A Sufficient Condition for Quasi-Claw-Free Hamiltonian Graphs

Jou-Ming Chang[1,*] and An-Hang Chen[1,2]

[1] Department of Information Management, National Taipei College of Business, Taipei, Taiwan, ROC
[2] Department of Information Management, National Taiwan University of Science and Technology, Taipei, Taiwan, ROC

Abstract. There are several well-known and important hamiltonian results for claw-free graphs but only a few are concerned with quasi-claw-free graphs. In this note, we provide a new sufficient condition for quasi-claw-free hamiltonian graphs.

Keywords: hamiltonian cycles; independent sets; claw-free graphs; quasi-claw-free graphs; asteroidal sets; AT-free graphs

1. Introduction

In this note, $G = (V, E)$ denotes a simple undirected graph (i.e., finite, loopless and without multiple edges) with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* and the *closed neighborhood* of a vertex $v \in V$ are the sets $N(v) = \{u \in V \mid uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. The *square* G^2 of G is the graph with the same vertex set as G and two vertices are adjacent if their distance in G is at most 2. With each pair (u, v) of vertices at distance 2 in G, we associate a set $J(u, v) = \{w \in N(u) \cap N(v) \mid N[w] \subseteq N[u] \cup N[v]\}$. For a subset $S \subset V(G)$, the subgraph of G induced by S is denoted by $G[S]$. Also, we use $G - S$ to denote the graph $G[V \setminus S]$ where $V \setminus S$ is the set $\{v \in V \mid v \notin S\}$. If H is a subgraph of G, then for simplicity, we sometimes write H to mean $V(H)$.

A set of vertices in a graph G is *independent* if no two of them are adjacent. The largest number of vertices in such a set is called the *independent number* of G and is denoted by $\alpha(G)$. The *connectivity* of G, denoted by $\kappa(G)$, is the minimum number of vertices whose removal from G results in

* Corresponding author. (Email: spade@mail.ntcb.edu.tw)

a disconnected or trivial graph. A graph G is *k-connected* if $\kappa(G) \geq k$. A cycle that contains every vertex of G exactly once is called a *hamiltonian cycle.* Thus, a graph G is *hamiltonian* if it possesses a hamiltonian cycle. A classical result, due to Chvátal and Erdös [6], in hamiltonian graph theory is the following

Theorem 1. [6] *For $k \geq 2$, a k-connected graph G is hamiltonian if $\alpha(G) \leq k$.*

A *claw* is a graph on four vertices such that one of them, called the *center*, is adjacent to the other three vertices which themselves are pairwise nonadjacent. A graph G is called *claw-free* if it has no claw as an induced subgraph. For a thorough treatment of claw-free graphs, we refer the reader to [9].

Ainouche, Broersma and Veldman in [3] showed the following analogue of the Chvátal-Erdös theorem for claw-free graphs.

Theorem 2. [3] *For $k \geq 2$, a k-connected claw-free graph G is hamiltonian if $\alpha(G^2) \leq k$.*

Later on, Ainouche [2] extended Theorem 2 to a wider class of graphs called quasi-claw-free graphs. A graph G is *quasi-claw-free* (*QC-free* for short) if each pair (x, y) of vertices at distance 2 satisfies the condition $J(x, y) \neq \emptyset$.

Theorem 3. [2] *For $k \geq 2$, a k-connected QC-free graph G is hamiltonian if $\alpha(G^2) \leq k$.*

In this note, we study the hamiltonicity of QC-free graphs and give a sufficient condition which is similar to Theorems 3 but using the asteroidal number of a graph instead of the independent number of its square. We now introduce the concept of the asteroidal number of graphs as follows.

An *asteroidal triple* (AT for short) of a graph is a set of three vertices such that any two of them are joined by a path avoiding the closed neighborhood of the third. A graph is called *asteroidal triple-free* (AT-free for short) if it does not contain an AT. Lekkerkerker and Boland [14] first introduced the concept of AT to characterize certain special class of graphs called interval graphs. Walter [16] generalized the notion of AT to the so-called asteroidal sets. A set of vertices $A \subseteq V(G)$ is an *asteroidal set* if for every vertex $a \in A$, the set $A \backslash \{a\}$ is contained in one connected component of $G - N[a]$. The *asteroidal number* of a graph G, denoted by $an(G)$, is the maximum size of an asteroidal set in G. Clearly, AT-free graphs are those graphs with asteroidal number at most 2. Also, every asteroidal set is an independent set, and thus we have $an(G) \leq \alpha(G)$ for every graph G. Kloks et al. [12] investigated the complexity of computing asteroidal number for certain special classes of graphs. In particular, it is shown that the polynomial algorithm obtained in [15] for computing the independent number of claw-free graphs can be also used to compute the asteroidal number. For more information about claw-free and/or AT-free graphs and the asteroidal number of graphs the reader can refer to [4],[5],[7],[8],[10],[11], and [13].

2. Main results

Before we introduce our main results, it is certainly important to distinguish the terms "QC-free" and "claw-free" for those graphs with bounded asteroidal number. Lemma 1 below shows that the two terms are equivalent for AT-free graphs. However, this is not true for graphs with asteroidal number at least three. Figure 1 shows a QC-free graph that is not AT-free or claw-free.

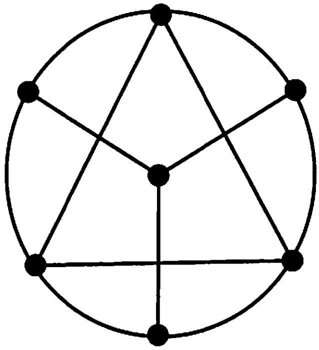
Lemma 1. *Let G be an AT-free graph. Then G is QC-free if and only if G is claw-free.*

Proof. Since every claw-free graph is QC-free, the "if part" is obvious. Conversely, we assume that G is both AT-free and QC-free but it contains a set of vertices $C = \{x_0, x_1, x_2, x_3\}$ that induces a claw with x_0 as the center vertex. Since $x_0 \notin J(x_i, x_j)$, for each pair of vertices x_i, x_j with $i, j \in \{1, 2, 3\}$ and $i \neq j$ there exists a vertex $y_{ij} \in J(x_i, x_j)$. Obviously, $y_{ij}x_k \notin E(G)$ where $k \in \{1, 2, 3\}$ and $k \neq i, j$ otherwise $x_k \in N[y_{ij}] \subseteq N[x_i] \cup N[x_j]$ which is a contradiction. Now, it is easy to check that $x_i y_{ij} x_j$ is a path connecting x_i and x_j in $G - N[x_k]$. This shows that $\{x_1, x_2, x_3\}$ is an AT in G, a contradiction. □

In this note, we show the following theorem whose proof will occur in Section 3.

Theorem 4. *For $k \geq 2$, a k-connected QC-free graph G is hamiltonian if $an(G) \leq k$.*

We note that Theorem 3 and Theorem 4 are incomparable in the sense that neither theorem implies the other. For instance, the cycle C_6 is a QC-free hamiltonian graph with $\alpha(C_6^2) = 2 = \kappa(C_6)$ and $an(C_6) = 3 > \kappa(C_6)$.



A QC-free graph contains an AT $\{x, y, z\}$ and a claw $\{w, x, y, z\}$.

Figure 1

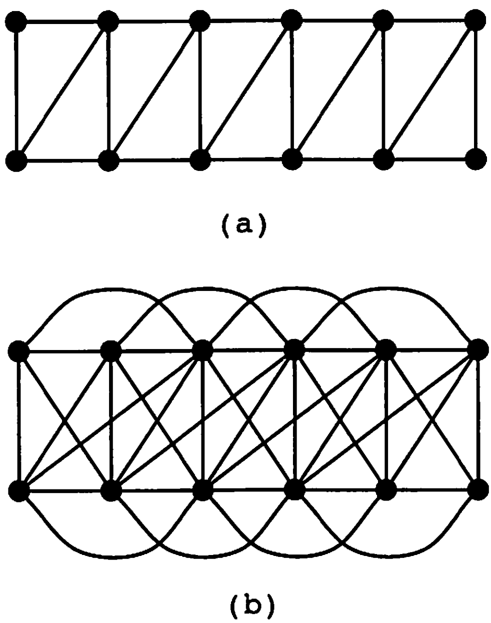
In contrast, Figure 2 depicts a QC-free hamiltonian graph G for which $an(G) = 2 = \kappa(G)$, whereas $\alpha(G^2) = 3 > \kappa(G)$. It is easy to exhibit that both examples can be extended to classes containing infinite number of graphs for which one theorem ensures the hamiltonicity of the graphs and the other one fails to draw the same conclusion. An immediate consequence of Theorems 3 and 4 is the following.

Corollary 1. *For $k \geq 2$, a k-connected QC-free graph G is hamiltonian if* $\min\{an(G), \alpha(G^2)\} \leq k$.

The combination of Lemma 1 and Theorem 4 implies that every 2-connected claw-free AT-free graph is hamiltonian. In fact, Brandstädt et al. [4] designed a linear time algorithm to construct a hamiltonian cycle in a 2-connected claw-free AT-free graph (see Corollary 5.9 in [4]).

3. Proofs

Let C be a longest cycle of a k-connected ($k \geq 2$), non-hamiltonian graph G. By Dirac's theorem, $|C| \geq 2k$. We fix an orientation on C and for $u \in V(C)$ we denote by u^+ and u^- the successor and the predecessor of u on C, respectively. If $u, v \in V(C)$ then uCv denotes the consecutive vertices



(a) A QC-free hamiltonian graph G with $an(G) = 2$; (b) The square G^2 contains an independent set $\{x, y, z\}$.

Figure 2

on C from u to v (including u and v) in the orientation of C. For a subgraph H of G, let $N_C(H)$ denote the set of neighbors of vertices of H that belong to C. Throughout the remainder of this section, we assume that H is a component of $G - C$ and x_0 is a vertex of H. Let $N_C(H) = \{d_1, \ldots, d_m\}$, where the subscripts of d_i's will be taken modulo m. Since G is k-connected, $k \leq m$. We assume that $d_1, \ldots, d_m$ occur on C in the order of their indices and set $C_i = d_i^+ C d_{i+1}^-$ for $i = 1, \ldots, m$. Given a path $P = v_1 v_2 \cdots v_p$ ($p \geq 2$) and a vertex $u \notin V(P)$, we say that u is *P-insertible* [17] if there exists $i \in \{1, \ldots, p-1\}$ such that v_i and v_{i+1} are both adjacent to u. For simplicity, a vertex $u \in V(C_i)$ is called *insertible* if it is $d_{i+1} C d_i$-insertible. We will use the following known results in our proof.

Lemma 2. [1] *Let G be a k-connected graph with $k \geq 2$, C be a longest cycle of G, and H be a component of $G - C$. Then*

(a) *For each* $i \in \{1, \ldots, m\}$, C_i *contains a non-insertible vertex.*
Let x_i *be the first non-insertible vertex along* C_i $(i = 1, \ldots, m)$ *and set* $X = \{x_0, x_1, \ldots, x_m\}$, *where* x_0 *is any vertex of* H. *Also set* $W_0 = V(H)$, $W_i = V(d_i^+ C x_i)$, *and for* $1 \le i \le m$ *choose* $u_i \in W_i$. *Then*
(b) $N(u_i) \cap W_0 = \emptyset$.
(c) *Any set* $W = \{w_i \in W_i \mid 0 \le i \le m\}$ *is an independent set. In particular,* X *is an independent set.*

Throughout, we refer to the set X defined in Lemma 2 as the set $X(G; C, x_0)$.

Lemma 3. [2] *Let* G *be a connected QC-free graph of order* n *and suppose that* G *contains a cycle* C *of length* r *where* $3 \le r < n$. *If* G *contains no cycle of length* $r + 1$, *then* $u^- u^+ \in E$ *for every vertex* u *of* C *that has neighbors outside of* C.

Lemma 4. *Let* G *be a* k*-connected* $(k \ge 2)$, *non-hamiltonian QC-free graph,* C *be a longest cycle of* G, H *be a component of* $G - C$, *and* x_0 *be any vertex of* H. *Then* $X(G; C, x_0)$ *is an asteroidal set of* G.

Proof. Set $X = X(G; C, x_0) = \{x_i \mid 0 \le i \le m\}$ where $m \ge k$. From Lemma 2(a), we have that each x_i is a non-insertible vertex. From Lemma 3, we have that $d_i^- d_i^+ \in E(G)$ for each $i \in \{1, 2, \ldots, m\}$. Thus $x_i \ne d_i^+$ for each $i \in \{1, 2, \ldots, m\}$. By Lemma 2(c), X is an independent set of G. For $i = 0, 1, \ldots, m$, let $G_i = G - N[x_i]$. We will show that $X \setminus \{x_i\}$ is contained in one connected component of G_i. Consider the following two cases.

Case 1: Suppose $i = 0$. Since $x_0 \in V(H)$, $N[x_0] \subseteq V(H) \cup N_C(H)$. Let $G' = G - (V(H) \cup N_C(H))$. Clearly, G' is a subgraph of G_0. Thus $d_j^-, d_j^+ \in V(G')$ and $d_j^- d_j^+ \in E(G')$ for each $j \in \{1, \ldots, m\}$. Therefore, G' contains a cycle that is obtained from C by replacing the edges $d_j^- d_j$ and $d_j d_j^+$ with $d_j^- d_j^+$ for all $j \in \{1, \ldots, m\}$. Since all vertices $x_1, \ldots, x_m$ occur on the cycle, it follows that $X \setminus \{x_0\}$ is contained in one connected component of G' and hence of G_0.

Case 2: Suppose $i > 0$. By Lemma 2(c), $N(x_i) \cap W_j = \emptyset$ for each $j \in \{0, \ldots, m\} \setminus \{i\}$. In the following proof, we always assume $j \in \{1, \ldots, m\} \setminus \{i\}$. By definition, $d_j^+ \in W_j$, $x_i \in W_i$, and from Lemma 3 we have $d_j^- d_j^+ \in E(G)$. Now we prove that $d_j \notin N(x_i)$. Suppose, to the contrary, that $d_j \in N(x_i)$. Let w be a neighbor of d_j in H. Since the distance between w and d_j^+ is two, there exists a vertex, say q, such that $q \in J(d_j^+, w)$. Obviously, $q \ne d_j$ since $x_i \in N[d_j]$ but $x_i \notin N[d_j^+] \cup N[w]$. Since $N(d_j^+) \cap V(H) = \emptyset$, q must be in $N_C(H)$, say d_k. Thus G has a cycle $w' d_k d_j^+ C d_k^- d_k^+ C d_j w H w'$

which is longer than C, where w' is a neighbor of d_k in H and wHw' denotes a path between w and w' in H, a contradiction. Let x_0Pd_j be a path from x_0 to d_j in G such that all internal vertices are contained in H. Then $x_0Pd_jd_j^+Cx_j$ is a path joining x_0 and x_j in G and x_i does not have any neighbor in this path. Since we have already pointed out that no vertices of this path are contained in $N[x_i]$, x_0 and x_j are connected by a path in G_i. This further implies that x_j and $x_{j'}$ are connected by a path in G_i for each pair $j, j' \in \{0, \dots, m\} \setminus \{i\}$. Therefor, $X \setminus \{x_i\}$ is contained in one connected component of G_i. □

Proof of Theorem 4. Suppose that G satisfies the conditions of Theorem 4 but it is not hamiltonian. Let C be a longest cycle of G and $x_0 \in V(G - C)$. Consider the independent set $X = X(G; C, x_0) = \{x_i \mid 0 \leq i \leq m\}$ where $m \geq k$. By Lemma 4, X is an asteroidal set of G and hence we have $an(G) \geq |X| = m + 1 > k$, a contradiction. □

Acknowledgements

The first author was supported by the National Science Council of Taiwan under the grant NSC92-2115-M-141-001. The authors would like to thank the anonymous referee for his careful reading of the paper and helpful suggestions.

References

1. A. Ainouche, An improvement of Fraisse's sufficient condition for hamiltonian graphs, *J. Graph Theory* 16 (1992) 529–543.
2. A. Ainouche, Quasi-claw-free graphs, *Discrete Math.* 179 (1998) 13–26.
3. A. Ainouche, H. J. Broersma and H. J. Veldman, Remarks on hamiltonian properties of claw-free graphs, *Ars. Combin.* 29C (1990) 110–121.
4. A. Brandstädt, F. F. Dragan and E. Köhler, Linear time algorithms for hamiltonian problems on (claw,net)-free graphs, *SIAM J. Comput.* 30 (2000) 1662–1677.
5. J. M. Chang, C. W. Ho and M. T. Ko, Powers of asteroidal triple-free graphs with applications, *Ars Combin.* 67 (2003) 161–173.
6. V. Chvátal and P. Erdös, A note on hamiltonian circuits, *Discrete Math.* 2 (1972) 111–135.
7. D. G. Corneil, S. Olariu and L. Stewart, Asteroidal triple-free graphs, *SIAM J. Discrete Math.* 10 (1997) 399–430.
8. D. G. Corneil, S. Olariu and L. Stewart, Linear time algorithms for dominating pairs in asteroidal triple-free graphs, *SIAM J. Comput.* 28 (1999) 1284–1297.

9. R. Faudree, E. Flandrin and Z. Ryjáček, Claw-free graphs — A survey, *Discrete Math.* 164 (1997) 87–147.
10. H. Hempel and D. Kratsch, On claw-free asteroidal triple-free graphs, *Discrete Appl. Math.* 121 (2002) 155–180.
11. T. Y. Ho, J. M. Chang and Y. L. Wang, On the powers of graphs with bounded asteroidal number, *Discrete Math.* 223 (2000) 125–133.
12. T. Kloks, D. Kratsch and H. Müller, Asteroidal sets in graphs, in: Proc. of WG'97 Conference, Lecture Notes in Computer Science, Vol. 1335, Springer-Verlag, New York, 1997, pp. 229–241.
13. T. Kloks, D. Kratsch and H. Müller, On the structure of graphs with bounded asteroidal number, *Graphs Combin.* 17 (2001) 295–306.
14. C. G. Lekkerkerker and J. C. Boland, Representation of a finite graph by a set of intervals on the real line, *Fund. Math.* 51 (1962) 45–64.
15. N. Sbihi, Algorithme de recherche d'un stable de cardinalite maximum dans un graphe sans etoile, *Discrete Math.* 29 (1980) 53–76.
16. J. R. Walter, Representations of chordal graphs as subtrees of a tree, *J. Graph Theory* 2 (1978) 265–267.
17. C. Q. Zhang, Hamilton cycles in claw-free graphs, *J. Graph Theory* 12 (1988) 209–216.