A Counting of the minimal realizations of the posets of dimension two

Pierre Ille* Jean-Xavier Rampon†

Abstract

The posets of dimension 2 are those posets whose minimal realizations have two elements, that is, which may be obtained as the intersection of two of their linear extensions. Gallai's decomposition of a poset allows for a simple formula to count the number of the distinct minimal realizations of the posets of dimension 2. As an easy consequence, the characterization of M. El-Zahar and of N.W. Sauer of the posets of dimension 2, with an unique minimal realization, is obtained.

Mathematics Subject Classifications (1991):

Key words: Counting; Dimension; Directed Graphs; Gallai's Partition; Indecomposable; Posets; Realization.

1 Introduction

A *digraph* D is an ordered pair $(V(D), E(D))$, where $V(D)$ is a finite set, called the *set of the vertices* of D, and $E(D)$ is a family of ordered pairs of distinct vertices of D, called the *set of the edges* of D. With each subset X of $V(D)$, is associated the *subdigraph* $(X, E(D) \cap (X \times X))$ of D *induced* by X, denoted by $D[X]$. Two of the usual examples of digraphs, on a given set of vertices V, are the *empty digraph* and the *complete digraph*, where the sets of the edges are respectively the empty set and the set of all of the ordered pairs of distinct elements of V.

*Institut de Mathématiques de Luminy CNRS-UPR 9016, 163 avenue de Luminy - Case 907, 13288 Marseille Cedex 09, France; ille@iml.univ-mrs.fr

†FST-Université de Nantes, 2 rue de la Houssinière, BP 92208, 44322 Nantes Cedex 3, France; Jean-Xavier.Rampon@irin.univ-nantes.fr

With any digraph D, is associated its *complement* and its *dual*, denoted respectively by $\overline{D}$ and $D^\star$, and defined by $E(\overline{D}) = (V(D) \times V(D)) - (E(D) \cup \{(x,x) \; ; \; x \in V(D)\})$ and by $E(D^\star) = \{(x,y) : \; (y,x) \in E(D)\}$. Given two digraphs D_1 and D_2 defined on the same set of vertices V, the *union* $D_1 \cup D_2$ of D_1 and of D_2 is the digraph $(V, E(D_1) \cup E(D_2))$, and the *intersection* $D_1 \cap D_2$ of D_1 and of D_2 is the digraph $(V, E(D_1) \cap E(D_2))$. In another respect, D_2 is an *extension* of D_1 if $E(D_1) \subseteq E(D_2)$, which is briefly denoted by $D_1 \subseteq D_2$. The operations of union and of intersection extend to any family $\mathcal{F} = \{D_i \; ; \; i \in I\}$ of digraphs defined on the same set of vertices V by $\bigcup \mathcal{F} = (V, \bigcup_{i \in I} E(D_i))$ and $\bigcap \mathcal{F} = (V, \bigcap_{i \in I} E(D_i))$.

A digraph D is a *symmetric* digraph or a *graph* provided that for all $x, y \in V(D)$, if $(x,y) \in E(D)$, then $(y,x) \in E(D)$. In another vein, a digraph D is *transitive* provided that for all $x, y, z \in V(D)$, if $(x,y), (y,z) \in E(D)$, then $(x,z) \in E(D)$. Since the edges of a digraph are constituted by distinct vertices, for all of the vertices x and y of a transitive digraph D, if $(x,y) \in E(D)$, then $(y,x) \notin E(D)$. A (strict) *partially ordered set* or *poset* is then a transitive digraph.

Let P be a poset. The *comparability graph* of P is the graph $G(P)$ defined on $V(G(P)) = V(P)$ as follows. For all $x, y \in V(G(P))$, $(x,y) \in E(G(P))$ if $(x,y) \in E(P)$ or if $(y,x) \in E(P)$. A graph G is then said to be a *comparability graph* if there exists a poset P such that $G = G(P)$. Furhermore, P is a *linear order* if $G(P)$ is a complete digraph. A poset L is a *linear extension* of P if $P \subseteq L$ and if L is a linear order. The set of all of the linear extensions of P is denoted by $\mathcal{L}(P)$. A *realization* of P is any subset $\mathcal{F} \subseteq \mathcal{L}(P)$ such that $\bigcap \mathcal{F} = \mathcal{P}$. The minimum cardinality of a realization of P is called the *dimension* of P and is denoted by $\dim(P)$. A *minimal realization* of P is a realization of P of cardinality $\dim(P)$. The set of all of the minimal realizations of P is denoted by $\mathcal{M}(P)$. The set of all of the ordered minimal realizations of P is denoted by $\mathcal{O}(P)$, that is, given $L_1, \ldots, L_n \in \mathcal{L}(P)$, $(L_1, \ldots, L_n) \in \mathcal{O}(P)$ if $\{L_1, \ldots, L_n\} \in \mathcal{M}(P)$ and if for all $i \neq j \in \{1, \ldots, n\}$, $L_i \neq L_j$. It follows that $|\mathcal{O}(P)| = \dim(P)! \times |\mathcal{M}(P)|$.

Given a poset P, a subset X of $V(P)$ is *convex* provided that for all $x, y, z \in V(P)$, if $(x,y), (y,z) \in E(P)$ and if $x, z \in X$, then $y \in X$. Clearly, $V(P)$ is convex and any intersection of convex subsets is convex. Consequently, for each subset Y of $V(P)$, the *convex hull* of Y, denoted by $\mathrm{conv}(Y)$, may be defined as the intersection of all of the convex subsets containing Y.

Given a graph G, the equivalence relation $\mathcal{C}$ is defined on $V(G)$ in the following manner. For all $x, y \in V(G)$, $x\mathcal{C}y$ if there exists a sequence $(z_1, \cdots, z_k)$ of vertices of G such that $z_1 = x$, $z_k = y$ and $(z_i, z_{i+1}) \in E(G)$ for every $1 \leq i < k$. The equivalence classes of $\mathcal{C}$ are called the *connected components* of G. A graph is then said to be *connected* if it admits a single connected component. For convenience, the connected components of $\overline{G}$ are called the *coconnected components* of G. Analogously, a graph is *coconnected* if its complement is connected. By extension, given a poset P, the connected (resp. coconnected) components of P are the connected (resp. coconnected) component of $G(P)$ and P is then said to be connected (resp. coconnected) if $G(P)$ is.

2 Preliminaries

Given a digraph D, a subset X of $V(D)$ is an *interval* (or an *autonomous* subset or a *homogeneous* subset or a *module*) of D provided that for all $u, v \in X$ and $x \in V(D) - X$, $(u, x) \in E(D)$ if and only if $(v, x) \in E(D)$, and $(x, u) \in E(D)$ if and only if $(x, v) \in E(D)$. Clearly, $\emptyset$, $V(D)$ and $\{x\}$, where $x \in V(D)$, are intervals of D, called *trivial* intervals. A digraph is then said to be *indecomposable* (or *prime*) if all of its intervals are trivial. Otherwise, it is said to be *decomposable*. To begin, the properties of the intervals of a digraph are reviewed.

Proposition 1 *Let X and Y be two intervals of a digraph D.*

(i) *$X \cap Y$ is an interval of D.*

(ii) *If $X \cap Y \neq \emptyset$, then $X \cup Y$ is an interval of D.*

(iii) *If $X - Y \neq \emptyset$, then $Y - X$ is an interval of D.*

(iv) *For every $Z \subseteq X$, Z is an interval of $D[X]$ if and only if Z is an interval of D.*

(v) *If $X \cap Y = \emptyset$, then for all $x, x' \in X$ and $y, y' \in Y$, $(x, y) \in E(D)$ if and only if $(x', y') \in E(D')$.*

The last above mentionned property allows for the following definition of the quotient. Given a digraph D, a partition of $V(D)$, all of the elements of which are intervals of D, is called an *interval partition* of D. For such a partition $\mathcal{P}$, is defined the *quotient* $D/\mathcal{P} = (\mathcal{P}, E(D/\mathcal{P}))$ of D by $\mathcal{P}$ as follows. For all $X \neq Y \in \mathcal{P}$, $(X, Y) \in E(D/\mathcal{P})$ if for $x \in X$ and $y \in Y$, $(x, y) \in E(D)$. The inverse operation of the quotient is the lexicographical sum defined in the following manner. Given a digraph D, with any $x \in V(D)$, is associated a digraph D_x so that for all $x \neq y \in V(D)$, $V(D_x) \cap V(D_y) = \emptyset$. The *lexicographical sum* of the D_x's under D is the digraph $D(D_x\ ;\ x \in V(D))$ defined on $\bigcup_{x \in V(D)} V(D_x)$ in the following way. For all $u, v \in \bigcup_{x \in V(D)} V(D_x)$, $(u, v) \in E(D(D_x\ ;\ x \in V(D)))$ if either $x = y$ and $(u, v) \in E(D_x)$ or $x \neq y$ and $(x, y) \in E(D)$, where x and y are the vertices of D such that $u \in V(D_x)$ and $v \in V(D_y)$.

To continue, the following strengthening of the notion of interval is introduced. Given a digraph D, a subset X of $V(D)$ is a *strong* interval (T. Gallai [3]) of D provided that X is an interval of D and for any interval Y of D, if $X \cap Y \neq \emptyset$, then $X \subseteq Y$ or $Y \subseteq X$. The properties of the strong intervals of a digraph, which follow from Proposition 1, are recalled.

Proposition 2 *Let D be a digraph.*

(i) For every interval X of D, if Y is a strong interval of D such that $Y \subseteq X$, then Y is a strong interval of $D[X]$.

(ii) For every strong interval X of D, if Y is a strong interval of $D[X]$, then Y is a strong interval of D.

The family of the maximal elements, with respect to the inclusion, among the proper strong intervals of a digraph D constitutes an interval partition of D. This family is called *Gallai's partition* of D and is denoted by $\mathrm{Gal}(D)$. Gallai's decomposition theorem characterizes the quotient $D/\mathrm{Gal}(D)$. It is stated only for the graphs and for the posets in what follows.

Theorem 1 (T. Gallai [3]) *Given a graph G such that $|V(G)| \geq 2$, one of the following holds.*

(i) If G is not connected, then $\mathrm{Gal}(G)$ *is the family of the connected components of G and $G/\mathrm{Gal}(G)$ is an empty graph.*

(ii) If G is not coconnected, then $\mathrm{Gal}(G)$ *is the family of the coconnected components of G and $G/\mathrm{Gal}(G)$ is a complete graph.*

(iii) If G is connected and coconnected, then $|\mathrm{Gal}(G)| \geq 4$ *and $G/\mathrm{Gal}(G)$ is indecomposable.*

Theorem 2 (T. Gallai [3]) *Given a poset P such that $|V(P)| \geq 2$, one of the following holds.*

(i) If P is not connected, then $\mathrm{Gal}(P)$ *is the family of the connected components of P and $P/\mathrm{Gal}(P)$ is an empty poset.*

(ii) If P is not coconnected, then $\mathrm{Gal}(P)$ *is the family of the coconnected components of P and $P/\mathrm{Gal}(P)$ is a linear order.*

(iii) If P is connected and coconnected, then $|\mathrm{Gal}(P)| \geq 4$ *and $P/\mathrm{Gal}(P)$ is indecomposable.*

Let P be a poset. By definition, if X and Y are strong intervals of P, then either $X \subseteq Y$ or $Y \subseteq X$ or $X \cap Y = \emptyset$. Hence, it is obtained the *tree decomposition* T_P of P defined on the set of the nonempty strong intervals of P as follows. For all $X, Y \in V(T_P)$, $(X,Y) \in E(T_P)$ if $X \subset Y$. It is noticed that, by Proposition 2, for every $X \in V(T_P)$, $V(T_{P[X]}) = \{Y \in V(T_P) \ : \ Y \subseteq X\}$ and, thus, $T_{P[X]} = T_P[V(T_{P[X]})]$. By using Gallai's decomposition, the labelling function λ_P from $V(T_P)$ to $\{\mathrm{e}, \mathrm{i}, \mathrm{l}\}$ is defined in the following manner, where $X \in V(T_P)$.

- $\lambda_P(X) = \mathrm{e}$ if $|X| \geq 2$ and $P[X]/\mathrm{Gal}(P[X])$ is an empty poset,
- $\lambda_P(X) = \mathrm{l}$ if either $|X| = 1$ or $|X| \geq 2$ and $P[X]/\mathrm{Gal}(P[X])$ is a linear order,
- $\lambda_P(X) = \mathrm{i}$ if $|\mathrm{Gal}(P[X])| \geq 4$ and $P[X]/\mathrm{Gal}(P[X])$ is indecomposable.

Now, it is presented a concise proof of the next result which emphasizes the importance of the strong intervals among the intervals when both a poset and its comparability graph are considered.

Theorem 3 (T. Gallai [3]) *Given a poset P, P and $G(P)$ share the same strong intervals.*

Proof. To begin, it is recalled that for any interval X of $G(P)$, X is an interval of P if and only if $X = \mathrm{conv}(X)$. Furthermore, it is assumed that there exists an interval X of $G(P)$ which is not an interval of P. By the definition of the convex hull, for every $y \in V(P)$, $y \in \mathrm{conv}(X) - X$ if and only if there are $x, x' \in X$ such that $(x,y), (y,x') \in E(P)$. In particular, it is deduced that $\mathrm{conv}(X)$ is an interval of P. Since X is an interval of $G(P)$, for any $x \in X$ and $y \in \mathrm{conv}(X) - X$, $(x,y) \in E(P)$ or $(y,x) \in E(P)$. As $X \subset \mathrm{conv}(X)$, $P[\mathrm{conv}(X)]$ is not coconnected. Moreover, given $y \in \mathrm{conv}(X) - X$, it is obtained that $X = X^- \cup X^+$, where $X^- = \{x \in X \ : \ (x,y) \in E(P)\}$ and $X^+ = \{x \in X \ : \ (y,x) \in E(P)\}$. By transitivity, for $u \in X^-$ and $v \in X^+$, $(u,v) \in E(P)$ and, hence, $P[X]$ is not coconnected. It follows from Theorem 2 that $\mathrm{Gal}(P[X])$ is the set of the coconnected components of $P[X]$. Given $Y \in \mathrm{Gal}(P[X])$, by which precedes, for all $y \in Y$ and $z \in \mathrm{conv}(X) - X$, $(y,z) \in E(P)$ or $(z,y) \in E(P)$. Consequently, Y is a coconnected component of $P[\mathrm{conv}(X)]$ as well. It results that for every interval X of $G(P)$, if X is not an interval of P, then $P[X]$ and $P[\mathrm{conv}(X)]$ are not coconnected, and $\mathrm{Gal}(P[X]) \subset \mathrm{Gal}(P[\mathrm{conv}(X)])$.

Let X be a strong interval of $G(P)$. Since all of the intervals of P are intervals of $G(P)$, it is sufficient to prove that X is an interval of P. By contradiction, it is supposed that X is not. By which precedes, there exists $Y \in \mathrm{Gal}(P[\mathrm{conv}(X)]) - \mathrm{Gal}(P[X])$. The set of the elements z of $\mathrm{conv}(X) - Y$, such that there is $y \in Y$ with $(z,y) \in E(P)$, is denoted by Y^-. As $P[\mathrm{conv}(X)]/\mathrm{Gal}(P[\mathrm{conv}(X)])$ is a linear order, $Y^- \cup Y$ is an interval of $P[\mathrm{conv}(X)]$. By Proposition 1, $Y^- \cup Y$ is an interval of P and, hence, of $G(P)$. In another vein, it is deduced from the previous characterization of the elements of $\mathrm{conv}(X) - X$ that the minimum element U and the maximum element U' of $P[\mathrm{conv}(X)]/\mathrm{Gal}(P[\mathrm{conv}(X)])$ belong to $\mathrm{Gal}(P[X])$. The contradiction then follows from $U \subseteq X \cap (Y^- \cup Y)$, $U' \subseteq X - (Y^- \cup Y)$ and $Y \subseteq (Y^- \cup Y) - X$.

Conversely, let Y be a strong interval of P. It suffices to establish that for every interval X of $G(P)$, which is not an interval of P, if $X \cap Y \neq \emptyset$ and $X - Y \neq \emptyset$, then $Y \subseteq X$. As $X \subset \mathrm{conv}(X)$, $\mathrm{conv}(X) \cap Y \neq \emptyset$ and $\mathrm{conv}(X) - Y \neq \emptyset$. Since $\mathrm{conv}(X)$ is an interval of P, $Y \subseteq \mathrm{conv}(X)$. By Proposition 2, Y is a strong interval of $P[\mathrm{conv}(X)]$. As $X - Y \neq \emptyset$, $Y \neq \mathrm{conv}(X)$ and, by the definition of $\mathrm{Gal}(P[\mathrm{conv}(X)])$, there exists $Z \in \mathrm{Gal}(P[\mathrm{conv}(X)])$ such that $Y \subseteq Z$. Since $\mathrm{Gal}(P[X]) \subset \mathrm{Gal}(P[\mathrm{conv}(X)])$ and since $\emptyset \neq X \cap Y \subseteq X \cap Z$, $Z \in \mathrm{Gal}(P[X])$ and, hence, $Y \subseteq X$. □

The following results from the three above theorems.

Corollary 1 (T. Gallai [3]) *Given a poset P, P is indecomposable if and only if $G(P)$ is indecomposable.*

The following theorem is also needed.

Theorem 4 (T. Gallai [3]) *Given an indecomposable poset P, for every poset Q, $G(Q) = G(P)$ if and only if $Q = P$ or $Q = P^\star$.*

3 A counting of the minimal realizations

Lemma 1 (B. Dushnik and E.W. Miller [1]) *Given a poset P,* $\dim(P) \leq 2$ *if and only if $\overline{G(P)}$ is a comparability graph. Moreover, if* $\dim(P) \leq 2$*, then for every element L of a minimal realization of P, $L \cap \overline{G(P)}$ is a poset. Conversely, if* $\dim(P) \leq 2$ *and if Q is a poset such that $G(Q) = \overline{G(P)}$, then $\{P \cup Q, P \cup Q^\star\}$ is an element of $\mathcal{M}(P)$.*

An immediate consequence of Corollary 1, of Lemma 1 and of Theorem 4 follows.

Corollary 2 (T. Gallai [3]) *Let P be a poset of dimension 2. If P is indecomposable, then $|\mathcal{M}(P)| = 1$.*

Now, it is presented a succinct proof of the following, which is very close to results of M. El-Zahar and N.W. Sauer [2], and of P. Winkler [4].

Corollary 3 *Let P be a poset of dimension 2. For every element L of a minimal realization of P,* Gal(P) *is an interval partition of L.*

Proof. By Lemma 1, $\underline{L} = \underline{P} \cup Q$, where Q is a poset such that $G(Q) = \overline{G(P)}$. Since $G(P)$ and $\overline{G(P)}$ share the same intervals and, thus, the same strong intervals, it follows from Theorem 3 that $\text{Gal}(P) = \text{Gal}(Q)$. Now, given $X \in \text{Gal}(P)$, an element x of $V(P) - X$ is considered. Since X is an interval of P, either $\{x\} \times X \subseteq E(P)$ or $X \times \{x\} \subseteq E(P)$ or $((\{x\} \times X) \cup (X \times \{x\})) \cap E(P) = \emptyset$. In the last sentence, as $G(Q) = \overline{G(P)}$, $((\{x\} \times X) \cup (X \times \{x\})) \cap E(Q) \neq \emptyset$ and, since X is an interval of Q, either $\{x\} \times X \subseteq E(Q)$ or $X \times \{x\} \subseteq E(Q)$. As $P \subseteq L$ and $Q \subseteq L$, it is always obtained that either $\{x\} \times X \subseteq E(L)$ or $X \times \{x\} \subseteq E(L)$. Consequently, X is an interval of L. □

The main result follows.

Theorem 5 *If P is a poset of dimension at most 2, then*

$$\dim(P) \times |\mathcal{M}(P)| = 2^{|\lambda_P^{-1}(\{i\})|} \times \prod_{X \in \lambda_P^{-1}(\{e\})} |\text{Gal}(P[X])|! \,.$$

Proof. The result is obvious for $|V(P)| = 2$ and an induction on $|V(P)|$ follows. It is considered the function Θ from $\mathcal{O}(P/\text{Gal}(P)) \times \prod_{Y \in \text{Gal}(P)} \mathcal{O}(P[Y])$ to $\mathcal{O}(P)$, which associates $(\underline{L}(L_Y\ ; Y \in \text{Gal}(P)), \underline{M}(M_Y\ ; Y \in \text{Gal}(P)))$ with $[(\underline{L}, \underline{M}), (L_Y, M_Y)_{Y \in \text{Gal}(P)}]$. It is easy to verify that Θ is well defined and is injective. Moreover, it follows from Corollary 3 that Θ is surjective. Thus, it is obtained that

$$|\mathcal{O}(P/\text{Gal}(P))| \times \prod_{Y \in \text{Gal}(P)} |\mathcal{O}(P[Y])| = |\mathcal{O}(P)| \,.$$

As previously noticed, for each $Y \in \text{Gal}(P)$, $V(T_{P[Y]}) = \{Z \in V(T_P) : Z \subseteq Y\}$. It is deduced that $\lambda_P / V(T_{P[Y]}) = \lambda_{P[Y]}$. It is also noted that for any poset Q, if $\dim(Q) \leq 2$, then $|\mathcal{O}(Q)| = \dim(Q) \times |\mathcal{M}(Q)|$. By induction hypothesis, it follows that

$$\dim(P) \times |\mathcal{M}(P)| =$$

$$|\mathcal{O}(P/\text{Gal}(P))| \times \prod_{Y \in \text{Gal}(P)} [2^{|\lambda_{P[Y]}^{-1}(\{i\})|} \times \prod_{X \in \lambda_{P[Y]}^{-1}(\{e\})} |\text{Gal}(P[X])|!]$$

and, hence,

$$\dim(P) \times |\mathcal{M}(P)| =$$

$$|\mathcal{O}(P/\text{Gal}(P))| \times 2^{|\lambda_P^{-1}(\{i\}) - \{V(P)\}|} \times \prod_{X \in (\lambda_P^{-1}(\{e\}) - \{V(P)\})} |\text{Gal}(P[X])|! \,.$$

In order to conclude, the following is observed.

- If $P/\text{Gal}(P)$ is a linear order, then $V(P) \notin \lambda_P^{-1}(\{\text{i},\text{e}\})$ and $|\mathcal{O}(P/\text{Gal}(P))| = 1$.
- If $P/\text{Gal}(P)$ is an empty poset, then $V(P) \in (\lambda_P^{-1}(\{\text{e}\}) - \lambda_P^{-1}(\{\text{i}\}))$ and
$|\mathcal{O}(P/\text{Gal}(P))| = |\text{Gal}(P)|!$.
- If $P/\text{Gal}(P)$ is indecomposable, then $V(P) \in (\lambda_P^{-1}(\{\text{i}\}) - \lambda_P^{-1}(\{\text{e}\}))$ and, by Corollary 2, $|\mathcal{O}(P/\text{Gal}(P))| = 2$.

□

As an easy consequence, the following characterization of the posets of dimension 2, with an unique minimal realization, is established.

Corollary 4 (M. El-Zahar and N.W. Sauer [2]) *Given a poset P of dimension 2, $|\mathcal{M}(P)| = 1$ if and only if P is decomposed into $Q[Q_x\ ;\ x \in V(Q)]$ with one of the following.*

(i) Q is indecomposable and for every $x \in V(Q)$, Q_x is a linear order.

(ii) Q is a linear order and all of the Q_x's admit a single vertex except for one which is either an indecomposable poset on at least 4 vertices or an empty poset on 2 vertices.

(iii) Q is an empty poset on 2 vertices and the two Q_x's are linear orders.

Proof. To begin, it is observed that, by Theorem 2, for every $X \in V(T_P)$ and for every $Y \in \text{Gal}(P[X])$ if $\lambda_P(X) \in \{\text{e},\text{l}\}$ and if $|Y| \geq 2$, then $\lambda_P(X) \neq \lambda_P(Y)$. Now, it is easy to verify that if P fulfils one of the three above assertions, then $|\mathcal{M}(P)| = 1$. Conversely, if P is a poset of dimension 2 such that $|\mathcal{M}(P)| = 1$, then, by Theorem 5, it is obtained that

$$2 = 2^{|\lambda_P^{-1}(\{\text{i}\})|} \times \prod_{X \in \lambda_P^{-1}(\{\text{e}\})} |\text{Gal}(P[X])|!\ .$$

Firstly, it is supposed that $\lambda_P^{-1}(\{\text{i}\}) \neq \emptyset$. Thus, $\lambda_P^{-1}(\{\text{i}\})$ contains an unique element X and $\lambda_P^{-1}(\{\text{e}\}) = \emptyset$. It is then obtained that for every $Y \in V(T_P) - \{X\}$, $\lambda_P(Y) = \text{l}$. If $X = V(P)$, then, by the previous observation, for every $Y \in \text{Gal}(P)$, $P[Y]$ is a linear order. Equivalently, P satisfies assertion (i). On the other hand, if $X \neq V(P)$, then $P/\text{Gal}(P)$ is a linear order. By the former observation, for each $Y \in \text{Gal}(P)$, if $|Y| \geq 2$, then $\lambda_P(Y) \neq \text{l}$ and, hence, $Y = X$. It follows that P fulfils assertion (ii).

Secondly, it is assumed that $\lambda_P^{-1}(\{\text{i}\}) = \emptyset$. Thus, $\lambda_P^{-1}(\{\text{e}\})$ contains an unique element X, which is reduced to a pair, and for every $Y \in V(T_P) - \{X\}$, $\lambda_P(Y) = \text{l}$. If $X = V(P)$, then, by the previous observation, P satisfies assertion (iii). On the other hand, if $X \neq V(P)$, then $P/\text{Gal}(P)$ is a linear order. By the former observation, for each $Y \in \mathrm{Gal}(P)$, if $|Y| \geq 2$, then $\lambda_P(Y) \neq 1$ and, hence, $Y = X$. It follows that P fulfils assertion (ii). □

References

[1] B. Dushnik and E.W. Miller, Partially ordered sets. Amer. J. Math. 63, 600-610 (1941).

[2] M. El-Zahar and N.W. Sauer, Asymptotic enumeration of two-dimensional posets. Order 5, No.3, 239-244 (1988).

[3] T. Gallai, Transitiv orientierbare Graphen. Acta Math. Acad. Sci. Hungar. 18 (1967), 25-66.

[4] P. Winkler, Random orders of dimension 2. Order 7, No.4, 329-339 (1991).