	$\emptyset$	$\{\lambda\}$
P_8	$\mathcal{B}(\Sigma^*)$	$\mathrm{bas}(x \cup y)$	$\{w_x * w_y \mid w_x \in x, w_y \in y\}$	$\emptyset$	$\{\lambda\}$

Table 1: Some well known path algebras.

The first four examples in Table 1 are "extremal" algebras dealing with real numbers, infinity symbols and conventional arithmetic operations. The algebras P_6 and P_8 use "linguistic" concepts, namely Σ denotes a finite alphabet, and Σ^* is the set of all words (finite sequences of letters) over Σ. Consequently, $\mathcal{P}(\Sigma^*)$ is the set of all languages (sets of words) over Σ. The operation $\vee$ is based on the set union $\cup$, and $\circ$ on word concatenation $*$. The symbol λ stands for the empty word, and $\emptyset$ is the empty language. The operator bas( ) extracts from a language all words that do not have abbreviations in that language. If $\mathrm{bas}(L) = L$, then L is a basic language. $\mathcal{B}(\Sigma^*)$ denotes the set of all basic languages over Σ.

Let X be a square matrix whose entries belong to a path algebra P. Then we consider the following expression:

$$\widehat{X} = X \vee X^2 \vee X^3 \vee \ldots \vee X^k \vee \ldots .$$

Here X^2 stands for $X \circ X$, X^3 for $X \circ X \circ X$, etc. The matrix operations $\vee$ and $\circ$ are derived from the corresponding scalar operations similarly as in ordinary linear algebra, provided that $\vee$ is analogous to the conventional addition and $\circ$ to the conventional multiplication. In most situations the involved matrix X is *stable*, meaning that the above expression becomes saturated after joining enough powers of X. Thus the matrix $\widehat{X}$, called the *closure* of X, is well defined and computable in a finite number of algebraic operations. The number of powers necessary to reach the closure is then called the *stability index* of X.

In this paper we consider directed graphs and explore their *paths*, i.e. non-empty sequences of consecutive arcs. A circular path is called a *cycle*. A path is *elementary* if it does not traverse any node more than once. A graph G is said to be *labeled* with a path algebra P if each arc (i, j) of G is assigned a non-zero label $l(i, j) \in P$. An n-node labeled graph G is fully described by its $n \times n$ *adjacency matrix* A over P, whose (i, j)-th entry is equal to $l(i,j)$ if the arc (i,j) exists, or ϕ otherwise. The label $l(\mu)$ of a path μ in G is computed as the product of its arc labels. G is *absorptive* if for any elementary cycle γ in G holds that $l(\gamma) \vee \epsilon = \epsilon$. The importance of absorptive graphs is stressed by the following theorem proved in [1].

Theorem 1. *The adjacency matrix A of an absorptive n-node graph G is always stable with the stability index $\leq n$. The (i,j)-th entry of the closure $\widehat{A}$ is then equal to the join of labels of all elementary paths from node i to node j (or ϕ if there are no such paths).*

Now we are ready to explain our algebraic approach to path problems. For a certain problem posed in a graph G, we choose a suitable path algebra P and assign appropriate arc labels $l(i,j) \in P$. We compute the closure $\widehat{A}$ of the adjacency matrix A of G and read from it the solution to the original problem. Feasibility and correctness of the whole procedure is usually guaranteed by Theorem 1. Note that each type of problem requires a different algebra, although the overall problem structure remains the same. Thus $\widehat{A}$ can be computed by general algorithms [1, 2, 5, 6, 8, 9] operating over an *arbitrary* algebra.

Examples of using the described algebraic approach with the particular path algebras from Table 1 can be found in [1, 5, 6, 8, 9]. It can be seen that by computing in extremal algebras, such as P_2, P_3, P_4 or P_5, one can find certain optimal values in a graph, e.g. shortest distances among nodes, maximum reliabilities, etc. Still, one cannot directly identify paths where those optimal values are achieved. On the other hand, by computing in linguistic algebras, such as P_6 or P_8, one can list all paths or all elementary paths. Still, there is no possibility to filter or sort the listed paths according to some optimality criterion.

3. Composite path algebras

Let $\bar{P}$ be a path algebra whose binary operations are $\bar{\vee}$ and $\bar{\circ}$. Let the zero element of $\bar{P}$ be $\bar{\phi}$ and the unit element $\bar{\epsilon}$. Suppose that

- the operation $\bar{\vee}$ is a "choice operation", i.e. for all $\bar{x}, \bar{y} \in \bar{P}$:

$$\bar{x} \mathbin{\bar{\vee}} \bar{y} = \bar{x} \quad \text{or} \quad \bar{x} \mathbin{\bar{\vee}} \bar{y} = \bar{y};$$

- the operation $\bar{\circ}$ has the "cancellation property", i.e. for all $\bar{x}, \bar{y}, \bar{z} \in \bar{P}$:

$$(\bar{x} \mathbin{\bar{\circ}} \bar{z} = \bar{y} \mathbin{\bar{\circ}} \bar{z} \quad \text{or} \quad \bar{z} \mathbin{\bar{\circ}} \bar{x} = \bar{z} \mathbin{\bar{\circ}} \bar{y}) \;\Rightarrow\; (\bar{x} = \bar{y} \quad \text{or} \quad \bar{z} = \bar{\phi}).$$

Let $\tilde{P}$ be any other path algebra whose binary operations, zero and unit element are denoted with $\tilde{\vee}$, $\tilde{\circ}$, $\tilde{\phi}$, $\tilde{\epsilon}$, respectively. Then we can construct a set $P = \bar{P} \otimes \tilde{P}$ and two binary operations, $\vee$ and $\circ$, in the following way.

- $\bar{P} \otimes \tilde{P} = \left\{ (\bar{x}, \tilde{x}) \,|\, \bar{x} \in \bar{P} \backslash \{\bar{\phi}\}, \tilde{x} \in \tilde{P} \right\} \cup \left\{ (\bar{\phi}, \tilde{\phi}) \right\}$.
- For all $(\bar{x}, \tilde{x}), (\bar{y}, \tilde{y}) \in \bar{P} \otimes \tilde{P}$:

$$
\begin{aligned}
(\bar{x}, \tilde{x}) \vee (\bar{y}, \tilde{y}) &= \begin{cases} (\bar{x}, \tilde{x}) & \text{if } \bar{x} \bar{\vee} \bar{y} = \bar{x} \neq \bar{y} \\ (\bar{y}, \tilde{y}) & \text{if } \bar{x} \bar{\vee} \bar{y} = \bar{y} \neq \bar{x} \\ (\bar{x} \bar{\vee} \bar{y}, \tilde{x} \tilde{\vee} \tilde{y}) & \text{if } \bar{x} = \bar{y} \end{cases}, \\
(\bar{x}, \tilde{x}) \circ (\bar{y}, \tilde{y}) &= (\bar{x} \bar{\circ} \bar{y}, \tilde{x} \tilde{\circ} \tilde{y}).
\end{aligned}
$$

The correctness of our construction is guaranteed by the following theorem.

Theorem 2. *The proposed* $\vee$ *and* $\circ$ *are correctly defined binary operations on the set* $\bar{P} \otimes \tilde{P}$. *With these operations,* $\bar{P} \otimes \tilde{P}$ *constitutes a path algebra, whose zero element is* $\phi = (\bar{\phi}, \tilde{\phi})$ *and unit element is* $\epsilon = (\bar{\epsilon}, \tilde{\epsilon})$.

Proof. In order to prove that $\vee$ and $\circ$ are correctly defined, we must check that a result of $\vee$ or $\circ$ cannot take the form $(\bar{\phi}, \tilde{z})$ where $\tilde{z} \neq \tilde{\phi}$. This checking is done quite easily, by taking into account special properties of $\bar{\vee}$ and $\bar{\circ}$.

In order to prove that $\bar{P} \otimes \tilde{P}$ constitutes a path algebra, we have to check that all properties listed in the definition of a path algebra are satisfied. First, it is obvious that $\vee$ is idempotent and commutative, and that $\circ$ is associative. Also, it is very easy to show that $\phi = (\bar{\phi}, \tilde{\phi})$ fulfills the properties of the zero element, and that $\epsilon = (\bar{\epsilon}, \tilde{\epsilon})$ is the unit element.

Next, we verify that $\vee$ is associative. Indeed, for all $(\bar{x_1}, \tilde{x_1})$, $(\bar{x_2}, \tilde{x_2})$, $(\bar{x_3}, \tilde{x_3}) \in \bar{P} \otimes \tilde{P}$ it holds:

$$
\begin{aligned}
\left((\bar{x_1}, \tilde{x_1}) \vee (\bar{x_2}, \tilde{x_2}) \right) \vee (\bar{x_3}, \tilde{x_3}) &= \left(\bar{x_1} \bar{\vee} \bar{x_2}, \tilde{\bigvee_{\substack{1 \le i \le 2 \\ \bar{x_i} = \bar{x_1} \bar{\vee} \bar{x_2}}}} \tilde{x_i} \right) \vee (\bar{x_3}, \tilde{x_3}) \\
&= \left(\bar{x_1} \bar{\vee} \bar{x_2} \bar{\vee} \bar{x_3}, \tilde{\bigvee_{\substack{1 \le i \le 3 \\ \bar{x_i} = \bar{x_1} \bar{\vee} \bar{x_2} \bar{\vee} \bar{x_3}}}} \tilde{x_i} \right) \\
\cdots &= (\bar{x_1}, \tilde{x_1}) \vee \left((\bar{x_2}, \tilde{x_2}) \vee (\bar{x_3}, \tilde{x_3}) \right) .
\end{aligned}
$$

For better understanding of the above identity, let us note that the choice operation $\bar{\vee}$ defines a total ordering in $\bar{P}$, such that $\bar{\vee}$ can be interpreted as choosing the maximum according to that ordering. Thus the identity simply says that locating maxima in a list of values according to a total ordering yields the same results, no matter in which order those values are compared.

To verify that $\circ$ is left-distributive over $\vee$, we compute the following two expressions for any $(\bar{x}_1,\tilde{x}_1),(\bar{x}_2,\tilde{x}_2),(\bar{x}_3,\tilde{x}_3)\in \bar{P}\otimes\tilde{P}$:

$$
\begin{aligned}
(\bar{x}_1,\tilde{x}_1)\circ\big(\,(\bar{x}_2,\tilde{x}_2)\vee(\bar{x}_3,\tilde{x}_3)\,\big) &= (\bar{x}_1,\tilde{x}_1)\circ\left(\bar{x}_2\,\bar{\vee}\,\bar{x}_3,\ \tilde{\bigvee_{\substack{2\le i\le 3\\ \bar{x}_i=\bar{x}_2\bar{\vee}\bar{x}_3}}}\ \tilde{x}_i\right)\\
&= \left(\bar{x}_1\bar{\circ}\bar{x}_2\,\bar{\vee}\,\bar{x}_1\bar{\circ}\bar{x}_3\ ,\ \tilde{\bigvee_{\substack{2\le i\le 3\\ \bar{x}_i=\bar{x}_2\bar{\vee}\bar{x}_3}}}\ \tilde{x}_1\tilde{\circ}\tilde{x}_i\right),
\end{aligned}
$$

$$
\begin{aligned}
(\bar{x}_1,\tilde{x}_1)\circ(\bar{x}_2,\tilde{x}_2)\vee(\bar{x}_1,\tilde{x}_1)\circ(\bar{x}_3,\tilde{x}_3) &= (\bar{x}_1\bar{\circ}\bar{x}_2,\tilde{x}_1\tilde{\circ}\tilde{x}_2)\vee(\bar{x}_1\bar{\circ}\bar{x}_3,\tilde{x}_1\tilde{\circ}\tilde{x}_3)\\
&= \left(\bar{x}_1\bar{\circ}\bar{x}_2\,\bar{\vee}\,\bar{x}_1\bar{\circ}\bar{x}_3\ ,\ \tilde{\bigvee_{\substack{2\le i\le 3\\ \bar{x}_1\bar{\circ}\bar{x}_i=\bar{x}_1\bar{\circ}\bar{x}_2\,\bar{\vee}\,\bar{x}_1\bar{\circ}\bar{x}_3}}}\ \tilde{x}_1\tilde{\circ}\tilde{x}_i\right).
\end{aligned}
$$

We can assume that $\bar{x}_1\neq\bar{\phi}$, since otherwise both considered expressions would become zero and equal. With this assumption, and thanks to the cancellation property of $\bar{\circ}$, we have:

$$
\begin{aligned}
\bar{x}_1\bar{\circ}\bar{x}_i=\bar{x}_1\bar{\circ}\bar{x}_2\,\bar{\vee}\,\bar{x}_1\bar{\circ}\bar{x}_3 \quad &\Leftrightarrow\quad \bar{x}_1\bar{\circ}\bar{x}_i=\bar{x}_1\bar{\circ}(\bar{x}_2\,\bar{\vee}\,\bar{x}_3)\\
&\Leftrightarrow\quad \bar{x}_i=\bar{x}_2\,\bar{\vee}\,\bar{x}_3.
\end{aligned}
$$

Consequently, the two expressions above have to be equal. The right-distributivity of $\circ$ over $\vee$ is checked analogously. □

The newly constructed path algebra $P=\bar{P}\otimes\tilde{P}$ will be called a *composite* path algebra with respect to $\bar{P}$ and $\tilde{P}$.

Let us now consider an n-node graph G labeled with both $\bar{P}$ and $\tilde{P}$. Thus each arc (i,j) of G is simultaneously assigned a label $\bar{l}(i,j)\in\bar{P}$ and a label $\tilde{l}(i,j)\in\tilde{P}$. These two values can obviously be interpreted as a composite label $l(i,j)=(\bar{l}(i,j),\tilde{l}(i,j))$ from $\bar{P}\otimes\tilde{P}$. The labeling in $\bar{P}\otimes\tilde{P}$ is correct since both $\bar{l}(i,j)$ and $\tilde{l}(i,j)$ are nonzero, thus $l(i,j)$ indeed cannot be zero. For any path μ in G we can compute its labels $\bar{l}(\mu)\in\bar{P}$, $\tilde{l}(\mu)\in\tilde{P}$, and $l(\mu)=(\bar{l}(\mu),\tilde{l}(\mu))\in\bar{P}\otimes\tilde{P}$. The next theorem establishes the necessary and sufficient conditions for absorptivity in the sense of $\bar{P}\otimes\tilde{P}$.

Theorem 3. *The considered graph G is absorptive in the sense of $P = \bar{P} \otimes \tilde{P}$ if and only if for any elementary cycle γ in G it simultaneously holds:*

- $\bar{l}(\gamma) \bar{\vee} \bar{\epsilon} = \bar{\epsilon}$,
- $\bar{l}(\gamma) \neq \bar{\epsilon}$ *or* $\tilde{l}(\gamma) \tilde{\vee} \tilde{\epsilon} = \tilde{\epsilon}$.

Proof. We show necessity. Suppose that G is absorptive in the sense of $\bar{P} \otimes \tilde{P}$, and let γ be any elementary cycle in G. By combining the definition of absorptivity with the definition of $\vee$, we obtain:

$$(\bar{\epsilon}, \tilde{\epsilon}) = (\bar{l}(\gamma), \tilde{l}(\gamma)) \vee (\bar{\epsilon}, \tilde{\epsilon}) = \begin{cases} (\bar{l}(\gamma), \tilde{l}(\gamma)) & \text{if } \bar{l}(\gamma) \bar{\vee} \bar{\epsilon} = \bar{l}(\gamma) \neq \bar{\epsilon} \\ (\bar{\epsilon}, \tilde{\epsilon}) & \text{if } \bar{l}(\gamma) \bar{\vee} \bar{\epsilon} = \bar{\epsilon} \neq \bar{l}(\gamma) \\ (\bar{l}(\gamma) \bar{\vee} \bar{\epsilon}\,,\, \tilde{l}(\gamma) \tilde{\vee} \tilde{\epsilon}) & \text{if } \bar{l}(\gamma) = \bar{\epsilon} \end{cases}.$$

This obviously cannot be true for the first line within the curly brace, so it is necessary that the conditions from the second or third line are valid. Or in other words, it is necessary that

$$\bar{l}(\gamma) \bar{\vee} \bar{\epsilon} = \bar{\epsilon}.$$

By inserting the above expression into the second and third line within the curly brace, and by insisting again that the values of the associated expressions are equal to $(\bar{\epsilon}, \tilde{\epsilon})$, we obtain that

$$\bar{l}(\gamma) \neq \bar{\epsilon} \text{ or } \tilde{l}(\gamma) \tilde{\vee} \tilde{\epsilon} = \tilde{\epsilon}.$$

Sufficiency can be proved by reading the above necessity proof in opposite direction. □

The first condition in Theorem 3 is simply absorptivity in the sense of $\bar{P}$. The second condition is a slight generalization of absorptivity in the sense of $\tilde{P}$. From Theorem 3 we directly derive two important corollaries, which give sufficient conditions for checking absorptivity in the sense of $\bar{P} \otimes \tilde{P}$.

Corollary 1. *If our graph G is absorptive both in the sense of $\bar{P}$ and in the sense of $\tilde{P}$, then G is also absorptive in the sense of $\bar{P} \otimes \tilde{P}$.*

Corollary 2. *Let our graph G be absorptive in the sense of $\bar{P}$. Suppose that, additionally, the label $\bar{l}(\gamma)$ of any elementary cycle γ in G is $\neq \bar{\epsilon}$. Then G is also absorptive in the sense of $\bar{P} \otimes \tilde{P}$.*

In many situations we can guarantee either by Corollary 1 or by Corollary 2 that our graph G is absorptive in the sense of $\bar{P} \otimes \tilde{P}$. Then we can apply the following direct re-statement of Theorem 1, to assure that computing in $\bar{P} \otimes \tilde{P}$ is feasible, and that the obtained results are useful.

Corollary 3. *Let our graph G be absorptive in the sense of $\bar{P} \otimes \tilde{P}$. Then the $n \times n$ adjacency matrix A of G over $\bar{P} \otimes \tilde{P}$ is stable, with the stability index $\leq n$. Also, the (i,j)-th entry $\hat{a}_{ij}$ of the closure $\widehat{A}$ has the following value:*

$$\hat{a}_{ij} = \left(\bar{\bigvee_{\mu \in T(i,j)}} \bar{l}(\mu) , \tilde{\bigvee_{\bar{\mu} \in \bar{T}(i,j)}} \tilde{l}(\bar{\mu}) \right) .$$

Here $T(i,j)$ denotes the set of all elementary paths μ in G from node i to node j, while $\bar{T}(i,j)$ denotes the set of all elementary paths $\bar{\mu}$ in G from node i to node j such that $\bar{l}(\bar{\mu}) = \bar{\bigvee}_{\mu \in T(i,j)} \bar{l}(\mu)$.

4. Applications of composite algebras

Composite path algebras can be applied for identification of optimal paths in graphs. The role of the first algebra $\bar{P}$ should be taken by an extremal path algebra, such as P_2, P_3 or P_4. The second algebra $\tilde{P}$ should be a linguistic path algebra, such as P_6 or P_8. By computing in the corresponding $\bar{P} \otimes \tilde{P}$, it is possible to obtain certain optimal values in a graph, together with identifiers of paths where those optimal values are achieved.

Composite path algebras can also be applied for multi-criteria optimization. Two different extremal path algebras should be chosen as $\bar{P}$ and $\tilde{P}$. The first algebra then implements the primary criterion of optimality, and the second algebra reflects the secondary criterion. By computing in the corresponding $\bar{P} \otimes \tilde{P}$ we can obtain the optimal value according to the first criterion computed on a certain set of paths in a graph. Simultaneously, we also obtain the optimal value according to the second criterion, computed on the subset of paths that are optimal according to the first criterion.

Multi-criteria optimization can be accomplished even better by iterating our construction of composite path algebras. Namely, we can use an algebra of the form $\bar{P} \otimes (\bar{\bar{P}} \otimes \tilde{P})$, where $\bar{P}$ and $\bar{\bar{P}}$ are two extremal path algebras implementing two optimization criteria, and $\tilde{P}$ is a linguistic path algebra. By computing in $\bar{P} \otimes (\bar{\bar{P}} \otimes \tilde{P})$ we obtain the same pair of optimal values as before, but this time with identifiers of paths where those optimal values are simultaneously achieved.

Some concrete applications are listed in Table 2. All proposed compositions are based on simple path algebras from Table 1. Table 2 describes how a graph should be labeled with a particular algebra, and what results can be obtained by computing the closure of the corresponding adjacency matrix in that algebra. If the given assumptions about the graph are satisfied, then feasibility and correctness of the involved computations can be guaranteed by Corollaries 1-3.

Composite path algebra P	Assumptions about the graph G	Components of the (i,j)-th entry of the adjacency matrix A	Components of the (i,j)-th entry of the closure matrix $\widehat{A}$
$P_2 \otimes P_6$	any elementary cycle has positive length	- length of arc (i,j) - identifier of arc (i,j)	- shortest distance from node i to node j - identifiers of all shortest paths from node i to node j
$P_2 \otimes P_8$	any elementary cycle has non-negative length	- length of arc (i,j) - identifier of arc (i,j)	- shortest distance from node i to node j - identifiers of all shortest paths from node i to node j
$P_3 \otimes P_6$	there are no cycles	- length of arc (i,j) - identifier of arc (i,j)	- longest distance from node i to node j - identifiers of all longest (critical) paths from node i to node j
$P_4 \otimes P_6$	any elementary cycle contains an arc with reliability < 1	- reliability of arc (i,j) - identifier of arc (i,j)	- maximum reliability of a path from node i to node j - identifiers of all most reliable paths from node i to node j
$P_2 \otimes (P_4 \otimes P_8)$	any elementary cycle has non-negative length	- length of arc (i,j) - reliability of arc (i,j) - identifier of arc (i,j)	- shortest distance from node i to node j - maximum reliability of a shortest path from node i to node j - identifiers of all shortest paths from node i to node j achieving maximum reliability
$P_4 \otimes (P_2 \otimes P_8)$	any elementary cycle has non-negative length	- reliability of arc (i,j) - length of arc (i,j) - identifier of arc (i,j)	- maximum reliability of a path from node i to node j - minimal length of a most reliable path from node i to node j - identifiers of all most reliable paths from node i to node j achieving minimal length

Table 2: Some applications of composite path algebras.

To explain the fifth application from Table 2 in more detail, let us consider the graph G in Figure 1, whose arcs are assigned lengths (integers), reliabilities (real numbers) and identifiers (letters from an alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots\}$). Then G can be treated as labeled with $P_2 \otimes (P_4 \otimes P_8)$. The corresponding adjacency matrix A and its closure $\widehat{A}$ are the following:

$$A = \begin{bmatrix} (\infty,0,\emptyset) & (\infty,0,\emptyset) & (3,0.9,\{\mathtt{a}\}) & (\infty,0,\emptyset) & (\infty,0,\emptyset) \\ (-1,0.7,\{\mathtt{b}\}) & (\infty,0,\emptyset) & (\infty,0,\emptyset) & (8,0.1,\{\mathtt{c}\}) & (\infty,0,\emptyset) \\ (9,0.4,\{\mathtt{d}\}) & (1,0.3,\{\mathtt{e}\}) & (\infty,0,\emptyset) & (\infty,0,\emptyset) & (4,0.8,\{\mathtt{f}\}) \\ (\infty,0,\emptyset) & (-5,1.0,\{\mathtt{g}\}) & (5,0.5,\{\mathtt{h}\}) & (\infty,0,\emptyset) & (\infty,0,\emptyset) \\ (\infty,0,\emptyset) & (\infty,0,\emptyset) & (-3,0.2,\{\mathtt{i}\}) & (2,0.6,\{\mathtt{j}\}) & (\infty,0,\emptyset) \end{bmatrix},$$

$$\widehat{A} = \begin{bmatrix} \begin{pmatrix} 3, \\ 0.3024, \\ \{\mathtt{afjgb}\} \end{pmatrix} & \begin{pmatrix} 4, \\ 0.432, \\ \{\mathtt{afjg}\} \end{pmatrix} & \begin{pmatrix} 3, \\ 0.9, \\ \{\mathtt{a}\} \end{pmatrix} & \begin{pmatrix} 9, \\ 0.432, \\ \{\mathtt{afj}\} \end{pmatrix} & \begin{pmatrix} 7, \\ 0.72, \\ \{\mathtt{af}\} \end{pmatrix} \\ \begin{pmatrix} -1, \\ 0.7, \\ \{\mathtt{b}\} \end{pmatrix} & \begin{pmatrix} 3, \\ 0.3024, \\ \{\mathtt{bafjg}\} \end{pmatrix} & \begin{pmatrix} 2, \\ 0.63, \\ \{\mathtt{ba}\} \end{pmatrix} & \begin{pmatrix} 8, \\ 0.3024, \\ \{\mathtt{bafj}\} \end{pmatrix} & \begin{pmatrix} 6, \\ 0.504, \\ \{\mathtt{baf}\} \end{pmatrix} \\ \begin{pmatrix} 0, \\ 0.336, \\ \{\mathtt{fjgb}\} \end{pmatrix} & \begin{pmatrix} 1, \\ 0.48, \\ \{\mathtt{fjg}\} \end{pmatrix} & \begin{pmatrix} 1, \\ 0.16, \\ \{\mathtt{fi}\} \end{pmatrix} & \begin{pmatrix} 6, \\ 0.48, \\ \{\mathtt{fj}\} \end{pmatrix} & \begin{pmatrix} 4, \\ 0.8, \\ \{\mathtt{f}\} \end{pmatrix} \\ \begin{pmatrix} -6, \\ 0.7, \\ \{\mathtt{gb}\} \end{pmatrix} & \begin{pmatrix} -5, \\ 1.0, \\ \{\mathtt{g}\} \end{pmatrix} & \begin{pmatrix} -3, \\ 0.63, \\ \{\mathtt{gba}\} \end{pmatrix} & \begin{pmatrix} 3, \\ 0.3024, \\ \{\mathtt{gbafj}\} \end{pmatrix} & \begin{pmatrix} 1, \\ 0.504, \\ \{\mathtt{gbaf}\} \end{pmatrix} \\ \begin{pmatrix} -4, \\ 0.42, \\ \{\mathtt{jgb}\} \end{pmatrix} & \begin{pmatrix} -3, \\ 0.6, \\ \{\mathtt{jg}\} \end{pmatrix} & \begin{pmatrix} -3, \\ 0.2, \\ \{\mathtt{i}\} \end{pmatrix} & \begin{pmatrix} 2, \\ 0.6, \\ \{\mathtt{j}\} \end{pmatrix} & \begin{pmatrix} 1, \\ 0.16, \\ \{\mathtt{if}\} \end{pmatrix} \end{bmatrix}.$$

To justify feasibility and correctness of the above computation, we first show that G is absorptive in the sense of $P_2 \otimes (P_4 \otimes P_8)$. This can be done in several ways: for instance, we can apply Corollary 1 two times; or more directly, we can note that the length of any elementary cycle in G happens to be strictly positive, which means that the conditions of Corollary 2 are fulfilled. After showing that G is absorptive, we apply Corollary 3. Thus according to Corollary 3, the adjacency matrix A of G over $P_2 \otimes (P_4 \otimes P_8)$ is stable, and the (i,j)-th entry of the corresponding closure $\widehat{A}$ contains exactly those three values that are listed in Table 2.

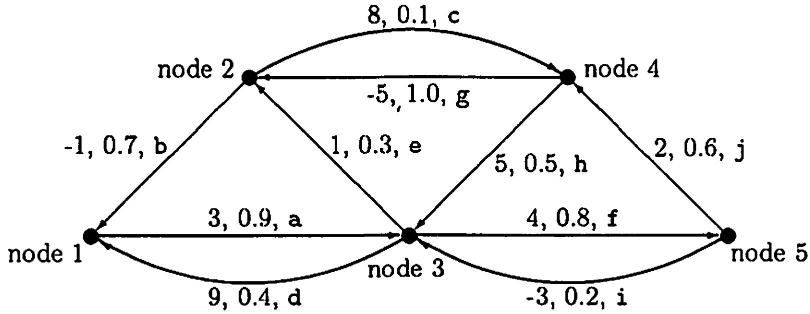

Figure 1: A graph G with arc lengths, arc reliabilities and arc identifiers.

5. Importance of the cancelation property

Let us now review once more our definition of a composite path algebra in order to see if it could be made more general. The definition of $\bar{P} \otimes \tilde{P}$ assumes that the operation $\bar{\vee}$ in $\bar{P}$ is a choice operation and that $\bar{\circ}$ in $\bar{P}$ has the cancellation property. There are no special requirements on $\tilde{P}$. The operation $\bar{\vee}$ being a choice operation is a natural assumption, which is essential in order that the whole construction of $\bar{P} \otimes \tilde{P}$ makes sense. However, the cancellation property of $\bar{\circ}$ is undesirable, since it decreases the number of examples where the construction can be applied. It would be fine if the cancellation property could be skipped or replaced by a weaker requirement.

Unfortunately, the cancellation property turns out to be virtually unavoidable. Namely, without this requirement it can happen very easily that $\bar{P} \otimes \tilde{P}$ is not a path algebra, as shown by the following consideration.

Let $\bar{P}$ be a path algebra whose operation $\bar{\vee}$ is a choice operation. Suppose that the operation $\bar{\circ}$ in $\bar{P}$ violates the cancellation property. Then we can choose $\bar{x}, \bar{y}, \bar{z} \in \bar{P}$ such that

$$\bar{x} \,\bar{\circ}\, \bar{z} = \bar{y} \,\bar{\circ}\, \bar{z} \quad \text{and} \quad \bar{z} \neq \bar{\phi} \quad \text{and} \quad \bar{x} \neq \bar{y}.$$

Moreover, in most cases $\bar{x}$ and $\bar{y}$ can be chosen as non-zero. Let $\tilde{P}$ be any path algebra with three elements $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ such that

$$\tilde{x} \,\tilde{\circ}\, \tilde{z} \neq \tilde{x} \,\tilde{\circ}\, \tilde{z} \;\tilde{\vee}\; \tilde{y} \,\tilde{\circ}\, \tilde{z} \neq \tilde{y} \,\tilde{\circ}\, \tilde{z}.$$

Then the pairs $(\bar{x}, \tilde{x})$, $(\bar{y}, \tilde{y})$, $(\bar{z}, \tilde{z})$ assembled from $\bar{x}$, $\bar{y}$, $\bar{z}$, $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ chosen above are elements of the set $\bar{P} \otimes \tilde{P}$. We show that in $\bar{P} \otimes \tilde{P}$ the operation $\circ$ is not right-distributive over $\vee$. Namely, the expressions

$$(\bar{x}, \tilde{x}) \circ (\bar{z}, \tilde{z}) \;\vee\; (\bar{y}, \tilde{y}) \circ (\bar{z}, \tilde{z})$$

and

$$(\,(\bar{x},\tilde{x}) \vee (\bar{y},\tilde{y})\,) \circ (\bar{z},\tilde{z})$$

are not equal. Indeed:

$$\begin{aligned}(\bar{x},\tilde{x})\circ(\bar{z},\tilde{z}) \vee (\bar{y},\tilde{y})\circ(\bar{z},\tilde{z}) &= (\bar{x}\,\bar{\circ}\,\bar{z},\tilde{x}\,\tilde{\circ}\,\tilde{z}) \vee (\bar{y}\,\bar{\circ}\,\bar{z},\tilde{y}\,\tilde{\circ}\,\tilde{z}) \\ &= (\bar{x}\,\bar{\circ}\,\bar{z}\,,\ \tilde{x}\,\tilde{\circ}\,\tilde{z}\ \tilde{\vee}\ \tilde{y}\,\tilde{\circ}\,\tilde{z}),\end{aligned}$$

while on the other hand:

$$\begin{aligned}(\,(\bar{x},\tilde{x}) \vee (\bar{y},\tilde{y})\,) \circ (\bar{z},\tilde{z}) &= (\,(\bar{x},\tilde{x}) \text{ or } (\bar{y},\tilde{y})\,) \circ (\bar{z},\tilde{z}) \\ &= (\bar{x}\,\bar{\circ}\,\bar{z},\tilde{x}\,\tilde{\circ}\,\tilde{z}) \text{ or } (\bar{x}\,\bar{\circ}\,\bar{z},\tilde{y}\,\tilde{\circ}\,\tilde{z}).\end{aligned}$$

We believe that the above procedure of assembling elements is general enough, so that it can be conducted for any practically relevant path algebra $\bar{P}$ whose $\bar{\vee}$ is a choice operation and $\bar{\circ}$ does not have the cancellation property. In this sense, we can regard the cancellation property as a necessary condition for correctness of working with composite path algebras.

In order to better understand the effects of not having the cancellation property, let us apply our general procedure to one particular path algebra. Take $\bar{P} = P_5$ from Table 1. Then $\bar{\vee}$ is max, thus a choice operation, while $\bar{\circ}$ is min. The three non-zero elements showing that $\bar{\circ}$ violates the cancellation property can be $\bar{x} = 2$, $\bar{y} = 4$, $\bar{z} = 1$. Namely,

$$\min\{2,1\} = \min\{4,1\} \ \text{ and } \ 1 \neq 0 \ \text{ and } \ 2 \neq 4.$$

As $\tilde{P}$ we can substitute P_8 with the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots\}$. The three elements from $\tilde{P}$ satisfying the required inequalities can be one-word languages $\tilde{x} = \{\mathtt{ab}\}$, $\tilde{y} = \{\mathtt{cd}\}$, $\tilde{z} = \{\mathtt{e}\}$. Indeed:

$$\{\mathtt{abe}\} \neq \{\mathtt{abe}, \mathtt{cde}\} \neq \{\mathtt{cde}\}.$$

The two expressions demonstrating violation of right-distributivity become:

$$\begin{aligned}(2,\{\mathtt{ab}\})\circ(1,\{\mathtt{e}\}) \vee (4,\{\mathtt{cd}\})\circ(1,\{\mathtt{e}\}) &= (1,\{\mathtt{abe}\}) \vee (1,\{\mathtt{cde}\}) \\ &= (1,\{\mathtt{abe},\mathtt{cde}\})\end{aligned}$$

and

$$\begin{aligned}(\,(2,\{\mathtt{ab}\}) \vee (4,\{\mathtt{cd}\})\,) \circ (1,\{\mathtt{e}\}) &= (4,\{\mathtt{cd}\})\circ(1,\{\mathtt{e}\}) \\ &= (1,\{\mathtt{cde}\}).\end{aligned}$$

The above two expressions can be interpreted as two ways of trying to solve a concrete path problem. Consider the graph G in Figure 2 whose arcs are given capacities and identifiers. Suppose that we want to identify all paths of maximum capacity from node 1 to node 5. Then:

- the first expression represents the correct solution, i.e. the maximum capacity is 1, and there exist two paths achieving that capacity: "abe" and "cde";
- the second expression is an incomplete solution identifying only one optimal path, which may be produced by some algorithms, for instance the Escalator method [5].

Thus the final consequence of not having the cancellation property is that some standard path-finding algorithms can fail.

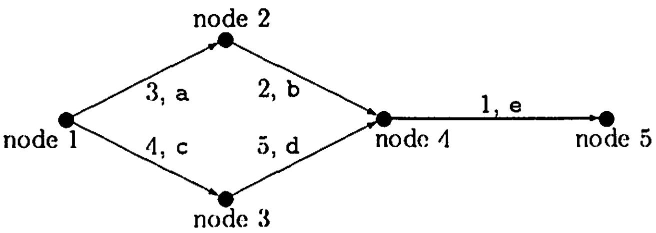

Figure 2: A graph G with given arc capacities and arc identifiers.

Our example with paths of maximum capacity can also be interpreted as follows: one optimal path is lost in the second expression since the maximum capacity problem does not satisfy Bellman's optimality principle. Namely, the path "abe" is an optimal path from node 1 to node 5 traversing node 4. However, the section of the same path from node 1 to node 4 is not an optimal path between node 1 and node 4! It is known [4] that Bellman's optimality principle is strongly related to the cancellation property.

6. Conclusion

The possibilities of simple path algebras are quite limited. To obtain more comprehensive solutions of path problems, the already known algebras should be combined. In this paper, we have proposed a suitable mechanism for combining simpler path algebras into composite algebras. We have proved that the proposed construction is correct and as general as it could be. By combining algebras from literature and by iterating the construction, it is possible to specify a large number of new algebras. Those composite algebras correspond to useful path problems, such as optimization with explicit identification of paths or multi-criteria optimization.

Composite path algebras are, first of all, attractive from the "aesthetic" point of view, since they considerably extend the applicability of our common algebraic framework. In addition to relatively simple problems, now it is possible to use the same algebraic formulation for more complex tasks.

Apart from its aesthetic appeal, the idea of putting a wider class of problems into the same algebraic framework brings some additional practical benefits. For instance, it becomes possible to solve that wider class of problems by already known and tested general algorithms.

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