On k-sets of type $(2, h)$ in a planar space

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Abstract

In this paper subsets of a three-dimensional locally projective planar space which meet every plane either in 2 or in h, $h > 2$, points are studied and classified.

1 Introduction

A *linear space* is a pair $(\mathcal{S}, \mathcal{L})$, where S is a non-empty set of *points* and $\mathcal{L}$ is a non-empty set of proper subsets of $\mathcal{S}$ called *lines*, such that through every pair of distinct points there is a unique line and every line has at least two points.

Let $(\mathcal{S}, \mathcal{L})$ be a finite linear space. For every point p of $\mathcal{S}$, the *degree* of p is the number $[p]$ of lines through p; for every line l, the *length* $[l]$ of l is its cardinality. The integer n defined by $n + 1 = \max\{[p] : p \in \mathcal{S}\}$ is the *order* of the linear space. A subset T of the point-set $\mathcal{S}$ of a linear space $(\mathcal{S}, \mathcal{L})$ is a *subspace* if it contains the line through any two of its points.

A *planar space* is a triple $(\mathcal{S}, \mathcal{L}, \mathcal{P})$, where $(\mathcal{S}, \mathcal{L})$ is a linear space and $\mathcal{P}$ is a non-empty family of proper subspaces of $(\mathcal{S}, \mathcal{L})$, called *planes*, satisfying the following properties:

(p_1) through any three non-collinear points there is a unique plane and it is the smallest subspace containing them;

(p_2) every plane contains at least three non collinear points.

Let $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ be a finite planar space. For every plane π of $\mathcal{P}$, denote by $\mathcal{L}_\pi$ the set of the lines of $\mathcal{L}$ contained in π and by $n(\pi)$ the order of the linear space $(\pi, \mathcal{L}_\pi)$. The integer $n = \{\max n(\pi) : \pi \in \mathcal{P}\}$, is the *order* of the planar space.

For any point x of $\mathcal{S}$, the *star of lines* with center x is the set of all lines through x. Let π be a plane of $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ and let x be a point of π: the

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pencil of lines with center x in π is the set of all lines through x contained in π. If every pencil of lines has at least three lines we have a *non-degenerate* planar space. Two *skew* lines are two non-coplanar lines of a planar space. In this paper v is the number of points, b is the number of lines and c is the number of planes of the planar space $(\mathcal{S}, \mathcal{L}, \mathcal{P})$.

A planar space $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is *embeddable* in a projective space $\mathbf{P}$ if there is an injection of S into the point set of $\mathbf{P}$ preserving the collinearities and coplanarities and non-collinearities and non-coplanarities.

A *three dimensional locally projective planar space* is a planar space $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ whose planes pairwise intersect either in the empty set or in a line. If $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is a non-degenerate planar space of order n, it is easy to see that the property that its planes pairwise intersect either in the empty set or in a line is equivalent to the property that for every point p of $\mathcal{S}$, the linear space $(\mathcal{L}_p, \mathcal{P}_p)$ whose points are the lines through p and whose lines are the pencils of lines with center p, is a projective plane of order n.

If for every point p of $\mathcal{S}$, the linear space $(\mathcal{L}_p, \mathcal{P}_p)$ is a projective space, then $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is a *locally projective* planar space. Throughout this paper $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is a non-degenerate finite planar space of order n satisfying the following property:

(i) *the planes of* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *pairwise intersect in a line.*

Hence $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is a three dimensional locally projective planar space of order n, and so $(\mathcal{L}_p, \mathcal{P}_p)$ is a finite projective plane of order n for every point p. It is not difficult to see that $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ satisfies the following properties:

(a) through every point there are $n^2 + n + 1$ lines and $n^2 + n + 1$ planes;

(b) in every plane the pencils have cardinality $n + 1$;

(c) through every line there are $n + 1$ planes;

(d) in every plane there are $n^2 + n + 1$ lines;

(e) the number of planes is $c = n^3 + n^2 + n + 1$;

(f) the number of lines is $b = (n^2 + 1)(n^2 + n + 1)$;

(g) the number v of points is at most $n^3 + n^2 + n + 1$.

Assume that in the planar space $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ there is a set $\mathbf{K}$ of points which meets every plane in either 2 or h ($h > 2$) points. A plane π meeting $\mathbf{K}$ in exactly two points is a *2-secant plane.* A plane meeting $\mathbf{K}$ in h points is a *h-secant plane.*

A set C of points meeting every line in at most two points is a *cap* of $(\mathcal{S}, \mathcal{L}, \mathcal{P})$. It is possible to prove (see [2]) that if $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is a non-degenerate planar space of order n ($n > 2$) satisfying Property (i), then every cap C has at most n^2+1 points. A cap of cardinality n^2+1 is an *ovoid* of $(\mathcal{S}, \mathcal{L}, \mathcal{P})$.

In order to study sets $\mathbf{K}$ of class $[2, h]$ with respect to planes of $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ it is useful to recall the following theorem contained in [1]:

Theorem I *Let $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ be a non-degenerate finite planar space of order n satisfying Property (i) and let $\mathbf{K}$ be a proper subset of $\mathcal{S}$ meeting every plane in either 1 or h $(h > 1)$ points. Then $\mathbf{K}$ is a line (of length $n+1$) or an ovoid of $(\mathcal{S}, \mathcal{L}, \mathcal{P})$.*

In the same paper [1] it is proved that if a set $\mathbf{K}$ meets every plane of $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ in exactly h points then $(\mathcal{S}, \mathcal{L} \mathcal{P})$ is PG$(3, n)$ and $\mathbf{K}$ is its point-set.

In this paper we prove the following theorem:

Theorem *Let $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ be a non-degenerate finite planar space of order n satisfying Property (i) and let $\mathbf{K}$ be a proper subset of $\mathcal{S}$ meeting every plane in either 2 or h $(h > 2)$ points. Then $\mathbf{K}$ is a pair of skew lines (both of length $n+1$) of $(\mathcal{S}, \mathcal{L}, \mathcal{P})$.*

2 Sets of type $(2, h)$ in $(\mathcal{S}, \mathcal{L}, \mathcal{P})$

Let $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ be a non-degenerate finite planar space of order n satisfying Property (i) and let $\mathbf{K}$ be a proper subset of the point set $\mathcal{S}$ meeting every plane in either 2 or h $(h > 2)$ points. Let k be the cardinality of $\mathbf{K}$. An *exterior* line is a line missing $\mathbf{K}$, a *tangent* line is a line meeting $\mathbf{K}$ in just one point, a 2*−secant* line is a line meeting $\mathbf{K}$ in exactly two points and a s*-secant* line is a line meeting $\mathbf{K}$ in s $(s \geq 3)$ points.

From the last part of Section one we may assume that there are both types of planes: 2-secant planes and h-secant planes. Let α be the number of 2-secant planes and let β be the number of h-secant planes. Let π be a 2-secant plane to $\mathbf{K}$ and let p and p' be the two points of $\mathbf{K}$ on π. Then the line pp' is a 2-secant line, any line of π containing neither p nor p' is an exterior line of $\mathbf{K}$ and a line of π containing p but not p' is a tangent line of $\mathbf{K}$. Let L be a 2-secant line and let $\{p, p'\} = \mathbf{K} \cap L$. Any point of $\mathbf{K} \setminus \{p, p'\}$ is on an h-secant plane through L. If μ denotes the number of h-secant planes through L then:

$$k = 2 + \mu(h-2). \tag{1}$$

The previous equation shows that μ is independent of the 2-secant line L and that

$$h - 2 \text{ divides } k - 2. \tag{2}$$

Let E be an exterior line of $\mathbf{K}$. Computing the cardinality of $\mathbf{K}$ via the planes through E we get

$$k \geq 2n + 2. \tag{3}$$

Let π be a 2-secant plane of $\mathbf{K}$ and let $\{p, p'\} = \mathbf{K} \cap \pi$. Let L be the line pp'. Computing k via the planes through L we get

$$k \leq n(h-2) + 2. \tag{4}$$

Let t be a tangent line of $\mathbf{K}$ at the point p. Denote by ρ the number of 2-secant planes through t. Then:

$$k = \rho + (n+1-\rho)(h-1) + 1. \tag{5}$$

From equation (5) we have

$$k - 2 = (n+1)(h-1) - \rho(h-2) - 1 = (n+1)(h-2) + n - \rho(h-2) \tag{6}$$

and since from (2) $h-2$ divides $k-2$ we get

$$h - 2 \text{ divides } n \tag{7}$$

hence

$$h \leq n + 2 \tag{8}$$

From (8) and (4) it follows:

$$k \leq n^2 + 2 \tag{9}$$

Since α is the number of 2-secant planes and β is the number of h-secant planes we have

$$2(n^3 + n^2 + n + 1) = 2\alpha + 2\beta. \tag{10}$$

Counting in two ways point-plane pairs (p, π) with $p \in \mathbf{K}$ and $p \in \pi$ we get

$$k(n^2 + n + 1) = 2\alpha + \beta h \tag{11}$$

Counting in two ways the pairs $(\{p, p'\}, \pi)$ with $p, p' \in \mathbf{K}$ and $p, p' \in \pi$ we get

$$k(k-1)(n+1) = 2\alpha + \beta h(h-1) \tag{12}$$

Subtracting (10) from (11) we get

$$k(n^2 + n + 1) - 2(n^2 + 1)(n + 1) = \beta(h - 2) \tag{13}$$

and subtracting (11) from (12) we get

$$k(k-1)(n+1) - k(n^2 + n + 1) = \beta h(h-2). \tag{14}$$

From (13)

$$k(k-1)(n+1) - k(n^2+n+1) - h[k(n^2+n+1) - 2(n^2+1)(n+1)] = 0. \tag{15}$$

Hence k satisfies the following equation of second degree in k

$$(n+1)k^2 - (h(n^2+n+1) + (n^2+2n+2))k + 2h(n^2+1)(n+1) = 0 \quad (16)$$

From (16) it follows that $k = 2n+2$ (its minimum value) if and only is $h = n+2$ (its maximum value). Indeed if $h = n+2$, then from (16) it follows $k = 2n+2$ or $k = (n^3+2n^2+n+2)/(n+1)$. Since $n \geq 2$ the element $(n^3+2n^2+n+2)/(n+1)$ is not an integer number and hence $k = 2n+2$. Viceversa if $k = 2n+2$, then from (16) it follows that $h = n+2$.

We can now prove the following

Proposition 2.1 *If* $\mathbf{K}$ *has* $2n+2$ *points, then* $\mathbf{K}$ *consists of two skew lines (both of length* $n+1$*).*

Proof : If $\mathbf{K}$ has $2n+2$ points, then from (16) it follows $h = n+2$. If $\mathbf{K}$ is a cap let π be a $(n+2)$-secant of $\mathbf{K}$ and let E of be an exterior line in π to the $(n+2)$-arc $\mathbf{K} \cap \pi$. Counting k via the planes through the line E we get $k \geq n+2+2n = 3n+2$, that is a contradiction. Hence there is at least a line t meeting $\mathbf{K}$ in at least three points. Let $s = |t \cap \mathbf{K}|$. Counting k via the planes through t, and since every such a plane is a $(n+2)$-secant plane, we get

$$k = 2n+2 = s + (n+1)(n+2-s). \quad (17)$$

From (17) it follows $s = n+1$ and hence the line t is contained in $\mathbf{K}$. If $\mathbf{K}' = \mathbf{K} \setminus t$, then $|\mathbf{K}'| = n+1$. Note that each plane meets $\mathbf{K}'$ in 1 or in $n+1$ points. From Theorem I it follows that also $\mathbf{K}'$ is a line (of length $n+1$) and hence the assertion. □

Because of the previous proposition we may assume that $h-2 < n$ and since $h-2$ divides n then $h-2 \leq n/2$ and hence $k \leq n^2/2+2$.

Assume that $\mathbf{K}$ is not a cap. Then there is a line r meeting $\mathbf{K}$ in more than two points. Let s be the number of common points of r and $\mathbf{K}$. Computing the cardinality of $\mathbf{K}$ via the planes through r we get:

$$k = s + (n+1)(h-s). \quad (18)$$

The previous equation shows that

$$s = h - \frac{k-h}{n}, \quad (19)$$

is a constant and so it is independent of the line r. Hence every line meeting $\mathbf{K}$ in more then two points meets $\mathbf{K}$ in a constant number s of points, thus the set $\mathbf{K}$ meets every line in $0, 1, 2$ or s points.

We can now prove the following lemma

Lemma 2.1 *Let $(\mathcal{S},\mathcal{L},\mathcal{P})$ be a finite non-degenerate planar space of order n satisfying Property (i) and let $\mathbf{K}$ be a proper subset of $\mathcal{S}$ meeting every plane in either 2 or h points. Then $\mathbf{K}$ is a pair of skew lines (both of length $n+1$) or a cap of $(\mathcal{S},\mathcal{L},\mathcal{P})$.*

Proof : Assume $\mathbf{K}$ is not a cap. Since $k = s + (n+1)(h-s)$, equation (16) becomes:

$$(n+1)h^2 - ((n^2+3n+1)s - (n^2-n-2))h + s(sn(n+1) + n^2 + 2n+2) = 0$$

The discriminant of this equation is:

$$\Delta = (s-1)^2n^4 + 2(s^2-4s-1)n^3 + (3s^2-4s-3)n^2 + 2(s^2-s+2)n + (s-2)^2$$

and since $s \geq 3$, we have $\Delta \geq 0$. Solving the previous equation in h gives:

$$h \in \left\{ \frac{(n^2+3n+1)s - (n^2-n-2) - \sqrt{\Delta}}{2(n+1)}, \frac{(n^2+3n+1)s - (n^2-n-2) + \sqrt{\Delta}}{2(n+1)} \right\}$$

Assume first $h = \dfrac{(n^2+3n+1)s - (n^2-n-2) + \sqrt{\Delta}}{2(n+1)}$.

We will show that

$$h = \frac{(n^2+3n+1)s - (n^2-n-2) + \sqrt{\Delta}}{2(n+1)} > n+2.$$

Indeed, $(n^2+3n+1)s - (n^2-n-2) + \sqrt{\Delta} > 2(n+1)(n+2)$ gives $(s-1)n^2 + (3s+1)n + s + 2 + \sqrt{\Delta} > 2n^2 + 4n + 2$ which is always true since $s \geq 3$. This gives a contradiction since $h-2|n$ so $h \leq n+2$.

We may now suppose that
$h = \dfrac{(n^2+3n+1)s - (n^2-n-2) - \sqrt{\Delta}}{2(n+1)}$.

In this case we will show that $h = s+1$. First we show that $h \leq s+2$. Indeed, $h = \dfrac{(n^2+3n+1)s - (n^2-n-2) - \sqrt{\Delta}}{2(n+1)} < s+3$ gives

$$\sqrt{\Delta} > (n^2+3n+1)s - (n^2-n-2) - 2(n+1)(s+3)$$

i.e.

$$\sqrt{\Delta} > (s-1)n^2 + (s-5)n - s - 4$$

which gives:

$$(s-3)n^3 + (s^2+3s-9)n^2 + (s^2-s-9)n - 3(s+1) > 0$$

and which is true since $s \geq 3$ and $n \geq 2$.

Assume $h = s + 2$. Then $2n^2 - (s^2 + s - 2)n + 2s = 0$, that is, $n = \frac{s^2 + s - 2 \pm \sqrt{\Delta'}}{4}$ where $\Delta' = s^4 + 2s^3 - 3s^2 - 20s + 4$.

Since $\Delta' = (s^2 + s - 2)^2 - 16s$ and Δ' is a square, we have $\Delta' \leq (s^2 + s - 3)^2$, that is, $2s^2 - 14s - 5 \leq 0$ so $s \leq 7$. It is easy to see that also for $3 \leq s \leq 7$ Δ' is never a square.

Then $h = s + 1$. If follows that $s = n + 1$. Hence **K** contains a line of length $n + 1$ and moreover $k = 2n + 2$. So, by Proposition 2.1, **K** is the union of two skew lines of length $n + 1$. □

It remains to study the case when **K** is a cap.

3 The case when K is a cap

From (16) we have the following equation of second degree in k

$$(n+1)k^2 - [h(n^2 + n + 1) + (n^2 + 2n + 2)]k + 2h(n^3 + n^2 + n + 1) = 0.$$

Moreover, by Proposition 2.1 and the remarks preceding Lemma 2.1, we know that if K is a cap then $3 \leq h \leq n/2 + 2$ and $2n + 3 \leq k \leq n^2/2 + 2$.

We can now prove the following lemma

Lemma 3.1 *In* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *there are no caps of type* $(2, h)$ *with respect to planes*

Proof : Assume **K** is a cap. Let r be a 2-secant of **K**. The number of 2-secant planes through r is: $\frac{k-2}{h-2}$. Counting line-plane pairs (r, π) with r a 2-secant line, π a h-secant plane and $r \subset \pi$, we have:

$$\frac{k-2}{h-2}\frac{k(k-1)}{2} = \frac{h(h-1)}{2}\beta,$$

with β the number of h-secant planes. It follows that $\beta = \frac{k(k-1)(k-2)}{h(h-1)(h-2)}$. On the other hand, from (14) we have $\beta = \frac{k(k-1)(n+1) - k(n^2+n+1)}{h(h-2)}$ comparing the two values of β so obtained we get

$$k(k-1)(k-2) = (h-1)(k(k-1)(n+1) - k(n^2+n+1))$$

and, dividing everything by k we get another equation:

$$k^2 - (hn + h + 2 - n)k - n^2 + hn^2 - 2n + 2h + 2hn = 0. \qquad (20)$$

solving this last equation in k we get

$$k = \frac{hn + h + 2 - n \pm \sqrt{(h^2 - 6h + 5)n^2 + 2(h^2 - 3h + 2)n + h^2 - 4h + 4}}{2}.$$

Moreover if

$$k = \frac{hn + h + 2 - n - \sqrt{(h^2 - 6h + 5)n^2 + 2(h^2 - 3h + 2)n + h^2 - 4h + 4}}{2}$$

then, for $h \geq 5$, we have $(h-5)n^2 + 6n - 2h > 0$ (since $n > h$), and so

$$k < \frac{hn + h + 2 - n - ((h-5)n + h + 2)}{2} = 2n + 2$$

which is a contradiction since $k \geq 2n + 3$. Hence for $h \geq 5$

$$k = \frac{hn + h + 2 - n + \sqrt{(h^2 - 6h + 5)n^2 + 2(h^2 - 3h + 2)n + h^2 - 4h + 4}}{2}$$

and so

$$k > \frac{hn + h + 2 - n}{2} + \frac{(h-5)n + h + 2}{2}. \tag{21}$$

On the other hand, equation (16) minus $n+1$ times equation (20) gives:

$$(2n^2 - hn + n)k - hn^3 - n^3 - 3n^2 + hn^2 + 2hn - 2n = 0. \tag{22}$$

Solving this last equation for k gives:

$$k = \frac{hn^3 + n^3 + 3n^2 - hn^2 - 2hn + 2n}{2n^2 - hn + n} = \frac{hn + n - h}{2} + \frac{h^2 + 5}{4} + \frac{(h^2 - 1)(h - 3)}{4(2n - h + 1)}$$

Comparing this last value of k with (21) gives

$$n - h + 1 + \frac{h^2 + 5}{4} + \frac{(h^2 - 1)(h - 3)}{4(2n - h + 1)} > \frac{(h-5)n + h + 2}{2}$$

i.e.

$$(h-7)n^2 - (h^2 - 7h + 2)n - h^2 + h < 0.$$

So, if $h > 7$, then:

$$n = \frac{h^2 - 7h + 2 + \sqrt{h^4 - 10h^3 + 21h^2 + 4}}{2(h-7)} < \frac{2h^2 - 12h}{2(h-7)} = h + 1 + \frac{7}{h-7}$$

which is a contradiction since $h \leq n/2 + 2$.

Hence $h \leq 7$.

For $h = 7$ we have $(2n^2 - 6n)k + 8n^3 + 4n^2 - 12n = 0$ so $k = 2\frac{2n^2-n-3}{n-3} = 4n + 10 + 24/(n-3)$ which is not an integer for $n > 27$.

For $h = 6$ we have $(2n^2 - 5n)k + 3n^2 - 7n^3 + 10n$ hence $k = 7/2n + 29/4 + 105/4(2n-5)$. Hence k is not integer for $2n - 5 > 105$, i.e. for $n > 55$.

For $h = 5$ we have $k = 3n + 5 + 3/(n-2)$. Hence k is not integer for $n > 5$.

For $h = 4$ we have $k = 5/2n + 13/4 + 14/4(2n-3)$. Hence k is not integer for $n > 9$.

For $h = 3$ we have $k = 2n+2$, which is a contradiction since $k \geq 2n+3$.

For $h = 7$ and $n \leq 27$, $h = 6$ and $n \leq 55$, $h = 5$ and $n \leq 5$ and for $h = 4$ and $n \leq 9$ we get a few cases in which k is a integer, but also in these cases using equation (20) we get a contradiction. □

From Lemma 2.1 and Lemma 3.1 we get the main Theorem:

Theorem 3.1 *Let $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ be a non-degenerate finite planar space of order n satisfying Property (i) and let* $\mathbf{K}$ *be a proper subset of $\mathcal{S}$ meeting every plane in either 2 or h points. Then* $\mathbf{K}$ *is a pair of skew lines (both of length $n+1$) of $(\mathcal{S}, \mathcal{L}, \mathcal{P})$.*

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