On Skolem-Graceful and Cordial Graphs

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Abstract

In this paper, we give a necessary condition for an odd degree graph to be Skolem-graceful and we prove that if G is a (p, q) pseudograceful graph such that $p = q + 1$, then $G \cup S_m$ is Skolem-graceful for all $m \geq 1$ and finally we give some variations on the definition of cordial graphs.

1. Introduction

Throughout this paper, all graphs are finite, simple and undirected. We follow the basic notations and terminology of graph theory as in [2]. A vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces for each edge xy a label depending on the vertex labels f(x) and f(y). Rosa [13] called a function f a *β-valuation* of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that the induced function f^* from the edges of G to the set $\{1, 2, \ldots, q\}$ defined by $f^*(xy) =$

$|f(x) - f(y)|$, for $xy \in E(G)$ is an injection. The image of f $(= I_m(f))$ is called the corresponding set of vertex labels. Golomb [7] subsequently called such labelings *graceful* and this is now the popular term. Labelings of graphs are interesting because of their applications to graph decomposition (see [3] and [13]). In particular, β-valuations originated as means of attacking the conjecture of Ringel [12] that K_{2n+1} can be decomposed into 2n+1 subgraphs that are all isomorphic to a given tree with n edges.

A graph G is graceful if it admits a graceful labeling. Rosa [13] has shown that if G is a graceful Eulerian graph with q edges, then

$q \equiv 0$ or 3 (mod 4). We call this condition the graceful parity condition.

A number of authors have invented analogues of graceful graphs by modifying the permissible vertex labels. For instance, Lee and Shee [10] call a graph G with p vertices and q edges *Skolem-graceful* if there is an injection from the set of vertices of G to the set $\{1, 2, \ldots, p\}$ such that the induced function f^* from the set of edges of G to the set $\{1, 2, \ldots, q\}$ defined by $f^*(xy) = |f(x) - f(y)|$, for $xy \in E(G)$ is an injection. A necessary condition for a graph to be Skolem-graceful is that $p \geq q+1$. Although the disjoint union of trees can not be graceful, Lee and Wui [11] proved that the disjoint union of 2 or 3 stars is Skolem-graceful if and only if at least one star has even size. Lee, Quach and Wang [9] showed that the disjoint union of the path P_n and the star S_m is Skolem-graceful if and only if $n = 2$ and m is even or $n \geq 3$ and $m \geq 1$. Frucht [5] has shown that $P_m \cup P_n$ is Skolem-graceful when $m+n \geq 5$. Harary and Hsu [8] studied Skolem-graceful graphs under the name node-graceful. For more details about Skolem-graceful graphs see Gallian [6].

For a (p, q) graph G with the property $p = q + 1$, Frucht [4] invented another analogue of graceful labelings. He calls a (p, q) graph G with $p=q+1$ *pseudograceful* if there exists an injection from $V(G)$ to the set $\{0, 1, 2, \ldots, q-1, q+1\}$ such that the induced function f^* from $E(G)$ to the set $\{1, 2, \ldots, q\}$ defined by $f^*(xy) = |f(x) - f(y)|$, for $xy \in E(G)$ is an injection. Frucht [4] proved that P_n $(n \geq 3)$, combs, sparklers (i.e., graphs obtained by joining an end vertex of a path to the center of a star), $C_3 \cup P_n$ $(n \neq 3)$, and $C_4 \cup P_n$ $(n \neq 1)$ is pseudograceful while S_n $(n \geq 3)$ is not. Seoud and Youssef [14] extended the definition of pseudograceful to all graphs with $p \leq q+1$. They proved that K_n is pseudograceful if and only if $n = 1, 3$ or 4. $K_{m,n}$ is pseudograceful if and only if $m, n \geq 2$ or $\{m, n\} = \{1, 2\}$. They also proved that if G is pseudograceful, then $G \cup K_{m,n}$ is graceful for $m, n \geq 2$ and $G \cup K_{m,n}$ is pseudograceful for $m, n \geq 2$ and $(m, n) \neq (2, 2)$. Youssef [15] proved that if G is a pseudograceful and if H

is an α-labeled graph, then $G \cup H$ can be graceful or pseudograceful under some conditions on the α-labeling function of H. See Gallian [6] for more details about α-lebelings. Youssef [16] proved that C_n is pseudograceful if and only if n $\equiv 0$ or 3 (mod 4).

2. Skolem-graceful labelings

In [15], Youssef proved the following result :

Theorem 2.1

If G is Skolem-graceful graph, then $G+\overline{K_n}$ is graceful for all $n \geq 1$.

An odd degree graph is a graph whose the degree of every vertex is odd. We give a necessary condition for an odd degree graph to be Skolem-graceful.

Theorem 2.2.

If G is a (p, q) odd degree Skolem-graceful graph, then $p + q \equiv 0$ or 3 (mod 4).

Proof

Let G be an odd degree Skolem-graceful graph, then by Theorem 2.1, $G + K_1$ is graceful and since $G + K_1$ is Eulerian graph of size p+q , then by the graceful parity condition, $p+q \equiv 0$ or 3 (mod 4).

Corollary 2.3.

If the disjoint union of n stars, each of which is of odd graceful, then n $\equiv 0$ or 1 (mod 4).

Proof

Immediate.

The above corollary shows that the non Skolem-graceful of the disjoint union of stars of odd size depends only on the number of stars.

Notice that for (p, q) graphs with $p = q+1$ the notions of graceful labelings and Skolem-graceful labelings coincide. Youssef [15] showed that if G is a (p, q) graph with $p = q+1$, then $G + S_m$ is graceful for all $m \geq 1$. The following result deals with the Skolem-graceful labeling of the disjoint union of such a graph G and the star S_m.

Theorem 2.4.

If G is a (p, q) pseudograceful graph with $p = q+1$, then $G \cup S_m$ is Skolem-graceful for all $m \geq 1$.

Proof

Let $V(S_m) = \{v_0, v_1, \ldots, v_m\}$ where v_0 is the center vertex of S_m. Let f be a pseudograceful labeling of G. Define a labeling function

$$g : V(G \cup S_m) \longrightarrow \{1, 2, \ldots, p+m+1\}$$

as follows

$$g(v_0) = 2$$

$$g(v_i) = p+1+i, \quad 1 \leq i \leq m$$

$$g|_{V(G)} = p+1 - f|_{V(G)}.$$

Clearly g and g^* are injections and g is a Skolem-graceful labeling as desired.

Corollary 2.5 [9]

For $n \geq 2, m \geq 1, P_n \cup S_m$ is Skolem-graceful if and only if $n = 2$ and m is even or $n \geq 3$ and $m \geq 1$.

Proof

If $n = 2$ and m is odd, then $P_2 \cup S_m$ is not Skolem-graceful by Corollary 2.3. If $n \geq 3$ and $m \geq 1$, then $P_n \cup S_m$ is Skolem-graceful by Theorem 2.4. It remains to show that $P_2 \cup S_m$ is Skolem-graceful when m is even : Let v be the center vertex of S_m. Define

$$f : V(P_2 \cup S_m) \rightarrow \{1, 2, \ldots, m+3\}$$

such that

$$f(v) = 1$$

$$f(V(S_m)) = \{1, 2, \ldots \tfrac{m}{2}+1; \tfrac{m}{2}+3, \tfrac{m}{2}+4, \ldots, m+2\}$$

$$f(V(P_2)) = \{\tfrac{m}{2}+2, m+3\},$$

then f is easily seen to be a Skolem-graceful labeling of $P_2 \cup S_m$.

3. Variations of cordial labelings

Let G be a simple graph. A vertex labeling $f : V(G) \rightarrow \{0, 1\}$ is called a binary labeling. A binary labeling f of G induces an edge labeling $f^* : E(G) \rightarrow \{0, 1\}$, defined by $f^*(xy) = |f(x) - f(y)|$, for every edge $xy \in E(G)$. For $i \in \{0, 1\}$, let $n_i(f) = |\{v \in V(G) : f(v) = i\}|$ and $m_i(f) = |\{e \in E(G) : f^*(e) = i\}|$. A binary labeling of G is said to be cordial if the following conditions hold :

(a) $|n_1(f) - n_0(f)| \leq 1$

(b) $|m_1(f) - m_0(f)| \leq 1$

Note that interchanging the vertex labels 0 and 1 will produce a new cordial labeling of G. A graph G is cordial if it admits a cordial labeling. Condition (b) may be restated as :

(b_1) $0 \leq m_1(f) - m_0(f) \leq 1$ or

(b_2) $0 \leq m_0(f) - m_1(f) \leq 1$.

The notion of a cordial labeling was first introduced by Cahit [1] as weaker version of graceful labeling. See Gallian's servey [6] for more details about cordial labelings.

The following result gives a connection between cordial labelings and Skolem-graceful labelings.

Theorem 3.1.

Every Skolem-graceful graph is cordial.

Proof

Let f be a Skolem-graceful labeling of G and define

$$g : V(G) \longrightarrow \{0, 1\}$$

by

$$g(v) = f(v) \pmod 2 , \quad \text{for every } v \in V(G) .$$

Since $g(v) \equiv f(v) \pmod 2$ and f is a bijection onto an interval of positive integers, then $| n_1(g) - n_0(g) | \leq 1$. Similarly, since $g^*(xy) \equiv f^*(xy) \pmod 2$ and f^* is also a bijection onto an interval of positive integers, then $|m_1(g) - m_0(g)| \leq 1$. Thus g is a cordial labeling of G .

We make some variations on the definition of cordial labelings. Our aim to find a connection between graceful labelings and such following labelings :

We call a graph G is semi cordial if condition (b) above is satisfied, but not necessarily condition (a). Clearly that every cordial graph is semi cordial and every graceful graph is semi cordial.

We call a graph G is odd (resp. even) semi cordial if condition (b_1) (resp. (b_2)) above is satisfied.

We give a necessary condition for an Eulerian graph to be an odd (resp. even) semi cordial.

Theorem 3.2.

If G is an odd (resp. even) semi cordial Eulerian graph of q edges, then $q \equiv 0$ or $3 \pmod 4$ (resp. $q \equiv 0$ or $1 \pmod 4$).

Proof

Let f be an odd (resp. even) semi cordial labeling of a graph G of q edges, then

$$\sum_{e \in E(G)} f^*(e) = \left\lceil \frac{q}{2} \right\rceil \left(\text{resp. } \left\lfloor \frac{q}{2} \right\rfloor \right)$$

Now, since G is Eulerian graph, we get $\sum_{e \in E(G)} f^*(e)$ is even and

$q \equiv 0$ or $3 \pmod 4$ (resp. $q \equiv 0$ or $1 \pmod 4$).

The proof of the following result parallels that for Theorem 3.1 and we omit it.

Theorem 3.3.

If G is a graceful graph of odd (resp. even) size, then G is odd (resp. even) semi cordial.

Determining whether or not a specific graph is semi cordial is more easier than checking that a graph is graceful or not. Theorem 3.3 is a good tool for showing that some specific graphs are not graceful.

Examples

(i) Since K_7 (resp. K_8) is not odd (resp. even) semi cordial, then K_7 (resp. K_8) is not graceful.

(ii) $2\,K_6$ is not even semi cordial and hence it is not graceful.

The following result gives a complete characterization of the odd (resp. even) semi cordialness of the complete graph K_n.

Theorem 3.4.

Let $n \geq 3$, then

(i) K_n is odd semi cordial if and only if $n = p^2$ or $n = p^2 + 2$ for some positive integer p.

(ii) K_n is even semi cordial if and only if $n = p^2$ or $n = p^2 - 2$ for some positive integer p.

Proof

(i) If $n = p^2$, we label $\binom{p}{2}$ vertices with the label 0 and we label $\binom{p+1}{2}$ vertices with the label 1. In this case, number of edges labeled 1 are $\binom{p}{2}\binom{p+1}{2}$ which is equal to half of the number of edges of K_n.

If $n = p^2 + 2$, we label $\binom{p}{2}+1$ vertices with the label 0 and $\binom{p+2}{2}+1$ vertices with the label 1. Clearly, number of edges labeled one are $\left(\binom{p}{2}+1\right)\left(\binom{p+1}{2}+1\right)=\left\lceil\frac{|E(K_n)|}{2}\right\rceil=\frac{p^2(p^2+3)}{4}+1$.

Conversely, let K_n be odd semi cordial, we have two cases to consider :

Case 1 : $n \equiv 0$ or $1 \pmod 4$)

In this case number of edges q of K_n is even. Let k_0 and k_1 be number of vertices labeled 0 and 1 respectively, then $k_0 + k_1 = n$ and $k_0 k_1 = \frac{q}{2}$.

Solving these two equations in k_0, we get

$$k_0 = \frac{n \pm \sqrt{n}}{2},$$

since k_0 is a positive integer, then n is a perfect square.

Case 2 : $n \equiv 2$ or $3 \pmod 4$)

In this case, $q = |E(K_n)|$ is odd. Let k_0 and k_1 as in Case 1, then $k_0 + k_1 = n$ and $k_0 k_1 = \left\lceil\frac{q}{2}\right\rceil = \frac{(n+1)(n-2)}{4}+1$.

Solving these two equations in k_0, we get

$$k_0 = \frac{n \pm \sqrt{n-2}}{2},$$

since k_0 is a positive integer, then $n-2$ is a perfect square. This proofs (i).

(ii) Similar as in (i).

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