The Counting Series for Unlabeled Linear Acyclic Hypergraphs *

Zhilong Shan † Bolian Liu ‡

Abstract

A hypergraph is a generalization of an ordinary graph, in which an edge is not limited to contain exactly two vertices, instead, it can contain an arbitrary number of vertices. A number of desirable properties of database schemes have been shown to be equivalent to hypergraphs. In addition, hypergraph models are very important for cellular mobile communication systems. By applying Pólya's Enumeration Theorem (PET) twice, the counting series is derived for unlabeled linear acyclic hypergraphs in this paper.

Keywords: Hypergraph; Linear hypergraph; Hypertree; Bipartite tree; Pólya's Enumeration Theorem

CLC: O157.5

1 Introduction

A hypergraph is an extension of a graph. Recent researches have shown that database schemes, which are usually collections of table skeletons, can be viewed as hypergraphs. Thus, the hypergraph models have proven to be a very useful structure in relational databases for computer science [1][2].

In a cellular network with limited spectral resources, cellular systems have hitherto been modeled mostly by graphs for the purpose of channel assignment. However, hypergraph modeling of cellular systems offers a significant advantage over graph modeling in terms of the total traffic carried by the system. Therefore, hypergraph models can be used to generate efficient fixed and dynamic channel allocation schemes, which outperform those designed using the graph model [3].

Definition 1 Let X be a finite set, ε be a family of subsets of X and $\bigcup_{E_i\in\varepsilon} E_i = X$. Then $H = (X, \varepsilon)$ is called a hypergraph with vertext set

*Supported by Natural Science Foundation of China (No. 10331020) and Guangdong province (No. 04010389).

†Department of Computer Science, South China Normal University, Guangzhou, Guangdong, 510631, P. R. China.

‡Corresponding author,Department of Mathematics, South China Normal University, Guangzhou, Guangdong, 510631, P. R. China.

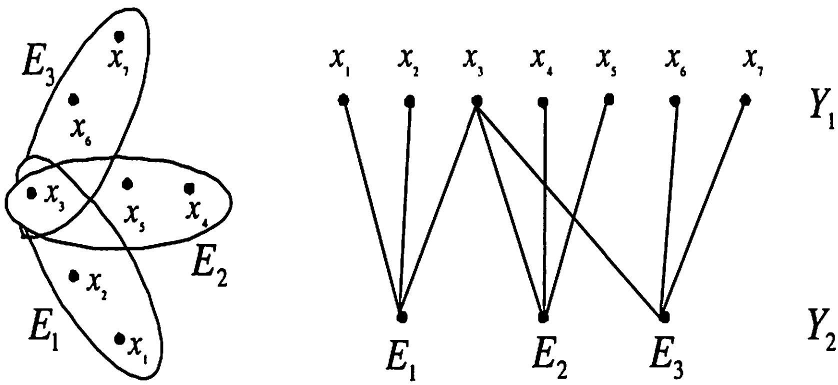

Figure 1: (a) A hypergraph H and (b) the corresponding bipartite graph $G(H)$

X and edge set ε . $\mid X \mid = p$ is called the order of H. If $\mid E_i \mid = M$ for each $E_i \in \varepsilon$, then H is an M-uniform hypergraph. If $E_i \in \varepsilon$ and $\mid E_i \mid = 1$, then E_i is called a loop.

In H, a vertex x is isolated if it belongs to only one edge. Suppose $E \in \varepsilon$, E is an ear of H if $E \neq \phi$ and there exists $E' \in \varepsilon \backslash E$ such that y is isolated for any $y \in E \backslash E'$. The Graham reduction of a hypergraph is the partial hypergraph obtained by removing ears until no more removals are possible [4][5]. H is an acyclic hypergraph if its Graham reduction is empty.

A hypergraph is said to be connected if there is an edge sequence $f_1, f_2, \cdots, f_k$ of ε such that $E_1 = f_1, E_2 = f_k$ for any $E_1, E_2 \in \varepsilon$ and $f_i \cap f_{i+1} \neq \phi, 1 \leq i \leq k-1$.

Definition 2 A hypergraph is called a hypertree if it is acyclic and connected. It is said to be linear if $\mid E_1 \cap E_2 \mid \leq 1$ for every pair of edges $E_1, E_2 \in \varepsilon$.

Definition 3 In a linear hypertree, E_i is called a suspending edge, if it has exactly one point adjacent to the other edges, and $\mid E_i \mid \geq 2$.

Definition 4 A hypergraph H has a corresponding bipartite graph $G(H) = (Y_1, Y_2, E)$, where $Y_1 = X$ and $Y_2 = \varepsilon$ are the vertex sets of the two parts of $G(H)$ respectively, and E is the edge set. That is, we take the edges of H as the vertex set Y_2 of $G(H)$. In $G(H)$, $x_i \in Y_1, E_j \in Y_2$ are adjacent if and only if $x_i \in E_j$ in H. A 3-uniform hypergraph H and its corresponding bipartite graph $G(H)$ are illustrated in Fig. 1 (a) and Fig. 1 (b) respectively.

Definition 5 The corresponding bipartite graph of a linear hypertree H is called a corresponding bipartite tree. A corresponding bipartite rooted tree has one of its points, called the root, distinguished from the others. A corresponding bipartite planted tree is a rooted tree in which the root has degree one.

Definition 6 H is called an unlabeled hypertree if H is a hypertree and its vertices and edges are not labeled.

Enumeration of hypergraphs has been studied by several authors. The fundamental work was done by N. G. de Bruijn and D. A. Klarner [6]. In [7] and [8], methods are proposed to enumerate the number of hypergraphs by applying PET. The number of maximum r-uniform acyclic hypergraphs are derived in [9]. We obtained the counting series for the $(k+1)$-uniform linear acyclic hypergraphs in [10]. Following this result, we derive the counting series for the unlabeled linear acyclic hypergraphs by applying PET twice in this paper.

2 Preliminaries

The following results have been obtained for the linear hypertree [11].

Lemma 1 H is a linear hypertree if and only if $G(H)$ is a tree.

Lemma 2 If H is a linear hypertree without loops and $|\varepsilon| \geq 2$, then H has at least two suspending edges.

If H is a linear hypertree without loops, then there exists a bijection between H and $G(H)$. In fact, we can get a unique $G(H)$ according to H. Conversely, it is possible to construct a unique H from a $G(H)$ as follows. Since H is a linear hypertree without loops, then Y_1 contains pendant vertices by Lemma 2 and Y_2 contains no pendant vertex. Thus the only way to construct a H from $G(H)$ is that the vertex set and the edge set of H correspond to Y_1 and Y_2 respectively. Therefore, there exists a bijection between H and $G(H)$ for linear hypertree without loops.

If H is a linear hypertree with loops, however, there is no bijection between H and $G(H)$. As discussed above, we can still get a unique $G(H)$ according to H. Since some vertices in Y_1 and Y_2 have degree one, we can get a new isomorphic $G'(H)$ in the following way. Take Y_1 and Y_2 of $G(H)$ as the edge set and the vertex set of $G'(H)$ respectively. Hence, we can get two nonisomorphic hypertrees from $G(H)$ and $G'(H)$. That is, a linear hypertree with loops, which has p vertices and q edges, corresponds to the same bipartite tree as a q vertices and p edges linear hypertree with loops does. Therefore, the former and the latter have the same number of hypertrees after counting the bipartite trees.

Using white and black, we color the corresponding bipartite trees of linear hypertrees, and get bicolor trees. Without loss of generality, suppose the vertices in Y_1 and Y_2 are colored by white and black respectively. Whether a linear hypertree H has loops or not, $G(H)$ has no symmetry edges. The number of the symmetry edges is equal to zero, because the endpoints of an edge are different for any edge of $G(H)$.

Lemma 3 If H is a p vertices and q edges linear hypertree without loops, then the degree of the vertex in Y_2 is less than $p-q+1$.

Proof By contradiction. Assume that the degree of some vertex in Y_2 is $p-q+2$, and the degree of the other vertices are two. Then the number of edges in $G(H)$ is $p-q+2+2(q-1)=p+q$. But $G(H)$ is a tree with $p+q$ vertices, there should be $p+q-1$ edges, a contradiction.

3 Counting linear acyclic hypergraphs without loops

For convenience, linear hypertrees without loops and linear acyclic hypergraphs without loops are simplified as linear hypertrees and linear hyperforests respectively in this section. The hypergraphs without loops have no edge containing the same vertices.

To enumerate linear hypertrees H, we need only to enumerate their corresponding bipartite trees $G(H)$ according to Lemma 1 . Once we obtain the counting series for H, then the counting series for linear hyperforests can be obtained consequently.

Since H has no loops, by Lemma 3, the degree of the vertex x in Y_2 satisfies that $2 \leq deg(x) \leq p-q+1$, and the degree of any vertex in Y_1 is q at most.

Let $\bar{H}_{a,b}, H_{a,b}, h_{a,b}$ denote the number of the corresponding bipartite planted trees, corresponding bipartite rooted trees and corresponding bipartite trees of H respectively, where a is the order of $G(H)$ and b is the number of edges in H. Let $\bar{H}(x,y), H(x,y), h(x,y)$ be the counting series for $\bar{H}_{a,b}, H_{a,b}, h_{a,b}$ respectively, where

$$\bar{H}(x,y) = \sum_{a,b} \bar{H}_{a,b} x^a y^b,$$

$$H(x,y) = \sum_{a,b} H_{a,b} x^a y^b,$$

$$h(x,y) = \sum_{a,b} h_{a,b} x^a y^b.$$

Let $|\ \varepsilon\ | = q, |\ X\ | = p$, then $|\ G(H)\ | = p+q$. Using Theorem 4, we can derive the counting series for H. Therefore, the coefficient of $x^s y^t$ in the function counting series of H is the number of such H which contains t edges and $(s-t)$ vertices.

Theorem 4 For corresponding bipartite planted trees, corresponding bipartite rooted trees and corresponding bipartite trees of H, the counting series $\bar{H}(x,y)$, $H(x,y)$, $h(x,y)$ satisfy

$$\bar{H}(x,y) = x^2 y \sum_{k=1}^{p-q} Z(S_k, \sum_{n=0}^{q-1} x Z(S_n, \frac{\bar{H}(x,y)}{x})), \tag{1}$$

$$H(x,y) = x\sum_{n=1}^{q} Z(S_n, \frac{\bar{H}(x,y)}{x}) + xy\sum_{k=2}^{p-q+1} Z(S_k, \sum_{n=0}^{q-1} xZ(S_n, \frac{\bar{H}(x,y)}{x})), \quad (2)$$

$$h(x,y) = H(x,y) - \frac{\bar{H}(x,y)}{x} \times \sum_{n=0}^{q-1} xZ(S_n, \frac{\bar{H}(x,y)}{x}). \quad (3)$$

Proof The counting series for planted trees will be derived firstly. For brevity, the point adjacent to the root is called the next root.

As mentioned previously, the colors of the vertices in Y_1 and Y_2 are white and black respectively, then the color of the root of a plant tree is white. n planted trees can construct a new tree by merging the white roots together. Hence, on applying PET to the symmetric group S_n with $\bar{H}(x,y)/x$ as the figure counting series, we obtain the cycle index $Z(S_n, \bar{H}(x,y)/x)$ as the configuration counting series. But the identified point has not yet been taken into account. Therefore the configuration counting series should be multiplied by x. Summing over all $n \geq 0$, we get

$$\sum_{n=0}^{\infty} xZ(S_n, \frac{\bar{H}(x,y)}{x}). \quad (4)$$

Formula (4) is denoted by $\bar{H}_1(x,y)$. Since the linear hypertrees considered here are not necessarily uniform, we conclude that such $k(1 \leq k \leq p-q)$ new trees can correspond to a new planted tree, where the next root is black with degree $(k+1)$ and the root is white with degree one. Applying PET once more, $Z(S_k, \bar{H}_1(x,y))$ enumerates these planted trees with $\bar{H}_1(x,y)$ as the figure counting series. The new white root and the identified next black root, however, have not yet been taken into account. The proper adjustment is made by multiplying by x^2y. Then sum over all $1 \leq k \leq p-q$. Noting that the number of the white vertices of a corresponding bipartite planted tree is less than q, then we need only to sum over all $0 \leq n \leq q-1$ in formula (4). Hence, formula (1) is obtained.

Next we verify (2). There are two cases of coloring the root of $G(H)$.

Case 1 The root is white in $G(H)$. In analogy to the discussion above, we can get that the figure counting series is $\bar{H}(x,y)/x$. By summing over all $n(1 \leq n \leq q)$, the counting series can be expressed as

$$x\sum_{n=1}^{q} Z(S_n, \frac{\bar{H}(x,y)}{x}). \quad (5)$$

Case 2 The root is black in $G(H)$. It is almost similar to the arguments of deriving the counting series for planted trees. Since the root is black, the configuration counting series is multiplied by xy. But the corresponding bipartite rooted trees have no loops, then the degree of the vertex in Y_2 is not one. Therefore, summing over all $k(2 \leq k \leq p-q+1)$, we get

$$xy \sum_{k=2}^{p-q+1} Z(S_k, \sum_{n=0}^{q-1} xZ(S_n, \frac{\bar{H}(x,y)}{x})). \tag{6}$$

Thus, combining (5) and (6), we have (2) as asserted.

Finally, we verify (3). Let $L(x,y)$ be the counting series for corresponding bipartite trees rooted at an unsymmetric edge. Subject to the result that the corresponding bipartite trees have no symmetry edge, a planted tree and a tree with a white root can correspond to a tree rooted at an unsymmetric edge. By joining the next root of the planted tree and the root of the other tree, then deleting the root of the planted tree, the 1-1 correspondence can be made. Therefore, $L(x,y)$ is

$$L(x,y) = \frac{\bar{H}(x,y)}{x} \times x \sum_{n=0}^{q-1} Z(S_n, \frac{\bar{H}(x,y)}{x}). \tag{7}$$

By Otter's formula, the counting series $h(x,y)$ for trees is expressed in terms of the series $H(x,y)$ for rooted trees and the counting series $L(x,y)$ for trees rooted at a nonsymmetry edge by $h(x,y) = H(x,y) - L(x,y)$ [12]. Hence, (3) is obtained.

Now we count the linear hyperforests.

Let $f_t(x,y)$ be the counting series for the number of corresponding bipartite forests of the linear hyperforests, each of which contains exactly t corresponding bipartite trees. Let $f(x,y)$ be the counting series for all of the general corresponding bipartite forests of the linear hyperforests. Then we have

$$f(x,y) = \sum_{t=1}^{q} f_t(x,y). \tag{8}$$

By PET,

$$f_t(x,y) = Z(S_t, h(x,y)). \tag{9}$$

Hence we have

Theorem 5

$$f(x,y) = \sum_{t=1}^{q} Z(S_t, h(x,y)). \tag{10}$$

Using Mathematica, we compute the following expressions as an example. Suppose $1 \leq p \leq 7$ and $1 \leq q \leq 3$, by Theorem 4 and formula (10), we have

$$\begin{aligned}
\bar{H}(x,y) &= x^3y + x^4y + x^5y + x^6y + x^7y + x^8y + \cdots \\
&\quad + x^5y^2 + 2x^6y^2 + 3x^7y^2 + 4x^8y^2 + 5x^9y^2 + \cdots \\
&\quad + 2x^7y^3 + 6x^8y^3 + 12x^9y^3 + 20x^{10}y^3 + \cdots, \\
H(x,y) &= 2x^3y + 2x^4y + 2x^5y + 2x^6y + 2x^7y + 2x^8y + \cdots \\
&\quad + 3x^5y^2 + 5x^6y^2 + 8x^7y^2 + 10x^8y^2 + 13x^9y^2 + \cdots \\
&\quad + 7x^7y^3 + 17x^8y^3 + 34x^9y^3 + 55x^{10}y^3 + \cdots, \\
h(x,y) &= x^3y + x^4y + x^5y + x^6y + x^7y + x^8y + \cdots \\
&\quad + x^5y^2 + x^6y^2 + 2x^7y^2 + 2x^8y^2 + 3x^9y^2 + \cdots \\
&\quad + 2x^7y^3 + 3x^8y^3 + 6x^9y^3 + 9x^{10}y^3 + \cdots, \\
f(x,y) &= x^3y + x^4y + x^5y + x^6y + x^7y + x^8y + \cdots \\
&\quad + x^5y^2 + 2x^6y^2 + 3x^7y^2 + 4x^8y^2 + 5x^9y^2 + \cdots \\
&\quad + 2x^7y^3 + 4x^8y^3 + 9x^9y^3 + 14x^{10}y^3 + \cdots.
\end{aligned}$$

The table below demonstrates the number of linear acyclic hypergraphs with p vertices and q edges when the hypergraphs have no loops, where the numbers in the first row indicate the orders of hypergraphs, and the numbers in the first column indicate the numbers of edges of hypergraphs.

	1	2	3	4	5	6	7
1	0	1	1	1	1	1	1
2	0	0	1	2	3	4	5
3	0	0	0	2	4	9	14

Table 1: The number of the linear acyclic hypergraphs without loops ($1 \leq p \leq 7$, $1 \leq q \leq 3$).

For linear acyclic hypergraphs with 6 points and 3 edges, their corresponding term in $f(x,y)$ is $9x^9y^3$. It can be checked that the number of the nonisomorphic linear acyclic hypergraphs without loops is 9.

4 Counting linear acyclic hypergraphs

In this section, we will discuss the number of linear acyclic hypergraphs, including hypergraphs without loops and hypergraphs with loops. The linear acyclic hypergraphs have no edge containing the same vertices, except the loops.

Since H has loops, the degree of the vertex x in Y_2 satisfies that $1 \leq deg(x) \leq p$, and the degree of any vertex in Y_1 is q at most.

Similar to the denotation described in Section 3, let $\bar{H}'_{a,b}, H'_{a,b}, h'_{a,b}$ denote the number of corresponding bipartite planted trees, corresponding bipartite rooted trees and corresponding bipartite trees of H respectively. Let $\bar{H}(x,y), H(x,y), h(x,y)$ be the counting series for $\bar{H}'_{a,b}, H'_{a,b}, h'_{a,b}$ respectively. Let $\bar{H}_x(x,y)$ and $\bar{H}_y(x,y)$ be the counting series for corresponding bipartite planted trees whose roots are vertices and edges in H respectively, where

$$\bar{H}'(x,y) = \sum_{a,b} \bar{H}'_{a,b} x^a y^b,$$

$$H'(x,y) = \sum_{a,b} H'_{a,b} x^a y^b,$$

$$h'(x,y) = \sum_{a,b} h'_{a,b} x^a y^b,$$

$$\bar{H}'(x,y) = \bar{H}_x(x,y) + \bar{H}_y(x,y). \tag{11}$$

Theorem 6 For corresponding bipartite planted trees, corresponding bipartite rooted trees and corresponding bipartite trees of H, the counting series $\bar{H}'(x,y)$, $H'(x,y)$, $H'(x,y)$ satisfy

$$\begin{aligned} \bar{H}'(x,y) \quad &= \quad x^2 y \sum_{k=0}^{p-1} Z(S_k, \sum_{n=0}^{q-1} xZ(S_n, \frac{\bar{H}_x(x,y)}{x})) \\ &\quad + x^2 y \sum_{k=0}^{q-1} Z(S_k, \sum_{n=0}^{p-1} xyZ(S_n, \frac{\bar{H}_y(x,y)}{xy})), \end{aligned} \tag{12}$$

$$H'(x,y) = x \sum_{n=1}^{q} Z(S_n, \frac{\bar{H}_x(x,y)}{x}) + xy \sum_{n=1}^{p} Z(S_n, \frac{\bar{H}_y(x,y)}{xy})), \tag{13}$$

$$h'(x,y) = H'(x,y) - \frac{\bar{H}_x(x,y)}{x} \times \frac{\bar{H}_y(x,y)}{xy}. \tag{14}$$

Proof Firstly, we take the counting series for planted trees whose roots are the vertices in H into consideration. The method is similar to that one deriving the formula (1). The difference is that we can sum over all $k(0 \le k \le p-1)$, since the degree of the vertex in Y_2 may be one or p.

Secondly, we consider the planted trees whose roots are the edges in H. Applying the method of deriving $\bar{H}_x(x,y)$, we can get the figure counting series $\bar{H}_y(x,y)/(xy)$ by applying PET once. The configuration counting series should be multiplied by xy, since the identified point in Y_2 has not yet been taken into account. From the discussion above and (11), (12) is obtained.

In analogy to the method in the third section, we derive the counting series for rooted trees. There are two cases of coloring the roots of $G(H)$.

Case 1 The root is white in $G(H)$. We can get the counting series without difficulties

$$x\sum_{n=1}^{q} Z(S_n, \frac{\bar{H}_x(x,y)}{x}). \tag{15}$$

Case 2 The root is black in $G(H)$. Similar arguments show that

$$xy\sum_{n=1}^{p} Z(S_n, \frac{\bar{H}_y(x,y)}{xy}). \tag{16}$$

(15) plus (16) is formula (13).

Finally, we attempt to get the counting series for corresponding bipartite trees. Similar to the deriving of $h'(x,y)$, let $L'(x,y)$ be the counting series for corresponding bipartite trees rooted at an unsymmetric edge. Knowing that the corresponding bipartite trees have no symmetry edge, a planted tree with a black root and a planted tree whose root is white can correspond to a tree rooted at an unsymmetric edge. By joining the next roots of the two planted trees, then deleting the roots of the two planted trees, the 1-1 correspondence can also be made. So $L'(x,y)$ is

$$L'(x,y) = \frac{\bar{H}_x(x,y)}{x} \times \frac{\bar{H}_y(x,y)}{xy}. \tag{17}$$

By Otter's formula $h'(x,y) = H'(x,y) - L'(x,y)$ again, (14) is obtained.

Now we count the linear acyclic hypergraphs.

Let $f'(x,y)$ denote the counting series of all the general corresponding bipartite hypergraphs of the linear acyclic hypergraphs. By adopting the same method of deriving linear hyperforests without loops, we have

Theorem 7

$$f'(x,y) = \sum_{t=1}^{q} Z(S_t, h'(x,y)). \tag{18}$$

As an example, we can also get the expressions for linear acyclic hypergraphs when $1 \le p \le 7$ and $1 \le q \le 3$. From Theorem 6 and formula (18), we have

$$\begin{aligned}\bar{H}'(x,y) \quad = \quad & 2x^2y + x^3y + x^4y + x^5y + x^6y + x^7y + x^8y + \cdots \\ & +x^3y^2 + 2x^4y^2 + 3x^5y^2 + 4x^6y^2 + 5x^7y^2 + 6x^8y^2 \\ & +7x^9y^2 + \cdots + x^4y^3 + 3x^5y^3 + 8x^6y^3 + 14x^7y^3 \\ & +23x^8y^3 + 33x^9y^3 + 46x^{10}y^3 + \cdots,\end{aligned}$$

$$
\begin{aligned}
H'(x,y) &= 2x^2y+2x^3y+2x^4y+2x^5y+2x^6y+2x^7y \\
&+2x^8y+\cdots+2x^3y^2+4x^4y^2+7x^5y^2+9x^6y^2 \\
&+12x^7y^2+14x^8y^2+17x^9y^2+\cdots+2x^4y^3 \\
&+7x^5y^3+18x^6y^3+34x^7y^3+55x^8y^3+81x^9y^3 \\
&+113x^{10}y^3+\cdots, \\
h'(x,y) &= x^2y+x^3y+x^4y+x^5y+x^6y+x^7y+x^8y+\cdots \\
&+x^3y^2+x^4y^2+2x^5y^2+2x^6y^2+3x^7y^2+3x^8y^2 \\
&+4x^9y^2+\cdots+x^4y^3+2x^5y^3+4x^6y^3+7x^7y^3 \\
&+10x^8y^3+14x^9y^3+19x^{10}y^3+\cdots, \\
f'(x,y) &= x^2y+x^3y+x^4y+x^5y+x^6y+x^7y \\
&+x^8y+\cdots+x^3y^2+2x^4y^2+3x^5y^2+4x^6y^2 \\
&+5x^7y^2+6x^8y^2+7x^9y^2+\cdots+x^4y^3+3x^5y^3 \\
&+7x^6y^3+12x^7y^3+18x^8y^3+26x^9y^3+35x^{10}y^3+\cdots.
\end{aligned}
$$

Likewise, the table for the number of the linear acyclic hypergraphs with p vertices and q edges can be presented below, where the numbers in the first row indicate the orders of hypergraphs, and the numbers in the first column indicate the numbers of edges of hypergraphs.

	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	3	7	12	18	26	35

Table 2: The number of the linear acyclic hypergraphs ($1 \leq p \leq 7$, $1 \leq q \leq 3$).

When $p = 6$ and $q = 3$, the corresponding item in $f(x,y)$ is $26x^9y^3$. We also can check that the number of the nonisomorphic linear acyclic hypergraphs is 26.

References

[1] Jianfang Wang and Tony T. Lee, Paths and Circles of Hypergraphs, Science in China(Series A), 42 (1999), No. 1,1-12.

[2] Jianfang Wang and Haizhu Li, Counting Acyclic Hypergraphs, Science in China(Series A), 44 (2001), No. 2, 220-224.

[3] S. Sarkar and K. N. Sivarajan, Hypergraph Models for Cellular Mobile Communication Systems, IEEE Trans. Veh. Technol., 47 (1998), No. 2, 460-471.

[4] D. Maier, The Theory of Relational Databases, Computer Science Press, Rockvile, 1983.

[5] C. Beeri, R. Fagin, D. Maier and M. Yannakakis, On the Desirability of Acyclic Database Schemes, Journal of the ACM (JACM), 30 (1983), 479-513.

[6] N. G. de Bruijin and D. A. Klarner, Enumeration of generalized graphs, Indag. Math., 31 (1969),1-9.

[7] M. Hegde and M. R. Sridharan, Enumeration of Hypergraphs. Discrete Math. 45 (1983), No. 2-3, 239-243.

[8] T. Ishihara, Enumeration of Hypergraphs. European J. Combin. 22 (2001), No. 4, 503-509.

[9] Jianfang Wang and Haizhu Li, Enumeration of Maximum Acyclic Hypergraphs. Acta Math. Appl. Sin. Engl. Ser. 18 (2002), No. 2, 215-218.

[10] Zhilong Shan and Bolian Liu, The Counting Series for $(k+1)$-uniform Linear Acyclic Hypergraphs, Chinese Science Bulletin, 3 (2001)197-200.

[11] Bolian Liu, Enumeration of Hypertrees, Appl. Math., A Journal of Chinese Universities, 9 (1988)359-363.

[12] F. Harary and E. M. Palmer, Graphical Enumeration, Academic Press, New York, 1973.