Domination in Planar Graphs with Small Diameter II

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ABSTRACT. MacGillivray and Seyffarth (J. Graph Theory **22** (1996), 213–229) proved that planar graphs of diameter three have domination number at most ten. Recently it was shown (J. Graph Theory **40** (2002), 1–25) that a planar graph of diameter three and of radius two has domination number at most six while every sufficiently large planar graph of diameter three has domination number at most seven. In this paper we improve on these results. We prove that every planar graph of diameter three and of radius two has total domination number (and therefore domination number) at most five. We show then that every sufficiently large planar graph of diameter three has domination number at most six and this result is sharp, while a planar graph of diameter three has domination number at most nine.

1 Introduction

In this paper we continue the study of the domination number of planar graphs with small diameter started by MacGillivray and Seyffarth [5] and continued in [2]. For diameter 2, MacGillivray and Seyffarth [5] proved that planar graphs have domination number at most 3. Thereafter, it was proven in [2] that there is a unique planar graph of diameter 2 with domination number 3. On the other hand, a tree of radius 2 and diameter 4 can obviously have arbitrarily large domination number.

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So the remaining question is what happens when the diameter is 3. MacGillivray and Seyffarth [5] proved that planar graphs of diameter 3 have domination number at most ten. It was proven in [2] that every planar graph of diameter 3 and of radius 2 has domination number at most 6 and that every sufficiently large planar graph of diameter 3 has domination number at most 7.

In this paper, we improve on the results of [2] and [5]. We prove that every planar graph of diameter 3 and radius 2 has domination number (and indeed total domination number) at most 5. We then show that every sufficiently large planar graph of diameter 3 has domination number at most 6 and this result is sharp, while a planar graph of diameter 3 has domination number at most 9. We use the same approach as in [2] but with more detailed analysis and with the use of a computer.

For notation and graph theory terminology we in general follow [1]. So, for a graph G, if $X, Y \subseteq V(G)$, then we say that X *dominates* (resp., *totally dominates*) Y if every vertex of $Y - X$ (resp., of Y) is adjacent to some vertex of X. In particular, if X dominates (resp., totally dominates) $V(G)$, then X is called a *dominating set* (resp., *total dominating set*) of G. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set, while the *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. Domination and its variations in graphs are now well studied (see [1, 3, 4]). To simplify the notation, if X dominates Y we write $X \succ Y$ while if X totally dominates Y we write $X \succ_t Y$. Further, if a vertex u is adjacent with a vertex v, we write $u \sim v$, while if u and v are nonadjacent, we write $u \not\sim v$. We denote the eccentricity of a vertex v in G by $\mathrm{ecc}_G(v)$, or simply $\mathrm{ecc}(v)$ if G is clear from the context. The subgraph induced by a subset $S \subseteq V(G)$ is denoted by $G[S]$.

2 Results

MacGillivray and Seyffarth [5] proved that planar graphs with diameter 3 have bounded domination numbers.

Proposition 1 ([5]) *A planar graph of diameter* 3 *has domination number at most* 10.

They gave an example (see Figure 1) of a planar graph of diameter 3 with domination number 6. The following three results were proven in [2].

Proposition 2 ([2]) *Every planar graph of diameter* 3 *and of radius* 2 *has domination number at most* 6.

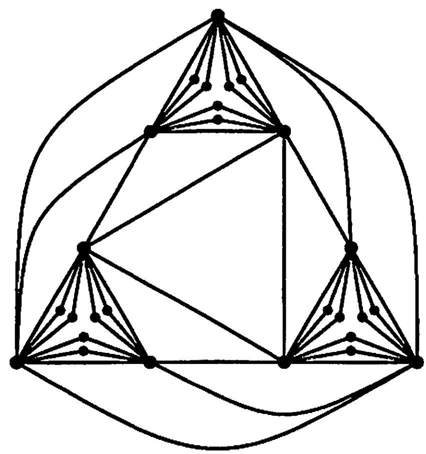

Figure 1: A planar graph with diameter 3 and domination number 6

Proposition 3 ([2]) *For a sufficiently large planar graph G of radius and diameter* 3, *there exists a planar graph G' of radius at most* 2 *and diameter at most* 3 *such that* $\gamma(G) \leq \gamma(G') + 1$.

Proposition 4 ([2]) *Every sufficiently large planar graph of diameter* 3 *has domination number at most* 7.

Our aim is to improve the bounds in Propositions 1, 2, and 4. We shall prove:

Theorem 1 *Every planar graph of diameter* 3 *and of radius* 2 *has total domination number at most* 5.

This is proven in Section 3. As an immediate consequence of Proposition 3 and Theorem 1, we have the following result.

Theorem 2 *Every sufficiently large planar graph of diameter* 3 *has domination number at most* 6, *and this bound is sharp.*

The sharpness is shown by the graph of Figure 1, which can be made arbitrarily large by duplicating any of the vertices of degree 2. Furthermore, by adding edges joining vertices of degree 2, it is possible to construct such a planar graph with minimum degree equal to 3. While this theorem shows that there are finitely many planar graphs of diameter 3 with domination number more than 6, we do not know of any.

As observed in [2, Section 7], the maximum domination number of a planar graph of diameter 3 is at most four more than the maximum for radius 2 and diameter 3. Hence the following result is an immediate consequence of Theorem 1, and improves on Proposition 1.

Theorem 3 *Every planar graph of diameter* 3 *has domination number at most* 9.

There are similar results for total domination number. The following result is proven in Section 4.

Theorem 4 *Every sufficiently large planar graph of diameter* 3 *has total domination number at most* 7.

The maximum total domination number of a planar graph of diameter 3 is at most five more than the maximum for radius 2 and diameter 3. For example, if G has diameter and radius 3, then shrink the open neighborhood of a vertex v of minimum degree to a single vertex x. The resultant graph G' has diameter at most 3 and radius 2. Also, $\gamma_t(G') \leq \gamma_t(G) + 5$ since x is in every minimum total dominating set S' of G', and so $(S' - \{x\}) \cup N[v]$ totally dominates G.

Hence the following result is an immediate consequence of Theorem 1.

Theorem 5 *Every planar graph of diameter* 3 *has total domination number at most* 10.

It is not known if the bound in Theorem 5 is sharp. We showed above that there are finitely many planar graphs of diameter 3 with total domination number more than 7. It is entirely possible that there is none.

3 Proof of Theorem 1

In this section we prove Theorem 1. We use the same approach used to prove Theorem 4 in [2], but with more detailed analysis and with the use of a computer.

The focus is on cut-cycles. Note that in a planar graph of diameter 3, there cannot on both sides of a cut-cycle be vertices not dominated by the cycle. We define a **basic cycle** as follows. Let vertex x have eccentricity 2 in G. Then a basic cycle C is an induced cycle $x, v_1, v_2, \ldots, v_r, x$ such that on both sides of the cycle there is a vertex whose neighbors on the cycle are a subset of the two consecutive vertices fartherest from x, specifically $v_{(r-1)/2}$ and $v_{(r+1)/2}$ if r is odd, and $v_{r/2}$ and $v_{r/2+1}$ if r is even. A **special basic cycle** is one with the added condition that there is on the dominated side of the cycle a vertex with only one neighbor on the cycle and that neighbor is not x.

Our strategy is as follows. In Subsection 3.1 we show the existence of a special basic cycle of length 3 or 4 or of a basic 5-cycle in G. Thereafter, in Subsection 3.2 we prove some lemmas about how to totally dominate vertices at distance 2 from two or more vertices, in particular the Divider Lemma. In Subsection 3.3 we use these lemmas to bound the total domination number when there exists a special basic cycle of length 3 or 4. In Subsection 3.4 we use a computer to complete the proof of Theorem 1 when there exists a basic 5-cycle.

3.1 Basic Cycles Exist

Let G be a plane graph of radius 2 and diameter 3 with central vertex x. We say that it is *edge-minimal* if for every edge e of G, $\text{diam}(G-e) > 3$ or $\text{ecc}(x) > 2$ in $G-e$. Clearly, we may assume that G is edge-minimal in proving Theorem 1 (since removing edges can only increase the total domination number).

The following improves considerably on Lemma 10 of [2].

Lemma 6 *Let G be an edge-minimal plane graph of radius 2 and diameter 3 with central vertex x. Then, $\gamma_t(G) \le 5$, or there exists a special basic triangle, special basic 4-cycle, or basic 5-cycle.*

Proof. Suppose there is neither a special basic cycle of length 3 or 4 nor a basic 5-cycle in G.

Let $Y = V(G) - N[x]$. Let M be a minimal subset of $N(x)$ that dominates Y. The set M exists since $\text{ecc}(x) = 2$. Let $|M| = m$. Since $\gamma_t(G) \le m+1$, we may assume $m \ge 5$.

Let the vertices of M be $n_0, n_1, \ldots, n_{m-1}$ in cyclic order (clockwise) around x in G. Let Y_i' be the set of vertices of Y whose only neighbor in M is n_i. By the minimality of M, each Y_i' is nonempty. Let $Y_0, Y_1, \ldots, Y_{m-1}$ be a partition of Y such that $Y_i \subseteq N(n_i)$ for each i. Necessarily, $Y_i' \subseteq Y_i$ for each i.

We now choose a vertex $y_i \in Y_i'$ for each i. If there is a vertex of Y_i' adjacent to both a vertex of Y_{i-1} and a vertex of Y_{i+1} (where addition is taken modulo m), then this vertex is unique by the planarity of G and we choose this as y_i. If there is no such vertex of Y_i', then we let y_i be any vertex of Y_i' adjacent to a vertex of Y_{i-1} or a vertex of Y_{i+1}, if such a vertex exists, failing which we let y_i be any vertex of Y_i'.

As in [2], we say that two neighbors u_1 and u_2 of x are **separated** if there is a vertex of M between u_1 and u_2 in both directions around x in the embedding of G. We define type-1, type-2 and type-3 edges as follows. A **type-1** edge joins vertices $u_1, u_2 \in N(x)$ such that u_1 and u_2 are separated. A **type-2** edge joins vertices $u_1 \in N(x)$ and $v_2 \in Y$ with v_2 dominated by a vertex u_2 of M such that u_1 and u_2 are separated. A **type-3** edge joins vertices $v_1, v_2 \in Y$ with v_1 and v_2 dominated by vertices u_1 and u_2 of M, respectively, such that u_1 and u_2 are separated.

Claim 6.1 *There is no type-*1*, type-*2 *or type-*3 *edge.*

Proof. Let e be an edge. Suppose $e = u_1u_2$ is a type-1 edge. Then there is a vertex n_i of M inside the cycle $C: x, u_1, u_2, x$ and a vertex n_j of M outside the cycle C. Since the vertices y_i and y_j are not dominated by x, C is a basic triangle.

Without loss of generality, C dominates its inside. By assumption, C is nonspecial. That is, every vertex of Y inside C is adjacent to both u_1 and u_2. By planarity, y_i is the only vertex of Y inside the triangle C, since each such vertex must be adjacent to all of u_1, u_2 and n_i. But then we can remove the edge n_iy_i, contradicting the minimality of G. Hence, G has no type-1 edge.

Suppose $e = u_1v_2$ is a type-2 edge. Then again there is a vertex n_i of M inside the cycle $C: x, u_1, v_2, u_2, x$ and a vertex n_j of M outside the cycle C with vertices y_i and y_j not dominated by $\{x, u_2\}$. Furthermore, since there is no type-1 edge, C is induced and hence a basic 4-cycle.

Without loss of generality, C dominates its inside. By assumption, C is nonspecial. That is, every vertex of Y inside C is adjacent to at least two vertices on the cycle. In particular, since $u_2 \in M$, y_i is adjacent to u_1 and v_2 (and not to u_2). Hence by planarity, each vertex of Y inside C is adjacent to at most one of u_1 and u_2, and therefore, since C is nonspecial, is adjacent to v_2. But then we can remove the edge n_iy_i, contradicting the minimality of G. Hence, G has no type-2 edge.

If $e = v_1v_2$ is a type-3 edge, then again there is a vertex y_i both inside and outside the cycle $C: x, u_1, v_1, v_2, u_2, x$ not dominated by $\{x, u_1, u_2\}$. Furthermore, since there is no type-1 or type-2 edge, C is induced and hence a basic 5-cycle, a contradiction. Hence, G has no type-3 edge. □

Claim 6.2 *If i, j are not consecutive, then*
(i) *there is no edge from Y_i to Y_j, and*
(ii) $d(n_i, y_j) = 3$.

Proof. If there is an edge from Y_i to Y_j, then the edge is type-3, contradicting Claim 6.1. Now, by the lack of type-2 edges, n_i and y_j are not adjacent. Suppose they have a common neighbor a. This creates a 5-cycle x, n_i, a, y_j, n_j, x. By Claim 6.1, it can be checked that this cycle is induced. (For example, if $x \sim a$ then either $n_i a$ is type-1 or $a y_j$ is type-2.) Thus it is a basic 5-cycle (the desired vertices are y_{i-1} and y_{i+1}). □

As in [2], for an index i, if $y_i \sim y_{i+1}$ we define the **lozenge** L_i as the region inside the 5-cycle $x, n_i, y_i, y_{i+1}, n_{i+1}, x$. We denote the set of vertices of Y inside lozenge L_i by X_i. The following result extends (and corrects) Claim 2 from [2].

Claim 6.3 (i) *If there is a path of length 2 from $y \in Y_i$ to $z \in Y_{i+2}$, then the intermediate vertex is y_{i+1} or n_{i+1}.*

(ii) *If $m \geq 6$, then the shortest y_i–y_{i+3} path is $y_i, y_{i+1}, y_{i+2}, y_{i+3}$, or if $m = 6$ is possibly $y_i, y_{i-1}, y_{i-2}, y_{i+3}$.*

(iii) *If $m \geq 6$ and $y \in X_i$, then the shortest y–y_{i+3} path is $y, y_{i+1}, y_{i+2}, y_{i+3}$ and the shortest y–y_{i-2} path is y, y_i, y_{i-1}, y_{i-2}. In particular, y is adjacent to both y_i and y_{i+1}.*

(iv) *If $m = 5$ and $y \in X_i$, then y is adjacent to y_i or y_{i+1}.*

(v) *A path of length at most 3 from y_i to y_{i+2} either uses a vertex of Y'_{i+1} or, if $m = 5$, is possibly $y_i, y_{i-1}, y_{i-2}, y_{i+2}$.*

Proof. (i) Assume the path is y, a, z. If $a \in N(x)$, then $a = n_{i+1}$ else one of the edges ay or az would be a type-2 edge. So assume $a \in Y$; then $a \in Y_{i+1}$ by Claim 6.2. Further, by Claim 6.1, vertex a cannot be placed in Y_i or Y_{i+2}; that is, $a \in Y'_{i+1}$, and by the definition of y_{i+1}, in fact $a = y_{i+1}$.

(ii) The same as the proof of part (ii) of Claim 2 in [2].

(iii) Assume the shortest y–y_{i+3} path is y, a, b, y_{i+3}. Then $a \notin \{n_i, n_{i+1}\}$, since each of n_i and n_{i+1} is distance at least 3 from y_{i+3} (by Claim 6.2). So $a = y_{i+1}$ and by part (i), $b = y_{i+2}$.

(iv) This follows by considering the distance between y and y_{i+3}.

(v) Consider a walk y_i, a, b, y_{i+2} (where possibly $a = b$). Then, $a, b \neq n_{i+1}$. If the walk uses a vertex of Y_{i-1}, then, by part (i), it must be the path $y_i, y_{i-1}, y_{i-2}, y_{i+2}$. So assume the walk does not use a vertex of Y_{i-1}.

Since there is no type-1 or type-2 edge, at least one of $a, b \in Y$, say b. Suppose $b \notin Y'_{i+1}$. Then, $b \sim n_i$ or $b \sim n_{i+2}$. However, since the edge

by_{i+2} is not a type-3 edge, $b \not\sim n_i$ and so $b \sim n_{i+2}$. If $b \in Y_{i+2}$, then by part (i), $a = y_{i+1}$. So assume $b \in Y_{i+1}$; in particular, $b \sim n_{i+1}$. If a were in $N(x)$, then by planarity a would lie between n_i and n_{i+1} in the ordering of neighbors of x and so edge ab would be type-2. So $a \in Y$. Since edge ab is not type-3, it follows that $a \not\sim n_i$, and so $a \in Y'_{i+1}$, as desired. □

By Claim 6.3, we may assume that $m \leq 7$, for otherwise $d(y_0, y_4) > 3$. So there are three cases.

Case 1: $m = 7$.

Then, by Claim 6.3, a shortest path from y_i to y_{i+3} must have as intermediate vertices y_{i+1} and y_{i+2}. That is, there is an edge from y_i to y_{i+1} for all i. Suppose there is another vertex of Y; say inside the lozenge L_0. Then it is too far from y_4, a contradiction. Hence, $Y = \{y_0, y_1, \ldots, y_6\}$. Thus, $\{x, n_0, y_0, y_3, y_4\}$ totally dominates G, and so $\gamma_t(G) \leq 5$.

Case 2: $m = 6$.

Consider the shortest y_0–y_3, y_1–y_4 and y_2–y_5 paths. Since by Claim 6.3(ii) each interior vertex on any such path is a vertex y_i for some i, it follows that there are at least five consecutive edges y_iy_{i+1}, starting at y_0 say.

First suppose that there are only five such consecutive edges, i.e., $y_0 \not\sim y_5$. Then, it follows by Claim 6.3(iii) that no vertex of Y is inside L_0, L_1, L_3 or L_4, while each vertex of X_2, if any, is adjacent to both y_2 and y_3. Hence, $\{x, n_0, n_5, y_2, y_3\}$ totally dominates G, and so $\gamma_t(G) \leq 5$.

Second suppose $y_0 \sim y_5$. By Claim 6.3(iii), each vertex of X_i, if any, is adjacent to both y_i and y_{i+1}. Hence, by planarity (recall that each vertex of X_i is adjacent to at least one of n_i or n_{i+1}), $|X_i| \leq 1$. Since $\text{diam}(G) = 3$, L_i and L_{i+3} cannot both contain elements of Y. Hence either (a) three consecutive X_i are empty, say X_0, X_1 and X_2, or (b) every alternate X_i is empty, say X_1, X_3 and X_5. If (a) holds, then $\{x, n_1, y_3, y_4, y_5\}$ totally dominates G, while if (b) holds, then $\{x, n_0, y_0, y_3, y_4\}$ totally dominates G. In any event, $\gamma_t(G) \leq 5$.

Case 3: $m = 5$.

We show that at least three of the edges y_iy_{i+1} are present. This result is clear if each y_i is adjacent to at least one of y_{i-1} and y_{i+1}. Suppose, then, there is some i such that $y_i \not\sim y_{i-1}$ and $y_i \not\sim y_{i+1}$. We may assume that y_2 is adjacent to neither y_1 nor y_3. By the way y_2 is chosen, each vertex of Y'_2 is adjacent to neither y_1 nor y_3. So by Claim 6.3(v), a shortest y_1–y_3 path must be the path y_1, y_0, y_4, y_3. Once again, at least three of the edges y_iy_{i+1} must be present.

Let q be the number of edges y_iy_{i+1} present. Then $3 \leq q \leq 5$. We consider four subcases depending on the value of q and the arrangement of these edges. If a vertex u is within distance 3 from a vertex v, we shall simply write that u reaches v.

Subcase 3.1: $q = 5$.

Then, $y_i y_{i+1}$ is an edge for all i and $y_0, y_1, \ldots, y_4, y_0$ is an induced cycle. If $y \in X_i$, then it is adjacent to one of n_i and n_{i+1} by the definition of M, and one of y_i and y_{i+1} by Claim 6.3(iv).

We say that the vertex y_i has a *private neighbor* in Y if it has a neighbor y in Y such that y is not adjacent to any other y_j. We claim that if y_i has a private neighbor in Y then X_{i+2} is empty. For suppose y_0 has a private neighbor y^*, in X_0 say, and $y' \in X_2$. Then a shortest path from y^* to y' must be the path y^*, n_1, n_2, y', and so each vertex in X_2 is adjacent to n_2. Let $y \in X_0$. If y lies inside the cycle y_0, y^*, n_1, y_1, y_0, then, by definition of M, $y \sim n_1$. On the other hand, if y lies inside the cycle x, n_0, y_0, y^*, n_1, x, then, in order to reach y', $y \sim n_1$. Hence every vertex of X_0 is adjacent to n_1. Every vertex of X_3 (resp., X_4) is adjacent to n_4 or y_4 in order to reach y^* (resp., y'). Hence, $\{x, n_1, n_2, n_4, y_4\} \succ_t V$ and $\gamma_t(G) \leq 5$. Thus the claim is established.

Now, if $\{x, n_i, y_i, y_{i-2}, y_{i+2}\} \succ_t V$, then $\gamma_t(G) \leq 5$. Hence we may assume that none of the five sets of that form is a total dominating set of G. It follows that for every i, y_{i-1} or y_{i+1} has a private neighbor in Y. Therefore at least three consecutive y_i's have private neighbors in Y, say y_0, y_1 and y_2. By the previous paragraph, $X_2 = X_3 = X_4 = \emptyset$. But then $\{x, n_1, y_0, y_1, y_2\} \succ_t V$, and so $\gamma_t(G) \leq 5$.

Subcase 3.2: $q = 4$.

We may assume $y_0 \not\sim y_4$. If $y \in X_1$, then $y \sim y_2$ in order to reach y_4. If $y \in X_2$, then $y \sim y_2$ in order to reach y_0. Hence, y_2 dominates $X_1 \cup X_2$. If $y \in X_0$ then $y \sim y_1$ in order to reach y_3, while if $y \in X_3$ then $y \sim y_3$ in order to reach y_1. Thus, y_1 dominates X_0 and y_3 dominates X_3. Hence if $X_0 = \emptyset$, then $\{x, n_0, n_4, y_2, y_3\} \succ_t V$, while if $X_3 = \emptyset$, then $\{x, n_0, n_4, y_1, y_2\} \succ_t V$. On the other hand, if both X_0 and X_3 are nonempty, then a shortest path from a vertex of X_0 to a vertex of X_3 must be via n_0 and n_4, and so $\{x, n_0, n_2, n_4, y_2\} \succ_t V$. In any event, $\gamma_t(G) \leq 5$.

Subcase 3.3: $q = 3$ and the edges $y_i y_{i+1}$ are not consecutive.

We may assume that $y_0 y_1$, $y_1 y_2$ and $y_3 y_4$ are edges. Then a shortest y_2–y_4 path has the form y_2, a, y_3, y_4 for some vertex a, while a shortest y_0–y_3 path has the form y_0, b, y_4, y_3 for some vertex b. The cycle $y_0, y_1, y_2, a, y_3, y_4, b, y_0$ is induced by the choice of the y_i and the above claims.

Since M dominates Y, there is no vertex outside the cycle $y_0, y_1, y_2, a, y_3, y_4, b, y_0$. Further, for $i \in \{0, 1, 3\}$, a shortest path from a vertex inside

L_i to the vertex y_{i+3} must be via x. Hence, X_0, X_1 and X_3 are empty.

Now, let C_a and C_b denote the cycles $x, n_2, y_2, a, y_3, n_3, x$ and $x, n_0, y_0, b, y_4, n_4, x$ respectively. If there is a vertex $y \in Y$ inside the cycle C_b, then a shortest path from y to y_2 must be the path y, y_0, y_1, y_2; thus y_0 dominates any vertices of Y inside the cycle C_b.

If $\{x, n_0, y_0, a, y_3\} \succ_t V$, then $\gamma_t(G) \leq 5$. Hence we may assume that $\{x, n_0, y_0, a, y_3\}$ does not totally dominate G. Thus there exists a vertex $y' \in Y$ inside the cycle C_a such that $y' \not\sim y_3$ and $y' \not\sim a$. Then a shortest b–y' path is b, n_4, n_3, y'. By planarity and the definition of M, it follows that $a \sim n_3$ and that every vertex inside the cycle C_a is adjacent to n_3 (to reach b or by definition of M). Furthermore, any vertex of Y inside the cycle C_b is adjacent to n_4 to reach y'. Hence, n_4 dominates b and every vertex inside the cycle C_b, while n_3 dominates a and every vertex inside the cycle C_a. Therefore, $\{n_1, n_3, n_4, y_1, x\} \succ_t V$, and so $\gamma_t(G) \leq 5$.

Subcase 3.4: *$q = 3$ and the edges y_iy_{i+1} are consecutive.*

Say the three edges y_iy_{i+1} are y_0y_1, y_1y_2 and y_2y_3. A shortest y_2–y_4 path has the form y_2, y_3, a, y_4 for some vertex a, while a shortest y_1–y_4 path has the form y_1, y_0, b, y_4 for some vertex b. By Claim 6.1 and Claim 6.3(i), $a \neq b$ and the cycle $y_0, y_1, y_2, y_3, a, y_4, b, y_0$ is induced. (The argument that $a \not\sim b$ uses the fact that by the choice of y_4, neither a nor b is in Y_4'.)

A shortest path from a vertex inside L_1 to y_4 must be via x, and so X_1 is empty. A shortest path from a vertex of X_2 to y_0 must be via y_2, and so y_2 dominates X_2. If there is a vertex $y \in Y$ inside the cycle C_a: $x, n_3, y_3, a, y_4, n_4, x$, then a shortest path from y to y_1 must be the path y, y_3, y_2, y_1, and so y_3 dominates any vertices of Y inside the cycle C_a. Hence if X_0 is empty, then $\{x, n_0, n_4, y_2, y_3\} \succ_t V$, and so $\gamma_t(G) \leq 5$. Thus we may assume that $|X_0| \geq 1$.

A shortest path from a vertex of X_0 to y_3 must be via y_1, and so y_1 dominates X_0. Since $|X_0| \geq 1$, every vertex of Y inside the cycle C_a, as well as the vertex a, must be adjacent to n_4 in order to reach a vertex of X_0. Hence, $\{x, n_0, n_4, y_1, y_2\} \succ_t V$, and so $\gamma_t(G) \leq 5$.

Consequently, $\gamma_t(G) \leq 5$. □

3.2 Outer Cycles and the Divider Lemma

We need the following strengthening of Lemma 11 from [2].

Lemma 7 *Consider a plane graph G with outer cycle C a triangle x, y, z, x. Let S be the set of vertices at distance 2 from each vertex of C. Then S is dominated by one vertex that is adjacent to a vertex of C or S is totally* *dominated by two vertices. Furthermore, if two vertices are needed to totally dominate S, then every pair of vertices of C have a common neighbor inside C and a common neighbor can be chosen for each pair such that any two dominate S.*

Proof. Let s_1 be a vertex of S; i.e., $d(s_1, x) = d(s_1, y) = d(s_1, z) = 2$. If a neighbor of s_1 is adjacent to each of x, y and z, then that vertex dominates S since it separates each vertex of S from one of x, y or z. If each neighbor of s_1 is adjacent to at most one of x, y and z, then s_1 is the unique vertex of S by the planarity of G.

So we may assume some neighbor of s_1, say u, is adjacent to two vertices of C, say x and y. Let v be a common neighbor of z and s_1. Then, $\{u, v\} \succ S$ by the planarity of G (the path x, u, s_1, v, z partitions the interior of C). Since $u, v \notin S$, $\{u, v\} \succ_t S$.

Furthermore, u dominates S unless there is a vertex s_2 of S nonadjacent to u. Without loss of generality, s_2 lies inside the region x, u, s_1, v, z, x. Therefore, $s_2 \sim v$ and $v \sim y$. Let w be a common neighbor of x and s_2. Now, v dominates S unless there is a vertex s_3 of S nonadjacent to v. The vertex s_3 must lie inside the region x, u, s_1, v, s_2, w, x and be adjacent to both u and w. Thus u, v and w are common neighbors for $\{x, y\}$, $\{y, z\}$ and $\{x, z\}$ respectively.

Now, w dominates S unless there exists a vertex s'_1 of S nonadjacent to w. The vertex s'_1 must lie inside the region u, y, v, s_2, w, s_3, u. Therefore, s'_1 must be adjacent to both u and v. Renaming if necessary, we may assume that $s'_1 = s_1$, as shown in Figure 2. Any two vertices of $\{u, v, w\}$ dominate S. □

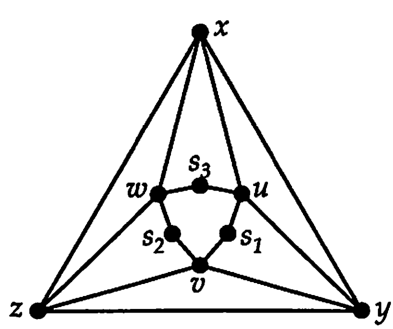

Figure 2: A possible subgraph

Lemma 8 *Consider a plane graph G with outer cycle C being a 4-cycle w, x, y, z, w. Let S be the set of vertices at distance 2 from each vertex of C. Then, S is dominated by a vertex that is adjacent to a vertex of C.*

The proof is identical to that of Lemma 12 from [2], even though the conclusion is slightly stronger.

Finally in this subsection we prove a slight strengthening of Lemma 13 from [2].

Lemma 9 (Divider Lemma) *Consider a planar graph G. Let x and y be distinct vertices such that $d_G(x, y) \leq 2$ and $G + xy$ is planar. Let S be the set of vertices at distance exactly 2 from both x and y. Suppose every pair of vertices in S are distance at most 3 apart. Then there exist a set D of at most three vertices such that $D \succ_t S$, and if x and y are not adjacent in G, then $D \succ_t \{x, y\}$ as well.*

Proof. Lemma 13 from [2] shows that a D of at most three vertices exists that dominates S. The proof of the extension is identical except that when the division into two cases is reached on page 17, one has to show that D totally dominates S, and $\{x, y\}$ as well if $d(x, y) = 2$.

Consider first Case 1 where at least one connector is long, say P. If the other connector Q is short, then take $D = W$ if M has length 1; $D = W - \{s\}$ if M has length 2; and $D = W - \{s, t\}$ if M has length 3.

On the other hand, assume both connectors are long (so that $M = W$). If M has length 1, then $S = \{s, t\}$ and take D to be the internal vertices of P; if M has length 2, then take D to consist of the interior vertex of M, say a, any common neighbor of a and x, and either any common neighbor of a and y if $a \in S$ or any common neighbor of x and y otherwise; if M has length 3, then take D to be the two vertices of $W - \{s, t\}$ and, if $d_G(x, y) = 2$, add any common neighbor of x and y.

Consider second Case 2 where both connectors are short. If M has length 1, then let $D = \{u, v\}$. If M has length 2 and the interior vertex a of M is in S^*, then take $D = \{u, s, v\}$; if M has length 2 and a is not in S^*, then take $D = \{u, a, v\}$. So assume M has length 3. If exactly one interior vertex of M is in S^*, then take $D = W - S^*$.

So suppose $M = s, m, n, t$ where $m, n \in S^*$. If m and n have a common neighbor z, then $D = \{u, v, z\}$ works. So we may assume that $N(m)$ and $N(n)$ are disjoint. In particular, shortest m–x and n–x paths are internally disjoint, as are shortest m–y and n–y paths. If $D_v = \{v, m, n\}$ dominates S then we are done. So we may assume there is a vertex $s' \in S$ not dominated by D_v. This vertex is adjacent to u. Similarly, we may assume there is a vertex t' not dominated by $\{u, m, n\}$, but adjacent to v.

Without loss of generality, s' is in the region bounded by W and x. In order for t' to reach s', it follows that t' is on the same side of W. Say m' and n' are internal vertices on shortest m–x and n–x paths, respectively. In order for s' to reach t, it follows that $s' \sim m'$, $m' \sim n'$ and $n' \sim t$. Similarly, $s \sim m'$ and $t' \sim n'$. If a vertex in $S \cap N(v)$ is adjacent to s or m, we have a previous case. It follows that $\{m', n'\}$ dominates S and $D = \{m', n', u\}$ works.

Finally, if neither interior vertex of M is in S^*, then proceed as before in [2]. □

As an immediate consequence of Lemma 9, we have the following result.

Lemma 10 *Consider a planar graph G of diameter 3. Let x and y be distinct vertices such that $G+xy$ is planar. If every vertex not dominated by $\{x, y\}$ is at distance 2 from both x and y, then there exists a total dominating set of G containing x and y of cardinality at most 5.*

3.3 Short Special Basic Cycles

In this section, we use Lemma 10 to show that if there exists a special basic 3-cycle or a special basic 4-cycle, then G has total domination number at most five.

Unless explicitly otherwise stated, given a special basic cycle C we draw G such that the vertices C does not dominate are inside C. A partner of a vertex v on C is a vertex outside C whose only neighbor on C is v. We will denote a partner of v by v' if it exists.

Lemma 11 *Let G be a plane graph of radius 2 and diameter 3 with central vertex x. If there exists a special basic triangle, then $\gamma_t(G) \leq 5$.*

Proof. Suppose $C: x, a, b, x$ is a basic triangle with vertex a' outside C adjacent only to a. Then every vertex inside C is within distance 2 from a (to reach a'), while every vertex outside C is dominated by C and is therefore within distance 2 from a. Hence, a has eccentricity 2. So we can apply Lemma 10 (with vertices x and a) to totally dominate G with five vertices. □

Lemma 12 *Let G be a plane graph of radius 2 and diameter 3 with central vertex x. If there exists a special basic 4-cycle, then $\gamma_t(G) \leq 5$.*

Proof. Suppose $C: x, a, b, c, x$ is a special basic 4-cycle. If $\mathrm{ecc}_G(a) = 2$, then we can apply Lemma 10 (with the edge ax) to totally dominate G

with five vertices. So we may assume that $\mathrm{ecc}_G(a) = 3$. Similarly, we may assume $\mathrm{ecc}_G(c) = 3$.

Let H be the graph induced by C and its inside. There are three cases.

Case 1: *Neither a nor c has a partner.*

Since C is special, the vertex b must have a partner b'. A length-2 b'–x path separates a and c; so there can be no vertex outside C whose neighbors on C are $\{a, c\}$. Hence, $\{x, b\}$ dominates the outside of C.

Let S be the set of vertices not dominated by $\{x, b\}$ (necessarily inside C). Every vertex of S is at distance 2 from both x (since $\mathrm{ecc}(x) = 2$) and b (in order to reach b'). By Lemma 10 applied to H (with the edge xb added outside C), there exists a total dominating set D of H containing x and b of cardinality at most 5. Thus D is a total dominating set of G, and so $\gamma_t(G) \leq 5$.

Case 2: *Both a and c have partners.*

Let a' and c' denote partners of a and c, respectively. Let S be the set of vertices inside C not dominated by $\{x, a, c\}$. Since every vertex of S reaches a' and c', every vertex of S is at distance exactly 2 from each of x, a and c. By Lemma 7 applied to H with the edge ac added outside C, we can totally dominate H with $\{a, c, x\}$ plus two additional vertices inside C (each of which is dominated by $\{a, c, x\}$).

So we are done unless there exists a partner b' of b. But then every vertex, if any, inside C at distance 3 from b is adjacent to x (to reach b'). It follows that every vertex inside C that is not dominated by C is at distance exactly 2 from each vertex of C. Hence, by Lemma 8, we can totally dominate G with $V(C)$ plus one additional vertex inside C (that is dominated by C). Thus, $\gamma_t(G) \leq 5$.

Case 3: *Exactly one of a and c has a partner.*

By symmetry, we may assume that vertex a has a partner a'. Since C dominates its outside and c has no partner, every vertex at distance 3 from a is inside C and is adjacent to c (in order to reach a'); further, every partner of a is at distance 2 from c (to reach the vertices at distance 3 from a).

Let S be the set of vertices not dominated by C. Then every vertex of S is at distance exactly 2 from each of x and a.

Subcase 3.1: Every vertex of S is at distance exactly 2 from b.

Apply Lemma 7 to H with the edge xb added outside C so that x, a, b, x is the outer triangle: either one can dominate S with one vertex that is adjacent to a vertex of C or S is totally dominated by two vertices. In the former case, the vertex that dominates S can be added to $V(C)$ to totally dominate G with five vertices. Hence we may assume the latter case.

Thus, by Lemma 7, it follows that the graph shown in Figure 2 is a subgraph of H except for a being called y, b being called z, and the edge $xz = xb$ subdivided by the vertex c. Since the 4-cycle $C^*: x, w, b, c, x$ does not dominate its outside (e.g. misses s_1), it must dominate its inside. If there were a vertex inside C^* whose only neighbor on C^* was c, then this vertex would be unable to reach s_1. Hence, $\{x, b, w\}$ dominates every vertex inside C^*, and so $\{x, a, b, v, w\}$ totally dominates H. Since c has no partner, this set also totally dominates G, and so $\gamma_t(G) \leq 5$.

Subcase 3.2: *There is a vertex of S at distance 3 from b.*

Then, b has no partner, since a partner of b is unable to reach a vertex of S at distance 3 from b. Let K be obtained from G by removing all neighbors of x and a outside C. Since neither b nor c has a partner, every vertex of K outside C is adjacent to both b and c. Thus both x and a are at distance 2 from every vertex in K outside C.

Let K' be obtained from K by adding the edge ac outside C. Then x has eccentricity 2 in K'. Since every vertex at distance 3 from a in H is adjacent to c (to reach a'), the vertex a also has eccentricity 2 in K'. Furthermore, K' is a plane graph of diameter at most 3. By Lemma 10 applied to K' (with vertices x and a), there exists a total dominating set D' of K' containing x and a of cardinality at most 5. Irrespective of whether or not c is in D', D' is a total dominating set of G, and so $\gamma_t(G) \leq 5$. This completes the proof of Lemma 12. □

3.4 Basic 5-cycles

In this section we outline the computer proof of the final result that if there is a basic 5-cycle then the total domination number is at most 5. Theorem 1 follows immediately from Lemmas 6, 11, 12 and 13.

Lemma 13 *If there exists a basic 5-cycle, then $\gamma_t(G) \leq 5$.*

Proof. Suppose $C : x, a, b, c, d, x$ is a basic 5-cycle. If C dominates G we are done, so we may assume that some vertex inside C is not dominated by C. Choose C such that there is a vertex inside not dominated by C and there is a minimum number of vertices inside C. Note that if a vertex outside C is not adjacent to any of $\{x, b, c\}$, then it cannot be adjacent to both a and d; for, the vertex f outside not adjacent to $\{x, a, d\}$—which is guaranteed by C being basic—must reach x in 2. There are therefore three cases.

Case 1: *There is a vertex a' outside adjacent only to a on C and a vertex d' outside adjacent only to d on C.*

Computer proof.

Case 2: *There is a vertex a' outside adjacent only to a on C and $\{x, a, b, c\}$ dominates the vertices outside C.*

Computer proof.

Case 3: *Every vertex outside C is dominated by $\{x, b, c\}$.*

Let S be the set of vertices not dominated by $\{x, b, c\}$; S is inside C. Note that if $S \cup \{x\}$ is totally dominated by two vertices we are done.

We follow the argument in the proof of Lemma 9. A connector now connects either b and x, or c and x. That is, for $s \in S$ if $N(s) \cap N(x) \cap (N(b) \cup N(c))$ is nonempty, then an s-connector is an x–b or x–c path of length 2 via a neighbor of s; otherwise, an s-connector consists of an x–s path of length 2 and either an s–b or s–c path of length 2. By a similar argument to that of the claim, there is again a leftmost and a rightmost connector.

Now, assume b and x have a common neighbor v inside C. Since $\{x, a, b, c, v\}$ does not totally dominate G, there is a vertex s inside C not dominated by $\{x, a, b, c, v\}$. We claim that the vertex s is not inside the 4-cycle C': x, a, b, v, x. If it is, then either $v \not\sim a$, in which case C' is a special basic 4-cycle, or s is in one of the triangles x, a, v, x or b, v, a, b and thus too far from either d or c.

It follows that if the leftmost connector is short, it starts at c. In fact it is c, d, x: if it is c, v, x, then replacing d by v contradicts the choice of C. Similarly, if the rightmost connector is short, then it is b, a, x. In particular, every element of S lies on or between the leftmost and rightmost connectors.

Let u, v, s and t be as in the proof of Lemma 9 and consider a shortest s–t path W (necessarily inside C). There are three cases.

Subcase 3.1. Both connectors are long. If W has length 2, the vertex f can only reach the middle vertex w of W via b or c and a vertex on one of the connectors. (A path such as f, b, z, w with z not on one of the connectors does not work since z must reach x in two.) Then $S \cup \{x\}$ is totally dominated by two vertices.

The case W has length 3 is handled by a computer proof.

Subcase 3.2. One connector short and one long. Computer proof.

Subcase 3.3. Both connectors short. Computer proof. □

3.4.1 Computer Proof

We have narrowed things down to proving a list of cases, all of which take the form: Let G be a plane graph of diameter 3 and radius 2 with central vertex x. Prove that if G contains a certain subgraph H then $\gamma_t(G) \leq 5$ or there is a contradiction.

The computer proof uses exhaustive search to try all possibilities looking for a counterexample. The basic algorithm we use is the following recursive procedure, which takes as parameter a plane subgraph of G:

> procedure Check(plane-subgraph H)
>
> Step 1: If H does not satisfy the distance constraints, then consider all possible completions H' that satisfy the distance constraints and call Check(H') recursively on each of them.
>
> Step 2: Else, if H has a total dominating set S of cardinality at most 5, then form H' by adding a new vertex not dominated by S, and call Check(H') recursively.
>
> Step 3: Else, H is a counterexample.

In order to improve efficiency and avoid infinite recursion, one can terminate a branch of the computation at any stage if either (1) the assumptions of the particular case are contradicted, or (2) it is impossible for the distance constraints to be satisfied. For example, we may have assumed that outside the cycle every vertex is adjacent to both b and c but this is violated in the current subgraph. Or, vertices r and s are distance 4 apart and cannot be any closer given the planarity and the edges assumed not to be present.

Step 1 is implemented as follows. If H does not obey the distance criteria, then there is a pair of vertices that are too far apart in H. So one tries all ways of fixing that pair, subject to the constraints of planarity and the assumptions so far. For example, a pair that should be at distance at most 3 can be fixed by (a) an edge joining them, (b) a path of length 2 that goes via an existing vertex, (c) a path of length 2 that goes via a new vertex, (d) a path of length 3 that goes via two existing vertices, (e) a path of length 3 that uses one existing vertex, and (f) a path of length 3 that uses two new vertices. Note that in all cases other than (a), the edge cannot be inserted later (to avoid duplication of cases).

An important idea for efficiency in Step 1 is to rectify only the distances of the current set of vertices, ignoring for the time being any new vertices introduced. Then Step 3 is modified so that if an alleged counterexample is found, the full distance constraint is then checked, and if it fails, the recursion continues.

The subgraph itself is represented by an object with fields that record its order, its embedding (the cyclic ordering of the edges at each vertex) and the status of each pair of vertices (adjacent, maybe adjacent, not adjacent, distance at least 3). Before an edge is added, it is checked that the two endpoints are in the same region, so a planar embedding is maintained throughout.

The computer program took about an hour to run on a typical PC and found no counterexample.

4 Proof of Theorem 4

In this section we prove Theorem 4. We use the same approach used to prove Theorem 5 in [2].

As in [2], we recall a family of graphs known as *lanterns* (or *theta graphs*). For $s \geq 3$, an *s-lantern* is a graph obtained from the complete bipartite graph $K(2, s)$ by subdividing each edge any number of times (including the possibility of none). The two vertices of degree more than 2, say x and y, we call the *hubs* of the lantern and the x–y paths of the lantern we call the *axes* of the lantern. A lantern with hubs x and y we also call an x-y *lantern*. A *region* of a lantern is a portion of the plane bounded by two consecutive axes in the lantern.

The following lemma establishes for any $s \geq 3$ the existence of an s-lantern in a graph of sufficiently large order.

Lemma 14 ([2]) *Let d be a positive integer. In a sufficiently large graph of diameter d and radius d there exists an arbitrarily large lantern.*

Lemma 15 *Let G be a planar graph of radius* 3 *and diameter* 3*. If G contains a* 10*-lantern L, then there is an axis of L whose contraction to a single vertex produces a planar graph G' of radius at most* 2 *and diameter at most* 3 *whose total domination number is at least $\gamma_t(G) - 2$.*

Proof. Let G' be the graph constructed exactly as in the proof of Lemma 10 from [2]. Then, G' is planar with diameter at most 3 and radius at most 2. Now, consider a minimum total dominating set D of G'. Let w be a vertex of D adjacent to a vertex of B. Let $D' = (D - \{w\}) \cup \{z\}$. Suppose some vertex s of G' is not adjacent to a vertex of D'. Then $s \in S$ and s is adjacent to w. However, w is at distance at most 1 from B so that s is at distance at most 2 from B, a contradiction. Hence D' is also a total dominating set of G'. Since z is adjacent to some vertex of D' in G', at least one of x and y is adjacent to some vertex of D' in G. Thus $(D - \{z\}) \cup \{x, y\}$ together with some neighbor of x or y is a total dominating set of G. Consequently, $\gamma_t(G) \leq |D| + 2 = \gamma_t(G') + 2$, as required. □

A consequence of the above two lemmas is:

Lemma 16 *For a sufficiently large planar graph G of radius and diameter 3, there exists a planar graph G' of radius at most 2 and diameter at most 3 such that $\gamma_t(G) \leq \gamma_t(G') + 2$.*

Theorem 4 now follows immediately from Theorem 1 and Lemma 16.

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