On the Cordiality of Corona Graphs

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Abstract: Let G_1, G_2 be simple graphs with n_1, n_2 vertices and m_1, m_2 edges respectively. The *Corona graph* $G_1 \circ G_2$ of G_1 with G_2 is obtained by taking one copy of G_1, n_1 copies of G_2 and then joining each vertex of G_1 to all the vertices of a copy of G_2.

For a graph G, by the *index of cordiality* $i(G)$ we mean $\min\{\mid e_f(0) - e_f(1) \mid\}$, where the minimum is taken over all the binary labelings of G with $\mid v_f(0) - v_f(1) \mid \leq 1$. In this paper, we investigate the cordiality of $G \circ \overline{K_t}$, $K_n \circ \overline{K_t}$ and $G \circ C_t$ where G is a graph with the index of cordiality k.

Introduction

Throughout this paper, all graphs are finite, simple and undirected. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G. A mapping $f : V(G) \longrightarrow \{0, 1\}$ is called a binary vertex labeling of G and $f(v)$ is called the label of the vertex v under f. For an edge $e = uv$, the induced edge labeling $\overline{f} : E(G) \longrightarrow \{0, 1\}$ is given by $\overline{f}(e) = \mid f(u) - f(v) \mid$. Let $v_f(0), v_f(1)$ be the number of vertices in G having labels 0 and 1 respectively under f. Let $e_f(0), e_f(1)$ be the number of edges having labels 0 and 1 respectively under $\overline{f}$.

Definition: For a graph G, by the *index of cordiality* $i(G)$ we mean $\min\{\mid e_f(0) - e_f(1) \mid\}$ where the minimum is taken over all the binary labelings of G with $\mid v_f(0) - v_f(1) \mid \leq 1$.

A graph G is called a *cordial graph* if $i(G) \leq 1$ and a binary labeling f of G is called a *cordial labeling* if $\mid v_f(0) - v_f(1) \mid \leq 1$ and $\mid e_f(0) - e_f(1) \mid \leq 1$.

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One can easily see that $i(K_{2t}) = t = i(K_{2t+1})$ and $i(C_{4t}) = 0, i(C_{4t+1}) = 1 = i(C_{4t+3}), i(C_{4t+2}) = 2$.

Cordial graphs were first introduced by Cahit as a weaker version of both graceful and harmonious graphs [7]. In the same paper, Cahit proved the following:

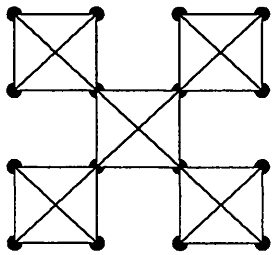
Theorem 1: If G is an Eulerian graph with e edges, where $e \equiv 2 (\bmod 4)$, then G has no cordial labeling.

In [1] several families of wheel related graphs were shown to be cordial. A fan F_m is a cycle C_m with $m-3$ concurrent chords. Shee and Ho [9] proved that the one point union $F_m^{(n)}$ of n copies of the fan (shell) F_m are cordial for all $m \geq 3, n \geq 1$. In [2], this result was generalized for one point union of fans of arbitrary sizes, that is, multiple shells. A t-uniform homeomorph $P_t(G)$ of a graph G is obtained by replacing each edge of the graph G by a path of length t. In [3], the cordiality of $P_t(G)$ was investigated while in [4], a necessary and sufficient conditions for the cordiality of the t-uniform homeomorphs of complete graphs were found. A ply is a vertex disjoint union of many paths having common end points. An elongated ply is a graph each of whose blocks is a ply and whose block-cut-vertex tree is a path. In [5], [6] it was proved that plys as well as elongated plys are cordial if and only if they are not ruled out by Theorem 1.

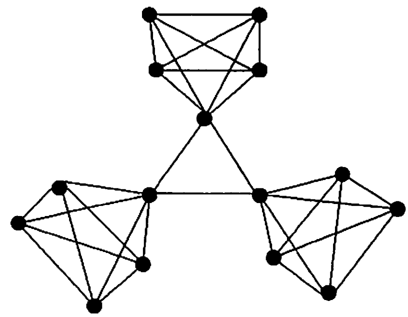
Definition: Let G_1, G_2 be two graphs with $|V(G_i)| = n_i, |E(G_i)| = m_i, i = 1, 2$. The *Corona* graph $G_1 \circ G_2$ of G_1 with G_2 is the graph obtained by taking one copy of G_1 and n_1 copies of G_2 and then joining the ith vertex of G_1 to all the vertices in the ith copy of G_2.

This term was introduced in [8]. Clearly this graph operation is not commutative and $|V(G_1 \circ G_2)| = n_1(1+n_2), |E(G_1 \circ G_2)| = m_1 + n_1(m_2 + n_2)$. In this paper, we investigate the cordiality of $G \circ \overline{K_t}$ and $G \odot C_t$. The following figure gives $K_4 \circ K_3$ and $K_3 \circ K_4$.

Let G be a graph with $|V(G)| = n$ and $i(G) = k$. Let $V(\overline{K_t}) =$



$K_4 \circ K_3$



$K_3 \circ K_4$

$\{v_1, \cdots, v_t\}$. The edge set $E(\overline{K_t})$ is empty. Let g be a binary labeling of G such that $|\ v_g(0) - v_g(1)\ | \leq 1$ and $|\ e_g(0) - e_g(1)\ | = k$. If necessary by interchanging the labels 0 and 1 in the labeling g, we can assume that $v_g(0) \geq v_g(1)$. Whenever possible, we extend this labeling g of G to a cordial labeling f of $G \circ \overline{K_t}$ and of $G \circ C_t$.

Method of Extension: There are $v_g(0)$ vertices of G which are assigned label 0 by G. In $G \circ \overline{K_t}$ there are $t.v_g(0)$ new vertices adjacent to these vertices. While constructing f, we will choose x number of these new vertices and assign them label 0. Similarly, there are $v_g(1)$ vertices of G which are assigned label 1 by G. In $G \circ \overline{K_t}$ there are $tv_g(1)$ new vertices adjacent to these vertices. While constructing f, we will choose y number of these new vertices and assign them label 0. All the remaining new vertices will be assigned the value 1 by the labeling f. The values x and y will be chosen suitably in each case so as to make the labeling cordial. It follows that for the labeling f, so constructed,

$$\begin{aligned} v_f(0) &= v_g(0) + x + y, \\ v_f(1) &= v_g(1) + (tv_g(0) - x) + (tv_g(1) - y), \\ e_f(0) &= e_g(0) + x + (tv_g(1) - y), \\ e_f(1) &= e_g(1) + (tv_g(0) - x) + y. \end{aligned} \qquad (**)$$

Cordiality of $G \circ \overline{K_t}$.

We split the analysis into two parts, according to the parity of t. Both the proofs are similar.

Theorem 2: Let g be a binary labeling of a graph G such that $i(G) = | e_g(0) - e_g(1) |$, then g can be extended (by the method of extension) to a cordial labeling of $G \circ \overline{K_{2m}}$ if and only if n is not even with $k \equiv 2(\bmod 4)$.

Proof: Here $t = 2m$. Let $k = 4p + r, 0 \leq r \leq 3$ and $n = 2q + s, s = 0, 1$.

Case 1: n is even, that is $n = 2q$ where q is even. This means $v_g(0) = q = v_g(1)$. Further $|V(G \circ \overline{K_{2m}})| = n + 2nm = 2q + 4mq$ and $|E(G \circ \overline{K_{2m}})| = |E(G)| + 2nm = |E(G)| + 4mq$. From the above it is clear that $|V(G \circ \overline{K_{2m}})|$is even, whereas $|E(G \circ \overline{K_{2m}})|$ is even or odd according as $|E(G)|$is even or odd. Thus $|E(G \circ \overline{K_{2m}})|$ is even if and only if k is even. Hence if f is to be a cordial labeling of $G \circ \overline{K_{2m}}$, then $v_f(0)$ must be equal to $v_f(1)$ and $e_f(0) = e_f(1)$ or $| e_f(0) - e_f(1) | = 1$ depending on whether k is even or odd. Note that for this case we are assuming that $k \not\equiv 2(\bmod 4)$. The following table gives the chosen values of x and y in various cases.

r	Case	x	y	$e_f(0)$	$e_f(1)$
0	$e_g(0) = e_g(1) + k$	$mq - p$	$mq + p$	$e_g(1) + 2mq$ $+2p$	$e_g(1) + 2mq$ $+2p$
	$e_g(1) = e_g(0) + k$	$mq + p$	$mq - p$	$e_g(0) + 2mq$ $+2p$	$e_g(0) + 2mq$ $+2p$
1	$e_g(0) = e_g(1) + k$	$mq - p$	$mq + p$	$e_g(1) + 2mq$ $+2p + 1$	$e_g(1) + 2mq$ $+2p$
	$e_g(1) = e_g(0) + k$	$mq + p$	$mq - p$	$e_g(0) + 2mq$ $+2p$	$e_g(0) + 2mq$ $+2p + 1$
3	$e_g(0) = e_g(1) + k$	$mq - p - 1$	$mq + p + 1$	$e_g(1) + 2mq$ $+2p + 3$	$e_g(1) + 2mq$ $+2p + 2$
	$e_g(1) = e_g(0) + k$	$mq + p + 1$	$mq - p - 1$	$e_g(0) + 2mq$ $+2p + 2$	$e_g(0) + 2mq$ $+2p + 3$

Using equations (**), one can easily see that in all these cases $v_f(0) =$

$q(1+2m) = v_f(1)$, that is, f is a cordial labeling.

Case 2: n is odd. that is $n = 2q+1$ and $s = 1$. This means $v_g(0) = q+1, v_g(1) = q$ and $| V(G \circ \overline{K_{2m}} | = (2q+1)(2m+1) = 4mq+2q+2m+1$ and $| E(G \circ \overline{K_{2m}} | =| E(G) | +2m(2q+1)$. Again, the following table gives the chosen values of x and y in various cases.

r	Case	x	y	$e_f(0)$	$e_f(1)$
0	$e_g(0) = e_g(1) + k$	$mq - p$ $+m$	$mq + p$	$e_g(1) + 2mq$ $+2p + m$	$e_g(1) + 2mq$ $+2p + m$
	$e_g(1) = e_g(0) + k$	$mq + p$ $+m$	$mq - p$	$e_g(0) + 2mq$ $+2p + m$	$e_g(0) + 2mq$ $+2p + m$
1	$e_g(0) = e_g(1) + k$	$mq - p$ $+m$	$mq + p$	$e_g(1) + 2mq$ $+2p + m + 1$	$e_g(1) + 2mq$ $+2p + m$
	$e_g(1) = e_g(0) + k$	$mq + p$ $+m$	$mq - p$	$e_g(0) + 2mq$ $+2p + m$	$e_g(0) + 2mq$ $+2p + m + 1$
2	$e_g(0) = e_g(1) + k$	$mq - p$ $+m - 1$	$mq + p$	$e_g(1) + 2mq$ $+2p + m + 1$	$e_g(1) + 2mq$ $+2p + m + 1$
	$e_g(1) = e_g(0) + k$	$mq + p$ $+m$	$mq - p$ -1	$e_g(0) + 2mq$ $+2p + m + 1$	$e_g(0) + 2mq$ $+2p + m + 1$
3	$e_g(0) = e_g(1) + k$	$mq - p - 1$ $+m$	$mq + p + 1$	$e_g(1) + 2mq$ $+2p + m + 1$	$e_g(1) + 2mq$ $+2p + m + 2$
	$e_g(1) = e_g(0) + k$	$mq + p + 1$ $+m$	$mq - p - 1$	$e_g(0) + 2mq$ $+2p + m + 2$	$e_g(0) + 2mq$ $+2p + m + 1$

Using equations (**), one can easily see that in all these cases f is a cordial labeling.

Now we assume that $k = 4p+2$ and n is even. In this case, we show that no values of x and y are possible such that the extension f of g is cordial. Suppose f is a cordial labeling for some x and y. Since $v_f(0) - v_f(1) = 2x + 2y - 4mq$, which is even, it must be zero. This gives $x + y = 2mq$. Also, either $e_g(0) = e_g(1) + k$ or $e_g(1) = e_g(0) + k$. Similarly, $e_f(0) - e_f(1) = 2x - 2y + 4p + 2$ or $e_f(0) - e_f(1) = 2x - 2y - 4p - 2$, which is again even. Hence, it must be zero. This gives either $y - x = 2p - 1$

or $x - y = 2p + 1$. Solving these equations, we get $x = \dfrac{2mq - 2p - 1}{2}$ or $x = \dfrac{2mq + 2p + 1}{2}$ which are not integers. Hence, no cordial extension f of g is possible.

Remark: If we take $e_f(0) - e_f(1) = 2$ instead of 0, one gets $x = mq - p, y = mq + p$ or $x = mq + p + 1, y = mq - p - 1$. We will thus get a binary extension f of g to $G \circ \overline{K_{2m}}$ such that $| e_f(0) - e_f(1) |= 2$. Thus, $i(G \circ \overline{K_{2m}}) \leq 2$, that is, this corona graph is either cordial or has index of cordiality 2.

Corollary: If G is a cordial graph on n vertices and g is a cordial labeling of G. then g can be extended to a cordial labeling of $G \circ \overline{K_{2m}}$.

Theorem 3: The corona $K_n \circ \overline{K_{2m}}$ is cordial if and only if $n \not\equiv 4(\text{mod}8)$.
Proof: We know that $i(K_n) = \left\lfloor \frac{n}{2} \right\rfloor$. Let g be a binary labeling of K_n which assigns the label 0 to $\left\lceil \frac{n}{2} \right\rceil$ vertices of K_n and the label 1 to $\left\lfloor \frac{n}{2} \right\rfloor$ vertices of K_n. Then $v_g(0) = v_g(1)$ or $v_g(0) = v_g(1) + 1$ and $e_g(1) = e_g(0) + k$, where $k = \left\lfloor \frac{n}{2} \right\rfloor$. From Theorem 2, we know that this labeling cannot be extended to a cordial labeling f of $K_n \circ \overline{K_{2m}}$ if and only if n is even with $k \equiv 2(\text{mod}4)$. Thus we get a cordial labeling of $K_n \circ \overline{K_{2m}}$ whenever n is not of the form $2k, k \equiv 2(\text{mod}4)$, that is, $n = 4q$ where q is odd. We note that $K_n \circ \overline{K_{2m}}$ has $n + 2nm$ vertices and $2q(4q - 1) + 2nm$ edges. Thus if f is to be a cordial labeling for $K_n \circ \overline{K_{2m}}$, then $v_f(0)$ must be equal to $v_f(1)$ and $e_f(0)$ must be equal to $e_f(1)$.

If possible, let f be a cordial labeling of labeling of $K_n \circ \overline{K_{2m}}$ where $n = 4q$, with q odd. Let g be the restriction of f to the vertices of K_n in this corona graph. Let $v_g(0) = a$ and $v_g(1) = n - a$. One can easily see that $e_g(0) = \dfrac{a(a-1)}{2} + \dfrac{(n-a)(n-a-1)}{2}$ and $e_g(1) = a(n-a)$. Then

$$
\begin{aligned}
v_f(0) &= a + x + y, \\
v_f(1) &= (n-a) + (2am - x) + (2m(n-a) - y) \\
&= n - a + amn - x - y, \\
e_f(0) &= \frac{a(a-1)}{2} + \frac{(n-a)(n-a-1)}{2} + x + (2m(n-a) - y), \\
e_f(1) &= a(n-a) + (2am - x) + y.
\end{aligned}
$$

The conditions on the vertex labels and edge labels given by f imply

$$
\begin{aligned}
x + y &= 4mq + 2q - a \\
x - y &= 4aq - a^2 + 2am - 4mq - 4q^2 + q.
\end{aligned}
$$

Solving these two equations, we get $x = 2aq + am - 2q^2 - \dfrac{a(a+1)}{2} + \dfrac{3q}{2}$. In this expression, all but the last term are integers and the last term $3q/2$ is not since q is odd, that is for no value of a do we have integral solutions x and y. Hence $K_{4q} \bigodot \overline{K_{2m}}$ cannot have a cordial labeling. □

The proof of the next theorem is similar to that of Theorem 2. Hence, we give only the values of x and y.

Theorem 4: Let g be a binary labeling of a graph G, such that $i(G) = \mid e_g(0) - e_g(1) \mid$, then g can be extended (by method of extensin) to a cordial labeling of $G \bigodot \overline{K_{2m+1}}$ if and only if G does not satisfy the following:

(a) $k \equiv 0 (\text{mod} 4)$, $n \equiv 2 (\text{mod} 4)$

(b) $k \equiv 1 (\text{mod} 4)$, $n \equiv 1 (\text{mod} 4)$ with $e_g(1) = e_g(0) + k$

(c) $k \equiv 1 (\text{mod} 4)$, $n \equiv 3 (\text{mod} 4)$ with $e_g(0) = e_g(1) + k$

(d) $k \equiv 2 (\text{mod} 4)$, $n \equiv 0 (\text{mod} 4)$

(e) $k \equiv 3 (\text{mod} 4)$, $n \equiv 1 (\text{mod} 4)$ with $e_g(0) = e_g(1) + k$

(f) $k \equiv 3 \bmod 4, n \equiv 3 \bmod 4$, with $e_g(1) = e_g(0) + k$.

Moreover, in those cases, $i(G \bigodot \overline{K_{2m+1}}) \leq 2$.

Proof: Let $k = 4p + r, n = 4q + s, 0 \leq r, s \leq 3$. The table below gives values of x and y for the cases in which f is a cordial labeling.

r	s	Case	x	y
0	0	$e_g(0) = e_g(1) + k$	$2mq + q - p$	$2mq + q + p$
0	0	$e_g(1) = e_g(0) + k$	$2mq + q + p$	$2mq + q - p$
0	1	$e_g(0) = e_g(1) + k$	$2mq + m + q - p$	$2mq + q + p$
0	1	$e_g(1) = e_g(0) + k$	$2mq + m + q + p$	$2mq + q - p$
0	3	$e_g(0) = e_g(1) + k$	$2mq + 2m + q + 1 - p$	$2mq + m + q + p$
0	3	$e_g(1) = e_g(0) + k$	$2mq + 2m + q + 1 + p$	$2mq + m + q - p$
1	0	$e_g(0) = e_g(1) + k$	$2mq + q - p$	$2mq + q + p$
1	0	$e_g(1) = e_g(0) + k$	$2mq + q + p$	$2mq + q - p$
1	1	$e_g(0) = e_g(1) + k$	$2mq + m + q - p$	$2mq + q + p$
1	2	$e_g(0) = e_g(1) + k$	$2mq + m + q - p$	$2mq + m + q + p + 1$
1	2	$e_g(1) = e_g(0) + k$	$2mq + m + q + 1 + p$	$2mq + m + q - p$
1	3	$e_g(1) = e_g(0) + k$	$2mq + 2m + q + 1 + p$	$2mq + m + q - p$
2	1	$e_g(0) = e_g(1) + k$	$2mq + m + q - p$	$2mq + q + p$
2	1	$e_g(1) = e_g(0) + k$	$2mq + m + q + p + 1$	$2mq + q - p - 1$
2	2	$e_g(0) = e_g(1) + k$	$2mq + m + q - p$	$2mq + m + q + p + 1$
2	2	$e_g(1) = e_g(0) + k$	$2mq + m + q + 1 + p$	$2mq + m + q - p$
2	3	$e_g(0) = e_g(1) + k$	$2mq + 2m + q - p$	$2mq + m + q + 1 + p$
2	3	$e_g(1) = e_g(0) + k$	$2mq + 2m + q + p + 1$	$2mq + m + q - p$
3	0	$e_g(0) = e_g(1) + k$	$2mq + q - p - 1$	$2mq + q + p + 1$
3	0	$e_g(1) = e_g(0) + k$	$2mq + q + p + 1$	$2mq + q - p - 1$
3	1	$e_g(1) = e_g(0) + k$	$2mq + q + p + 1$	$2mq + q - p - 1$
3	2	$e_g(0) = e_g(1) + k$	$2mq + m + q - p$	$2mq + m + q + p + 1$
3	2	$e_g(1) = e_g(0) + k$	$2mq + m + q + 1 + p$	$2mq + m + q - p$
3	3	$e_g(0) = e_g(1) + k$	$2mq + m + q - p$	$2mq + m + q + p + 1$

It can be shown, by proving that the only possible values of x, y for cordiality are not integers, that the cordial extension f of g is not possible in the remaining cases which are listed in the statement of the Theorem. As before, it can be shown that in all these cases, $i(G \circ \overline{K_{2m+1}}) \leq 2$.

Corollary: Let G be a cordial graph on n vertices. Any cordial labeling g of G can be extended to a cordial labeling of $G \circ \overline{K_{2m+1}}$ if and only if G does not satisfy any of the following:

(a)G has an even number of edges and $n \equiv 2(\bmod 4)$.

(b)G has an odd number of edges and either $n \equiv 1(\bmod 4)$, with $e_g(1) = e_g(0) + 1$ or $n \equiv 3(\bmod 4)$, with $e_g(0) = e_g(1) + 1$.

Theorem 5: The corona graph $K_n \circ \overline{K_{2m+1}}$ is cordial if and only if $n \not\equiv 7(\bmod 8)$.

Proof: As before $i(K_n) = \lfloor \frac{n}{2} \rfloor = k$. By Theorem 4, any binary labeling g of K_n with $v_g(0) = \lceil \frac{n}{2} \rceil, v_g(1) = \lfloor \frac{n}{2} \rfloor$ and $e_g(1) = e_g(0) + k$, can be extended to a cordial labeling of $K_n \circ \overline{K_{2m+1}}$ in all except six cases. We now check which of the six cases apply in this context.

Case (i) If n is even, write $n = 4q + s, s = 0, 2$.

If $s = 0$, then $n = 4q, k = 2q$ for some integer q. If q is even, then $k \not\equiv 2(\bmod 4)$. If q is odd, then $k \equiv 2(\bmod 4)$. Moreover, this also implies $n \equiv 4(\bmod 8.)$. Thus the condition (d) in Theorem 4 applies and an extension of g to a cordial labeling of $K_n \circ \overline{K_{2m+1}}$ is not possible. If $s = 2$, then $k = 2q + 1$ that is k is odd. Hence, $k \not\equiv 0(\bmod 4)$. Thus condition (a) does not apply.

Case (ii) If n is even write $n = 4q + s, s = 1, 3$. Clearly conditions (c) and (d) of Theorem 4 do not apply since in those cases $e_g(0) = e_g(1) + k$. Hence we look at only conditions (b) and (f).

If $s = 1$, then $n = 4q+1, k = 2q$ for some integer q. Clearly k is even and hence not equivalent to 1 modulo 4, that is condition (b) does not apply. If $s = 3$, then $n = 4q + 3$ and $k = 2q + 1$. If q is even, then $k \equiv 1(\bmod 4)$, that is condition (f) does not apply. If q is odd, then $k \equiv 3(\mod 4)$ and $n = 4q+3$ implies $n \equiv 7(\mod 8)$. Thus condition (f) applies if and only if $n \equiv 7(\mod 8)$.

This means that $K_n \circ \overline{K_{2m+1}}$ is cordial whenever $n \not\equiv 4, 7(\mod 8)$. We consider these cases separately.

Case 1: $n = 8z + 4$. Let g be a binary labeling of K_n with $v_g(0) = 4z+3, v_g(1) = 4z+1$. Then $e_g(0) = (4z+3)(2z+1)+(4z+1)(2z), e_g(1) = (4z+3)(4z+1)$. Let x, y be as explained earlier. Then

$$\begin{aligned}
v_f(0) &= (4z+3)+x+y, \\
v_f(1) &= (4z+1)+(2m+1)(4z+3)-x+(2m+1)(4z+1)-y \\
&= 16mz+8m+12z+5-x-y, \\
e_f(0) &= (4z+3)(2z+1)+(4z+1)(2z)+x+(2m+1)(4z+1)-y \\
&= 16z^2+8mz+2m+16z+4+x-y, \\
e_f(1) &= (4z+3)(4z+1)+(2m+1)(4z+3)-x+y \\
&= 16z^2+8mz+6m+20z+6-x+y.
\end{aligned}$$

Now, $v_f(0) = v_f(1)$ implies $x+y = 8mz+4m+4z+1$ and $e_f(0) = e_f(1)$ implies $x-y = 2m+2z+1$. Solving these two equations we get $x = 4mz+3m+3z+1, y = 4mz+m+z$. One can easily verify that $v_f(0) = 8mz+4m+8z+4 = v_f(1)$ and $e_f(0) = 16z^2+8mz+18z+4m+5 = e_f(1)$. Thus we obtain a cordial labeling of $K_n \circ \overline{K_{2m+1}}$ even in the case when $n \equiv 4(\mod 8)$.

Case 2: $n = 8z + 7$. We prove that no binary labeling of K_n can be extended to a cordial labeling of $K_n \circ \overline{K_{2m+1}}$. Let g assign the label 0 to *a* of the vertices and the label 1 to the remaining $n-a$ vertices of K_n. Then

$$\begin{aligned}
v_g(0) &= a, \\
v_g(1) &= n-a, \\
e_g(0) &= a(a-1)/2+(n-a)(n-a-1)/2, \\
e_g(1) &= a(n-a).
\end{aligned}$$

Let f, x and y be as described earlier. Then

$$\begin{aligned}
v_f(0) &= a+x+y. \\
v_f(1) &= (n-a)+[(2m+1)a-x]+[(2m+1)(n-a)-y] \\
&= 2mn+2n-a-x-y \\
e_f(0) &= a(a-1)/2+(n-a)(n-a-1)/2+x+[(2m+1)(n-a)-y], \\
e_f(1) &= a(n-a)+[(2m+1)a-x]+y.
\end{aligned}$$

Now, $v_f(0)=v_f(1)$ implies that $x+y=mn+n-a$ and $e_f(0)=e_f(1)$ implies that $x-y=2am-mn+an-a^2+a-n/4-n^2/4$. Solving these two equations, we get $x=am+4az+4a-a/2(a+1)+3z+22/8-8q^2-14q-6$. Every term on the right hand side of this equation except 22/8 is an integer. Consequently, x cannot be an integer. Hence in this case, $K_n \circ \overline{K_{2m+1}}$ is not cordial when $n \equiv 7 \bmod 8$. □

Remark: Even in the cases where an extension of g is not possible, $G \circ \overline{K_t}$ may very well be cordial.

Cordiality of $G \circ C_t$

Let G be a cordial graph. Then there exists a labeling g of G such that $\mid v_g(0) - v_g(1) \mid \leq 1$ and $\mid e_g(0) - e_g(1) \mid \leq 1$. Whenever possible we extend this labeling to a cordial labeling of $G \circ C_t$.

Theorem 6: Let G be a cordial graph on n vertices with a cordial labeling g. Then g can be extended to a cordial labeling of $G \circ C_t$ if $t \neq 4m + 3$, with n odd and $e_g(0) = e_g(1)$.

Proof: Let $\mid V(G) \mid = n$ and $V(C_t) = \{v_1, v_2, \cdots, v_t\}$, $E(C_t) = \{v_i v_{i+1}, v_t v_1 \mid 1 \leq i \leq t - 1.\}$. Clearly $\mid V(G \circ C_t) \mid = n + nt = n(1 + t)$ and $\mid E(G \circ C_t) = \mid E(G) \mid + 2nt$.

The edges which connect the ith vertex of G to each of the vertices of the ith copy of C_t will henceforth be called the **connecting edges**. Note that there will be nt such connecting edges in $G \circ C_t$. Let $t = 4m + r, 0 \leq r \leq 3$.

Case 1: $r = 0$, that is $t = 4m$. Let f be a binary labeling of $G \circ C_t$ defined as follows: $f(u) = g(u)$ for each $u \in V(G)$. In each copy of C_t, label the vertices as $1, 1, 0, 0, \cdots$, that is in each copy of C_t, let $f(v_i) = 1, i \equiv 1, 2 (\text{mod} 4)$ and $f(v_i) = 0, i \equiv 0, 3 (\text{mod} 4)$.

It is clear that in each copy of C_t, $2m$ edges as well as $2m$ vertices receive the label 0 and $2m$ edges as well as $2m$ vertices receive the label 1. Of the $4mt$ connecting edges, $2mt$ will have the label 0 and $2mt$ will have the label 1. Thus,

$$\mid v_f(0) - v_f(1) \mid = \mid v_g(0) - v_g(1) \mid \leq 1,$$
$$\mid e_f(0) - e_f(1) \mid = \mid e_g(0) - e_g(1) \mid \leq 1.$$

Hence f is a cordial labeling.

Case 2: $r = 1$, that is $t = 4m + 1$. Let f be a binary labeling of $G \circ C_t$ defined as follows:

Define $f(u) = g(u)$ when u belongs to $V(G)$. In the $v_g(0)$ copies of C_t adjacent to the vertices labeled 0 in G, use the labeling $1,1,0,0,1,1,0,0,\cdots 1,1,0,0,1$ that is, define $f(u_i)=1, i\equiv 1,2(\text{mod}4)$ and $f(u_i)=0, i\equiv 0,3(\text{mod}4)$.

On each such copy of C_t, $2m$ vertices receive the label 0 and $2m+1$ vertices receive the label 1. Further, on each such copy of C_t, $2m+1$ edges receive the label 0 and $2m$ receive the label 1. Among the connecting edges connecting any one such vertex of G to a copy of C_t, $2m$ edges receive the label 0 and $2m+1$ receive the label 1.

In the $v_g(1)$ copies of C_t adjacent to the vertices labeled 1 in G use the labeling $0,0,1,1,0,0,1,1,\cdots,0,0,1,1,0$ that is

$$
\begin{aligned}
v_f(0) &= v_g(0)+2mv_g(0)+(2m+1)v_g(1)\\
&= (2m+1)[v_g(0)+v_g(1)]\\
&= (2m+1)n.\\
v_f(1) &= v_g(1)+2mv_g(1)+(2m+1)v_g(1)\\
&= (2m+1)n.\\
e_f(0) &= e_g(0)+v_g(0)[2m+(2m+1)]+v_g(1)[(2m+1)+2m]\\
&= e_g(0)+(4m+1)n.\\
&= e_g(0)+nt.\\
e_f(1) &= e_g(1)+v_g(0)[(2m+1)+2m]+v_g(1)[2m+(2m+1)]\\
&= e_g(1)+(4m+1)n\\
&= e_g(1)+nt.
\end{aligned}
$$

We thus have

$$\mid e_f(0)-e_f(1)\mid=\mid e_g(0)-e_g(1)\mid\leq 1.$$

Hence f is a cordial labeling.

Case 3: $r=2$, that is $t=4m+2$.

Case 3A: Let n be even.Then $v_g(0)=v_g(1)=n/2$. Let f be a binary labeling of $G\circ C_t$ defined as follows:

Define$f(u) = g(u)$ when u belongs to $V(G)$. In the $v_g(0)$ copies of C_t adjacent to the vertices labeled 0 in G, use the labeling $0, 0, 1, 1, 0, 0, 1, 1, \cdots 0, 0, 1, 1, 0, 0, 1, 0, 0, 1$

that is, for $1 \leq i \leq 4m - 1$

$$\begin{aligned} f(v_i) &= 0, i \equiv 1, 2(\text{mod}4) \\ &= 1, i \equiv 0, 3(\text{mod}4) \\ &= 0, i = 4m, 4m + 1 \\ &= 1, i = 4m + 2. \end{aligned}$$

In each such copy of C_t, $2m + 2$ vertices receive the label 0 and $2m$ vertices the label 1. Further, in each such copy, $2m$ edges receive the label 0 and $2m + 2$ the label 1. Amongst the connecting edges connecting any one such vertex of G to a copy of C_t, $2m + 2$ edges receive the label 0 and $2m$ receive the label 1.

In the $v_g(1)$ copies of C_t adjacent to the vertices labeled 1 in G, use the labeling $1, 1, 0, 0, 1, 1, 0, 0, \cdots 1, 1, 0, 0, 1, 1, 0, 1, 1, 0$

that is for $1 \leq i \leq 4m - 1$

$$\begin{aligned} f(v_i) &= 1, i \equiv 1, 2(\text{mod}4) \\ &= 0, i \equiv 0, 3(\text{mod}4) \\ &= 1, i = 4m, 4m + 1 \\ &= 0, i = 4m + 2. \end{aligned}$$

In each such copy of C_t, $2m$ vertices receive the label 0 and $2m + 2$ vertices the label 1. Further, in each such copy, $2m$ edges receive the label 0 and $2m + 2$ the label 1. Amongst the connecting edges connecting any one such vertex of G to a copy of C_t, $2m + 2$ edges receive the label 0 and $2m$ receive the label 1. Hence

$$\begin{aligned}
v_f(0) &= v_g(0) + v_g(0)(2m+2) + v_g(1)2m \\
&= (n/2)(2m+3) + (n/2)2m \\
&= (n/2)(4m+3), \\
v_f(1) &= v_g(1) + v_g(0)(2m) + v_g(1)(2m+2) \\
&= (n/2)(4m+3). \\
e_f(0) &= e_g(0) + v_g(0)[2m + (2m+2)] + v_g(1)[2m + (2m+2)] \\
&= e_g(0) + n(4m+2) \\
&= e_g(0) + nt, \\
e_f(1) &= e_g(1) + v_g(0)[(2m+2) + 2m] + v_g(1)[(2m+2) + 2m] \\
&= e_g(1) + n(4m+2) \\
&= e_g(1) + nt.
\end{aligned}$$

Thus $v_f(0) = v_f(1)$ and $\mid e_f(0) - e_f(1) \mid \leq 1$, that is, f is a cordial labeling.

Case 3B: Let n be odd. Then, without loss of generality we can assume that $v_g(0) = (n+1)/2, v_g(1) = (n-1)/2$. In the $(n-1)/2$ copies of C_t, adjacent to the vertices labeled 0 in G as well as the $(n-1)/2$ copies of C_t adjacent to the vertices labeled 1 in G, f assigns labels as in Case 3(A). For the one remaining copy of C_t adjacent to a vertex labeled 0 in G, use the labeling $1, 1, 0, 0, \cdots, 1, 1$. On this copy, $2m$ vertices receive the label 0 and $2m+2$ vertices receive the label 1. Further, on this copy, $2m+2$ edges receive the label 0 and $2m$ edges receive the label 1. Amongst the connecting edges associated with this copy, $2m$ edges receive the label 0 and $2m+2$ edges receive the label 1.

$$\begin{aligned}
v_f(0) &= v_g(0)+(2m+2)(n-1)/2+2m(n-1)/2+2m \\
&= (n+1)/2+(n-1)(4m+2)/2+2m \\
&= (n+1)/2+(n-1)(2m+1)+2m, \\
v_f(1) &= v_g(1)+(n-1)2m/2+(n-1)(2m+2)/2+(2m+2) \\
&= (n-1)/2+(n-1)(2m+1)+2m+2 \\
&= (n+1)/2+(n-1)(2m+1)+2m+1. \\
e_f(0) &= e_g(0)+(n-1)[2m+(2m+2)]/2 \\
&\quad +(n-1)[2m+(2m+2)]/2+4m+2. \\
e_f(1) &= e_g(1)+(n-1)[(2m+2)+2m]/2 \\
&\quad +(n-1)[(2m+2)+2m]/2+4m+2.
\end{aligned}$$

Thus $\mid v_f(0)-v_f(1)\mid = 1$ and $\mid e_f(0)-e_f(1)\mid = \mid e_g(0)-e_g(1) \leq 1$, that is, f is a cordial labeling.

Case 4: $r=3$, that is $t=4m+3$.

Case 4A: Let n be even.Then $v_g(0)=v_g(1)=n/2$. In the $n/2$ copies of C_t connected to vertices having the label 0 in G assign the labels as $0,0,1,1,\cdots,0,0,1$, that is $f(v_i)=0$ for $i\equiv 1,2(\text{mod}4)$ and $f(v_i)=1$ for $i\equiv 0,3(\text{mod}4)$. In such copies of C_t, $(2m+2)$ vertices receive the label 0 and $(2m+1)$ vertices receive the label 1. Of the connecting edges associated with each such copy, $(2m+2)$ edges receive the label 0 and $(2m+1)$ edges receive the label 1. In the $n/2$ copies of C_t connected to vertices having the label 1 in G assign the labels as $1,1,0,0,\cdots,1,1,0$, that is $f(v_i)=1$ for $i\equiv 1,2(\text{mod}4)$ and $f(v_i)=0$ for $i\equiv 0,3(\text{mod}4)$. In such copies of C_t, $(2m+2)$ vertices receive the label 1 and $(2m+1)$ vertices receive the label

0. Then,

$$\begin{aligned}
v_f(0) &= v_g(0) + v_g(0)(2m+2) + v_g(1)(2m+1) \\
&= n(4m+4)/2, \\
v_f(1) &= v_g(1) + v_g(0)(2m+1) + v_g(1)(2m+2) \\
&= n(4m+4)/2, \\
e_f(0) &= e_g(0) + v_g(0)(4m+3) + v_g(1)(4m+3) \\
&= e_g(0) + nt. \\
e_f(1) &= e_g(1) + v_g(0)(4m+3) + v_g(1)(4m+3) \\
&= e_g(1) + nt.
\end{aligned}$$

Thus $v_f(0) = v_f(1)$ and $\mid e_f(0) - e_f(1) \mid = \mid e_g(0) - e_g(1) \leq 1$, that is, f is a cordial labeling.

Case 4B: Let n be odd. Again without loss of generality, we assume that $v_g(0) = v_g(1) + 1$.

We first consider the case $e_g(0) = e_g(1)+1$. For $(n-1)/2$ copies adjacent to the vertices labeled 0 in G and $(n-1)/2$ copies adjacent to the vertices labeled 1 in G we use the labeling as described in Case A. For the one remaining copy of C_t, adjacent to a vertex labeled 0 in G, use the labeling $1, 1, 0, 0, \cdots, 1, 1, 0$. Then on this copy of C_t, $(2m+1)$ vertices receive the label 0 and $(2m+2)$ vertices receive the label 1, whereas $2m+1$ edges receive the label 0 and $2m+2$ edges receive the label 1. Amongst the connecting edges with this copy, $2m+1$ edges receive the label 0 and $2m+2$ edges receive the label 1. Thus,

$$
\begin{aligned}
v_f(0) &= v_g(0) + (n-1)(2m+2)/2 + (n-1)(2m+1)/2 + 2m+1 \\
&= (n+1)/2 + (n-1)(4m+3)/2 + 2m+1 \\
&= (n-1)(4m+4)/2 + 2m+2 \\
&= n(2m+2), \\
v_f(1) &= v_g(1) + (n-1)(2m+1)/2 + (n-1)(2m+2)/2 + 2m+2 \\
&= (n-1)(4m+4)/2 + 2m+2 \\
&= n(2m+2), \\
e_f(0) &= e_g(0) + (n-1)(4m+3)/2 + (n-1)(4m+3)/2 + 4m+2 \\
&= e_g(1) + (n-1)(4m+3) + 4m+3, \\
e_f(1) &= e_g(1) + (n-1)(4m+3)/2 + (n-1)(4m+3)/2 + 4m+4 \\
&= e_g(1) + (n-1)(4m+3) + 4m+4.
\end{aligned}
$$

Thus $v_f(0) = v_f(1)$ and $e_f(0) + 1 = e_f(1)$, that is, f is a cordial labeling. Next consider the case $e_g(0) + 1 = e_g(1)$. For $(n-1)/2$ copies adjacent to the vertices labeled 0 in G and for $(n-1)/2$ copies adjacent to the vertices labeled 1 in G, we use the labeling as described in Case A. For the one remaining copy of C_t adjacent to a vertex labeled 0 in G, use the labeling $1, 1, 0, 0, \cdots 1, 1, 0, 0, 0, 1, 1$. Then on this copy of $C_t, 2m+1$ vertices receive the label 0 and $2m+2$ vertices receive the label 1, whereas $2m+3$ edges receive the label 0 and $2m$ edges receive the label 1. Amongst the connecting edges with this copy, $2m+3$ edges receive the label 0 and $2m$ edges receive the label 1. Thus,

$$
\begin{aligned}
v_f(0) &= v_g(0) + (n-1)(2m+2)/2 + (n-1)(2m+1)/2 + 2m+1 \\
&= n(2m+2), \\
v_f(1) &= v_g(1) + (n-1)(2m+1)/2 + (n-1)(2m+2)/2 + 2m+2 \\
&= n(2m+2). \\
e_f(0) &= e_g(0) + (n-1)(4m+3)/2 + (n-1)(4m+3)/2 + (2m+1) + (2m+3) \\
&= e_g(0) + (n-1)(4m+3) + 4m+4, \\
e_f(1) &= e_g(1) + (n-1)(4m+3)/2 + (n-1)(4m+3)/2 + (2m+2) + 2m \\
&= e_g(0) + (n-1)(4m+3) + 4m+3.
\end{aligned}
$$

Thus $v_f(0) = v_f(1)$ and $e_f(0) = e_f(1)$, that is, f is a cordial labeling. □

Remark: If n is odd, $e_g(1) = e_g(0)$ and $t = 4m+3$, then we can prove that $i(G \circ C_t) \leq 2$. For $(n-1)/2$ copies adjacent to the vertices labeled 0 in G and $(n-1)/2$ copies adjacent to the vertices labeled 1 in G, we use the labeling as described in Case A. For the one remaining copy of C_t, adjacent to a vertex labeled 0 in G use the labeling $1, 1, 0, 0, \cdots, 1, 1, 0$. Then on this copy of C_t, $(2m+1)$ vertices receive the label 0 and $(2m+2)$ vertices receive the label 1, whereas $2m+1$ edges receive the label 0 and $2m+2$ edges receive the label 1. Amongst the connecting edges with this copy, $2m+1$ edges receive the label 0 and $2m+2$ edges receive the label

1. Thus,

$$\begin{aligned}
v_f(0) &= v_g(0) + (n-1)(2m+2)/2 + (n-1)(2m+1)/2 + 2m+1 \\
&= (n+1)/2 + (n-1)(4m+3)/2 + 2m+1 \\
&= (n-1)(4m+4)/2 + 2m+2 \\
&= n(2m+2), \\
v_f(1) &= v_g(1) + (n-1)(2m+1)/2 + (n-1)(2m+2)/2 + 2m+2 \\
&= (n-1)(4m+4)/2 + 2m+2 \\
&= n(2m+2), \\
e_f(0) &= e_g(0) + (n-1)(4m+3)/2 + (n-1)(4m+3)/2 + 4m+2, \\
e_f(1) &= e_g(1) + (n-1)(4m+3)/2 + (n-1)(4m+3)/2 + 4m+4.
\end{aligned}$$

From the above it follows that $e_f(0) + 2 = e_f(1)$. Hence $i(G \circ C_t) \leq 3$. □

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