Some $_3\psi_3$ transformations formulas related to Bailey's $_2\psi_2$

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Abstract

In this paper, using the q-exponential operator technique to Bailey's $_2\psi_2$ transformation, we obtain some interesting $_3\psi_3$ transformation formulae and summation theorems.

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1 Introduction

Throughout this paper, let $0 < q < 1$ and we will use the following equations frequently ([4]):

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \tag{1}$$

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) = (a;q)_n(aq^n;q)_\infty; \tag{2}$$

ARS COMBINATORIA 78(2006), pp. 257-265

$$(q/a;q)_n = (-a)^{-n} q^{\binom{n+1}{2}} \frac{(q^{-n}a;q)_\infty}{(a;q)_\infty}, \tag{3}$$

$$(q^{-n}a;q)_\infty = (-a)^n q^{-\binom{n+1}{2}} (q/a;q)_n (a;q)_\infty. \tag{4}$$

We also adopt the following notation for multiple q-shifted factorial:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \ldots (a_m;q)_n,$$

where n is an integer or ∞.

The bilateral basic hypergeometric series ${}_r\psi_r$ is given by

$${}_r\psi_r \left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_r \end{matrix} ; q, z \right) = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} z^n.$$

In [1], applying Ramanujan's ${}_1\psi_1$ summation and elementary manipulations of series, Bailey derived the following transformation

$${}_2\psi_2 \left(\begin{matrix} a, b \\ c, d \end{matrix} ; q, z \right) = \frac{(az, \frac{d}{a}, \frac{c}{b}, \frac{dq}{abz}; q)_\infty}{(z, d, \frac{q}{b}, \frac{cd}{abz}; q)_\infty} \, {}_2\psi_2 \left(\begin{matrix} a, \frac{abz}{d} \\ az, c \end{matrix} ; q, \frac{d}{a} \right) \tag{5}$$

where $\max(|z|, |cd/abz|, |d/a|, |c/b|) < 1$.

Bailey's ${}_2\psi_2$ transformation (5) can be iterated. The result is

$${}_2\psi_2 \left(\begin{matrix} a, b \\ c, d \end{matrix} ; q, z \right) = \frac{(az, bz, \frac{cq}{abz}, \frac{dq}{abz}; q)_\infty}{(\frac{q}{a}, \frac{q}{b}, c, d; q)_\infty} \, {}_2\psi_2 \left(\begin{matrix} \frac{abz}{c}, \frac{abz}{d} \\ az, bz \end{matrix} ; q, \frac{cd}{abz} \right) \tag{6}$$

where $\max(|z|, |cd/abz|) < 1$.

If $d = bq$ and $z = q/a$ in (5), then the series on the right side reduces just to one term, 1, and we have the summation

$${}_2\psi_2 \left(\begin{matrix} a, b \\ c, bq \end{matrix} ; q, \frac{q}{a} \right) = \frac{(q, q, bq/a, c/b; q)_\infty}{(q/a, bq, q/b, c; q)_\infty} \tag{7}$$

where $\max(|q/a|, |c|) < 1$.

In this paper, using the q-exponential operator technique to these ${}_2\psi_2$ transformation and summation formulas, we obtain some interesting ${}_3\psi_3$ transformation and summation formulas.

To make the the paper self-contained, the q-exponential operator due to Chen and Liu (see [2], [3] and [5]) can be restated as follows:

The q-difference operator and the q-shift operator η are defined by

$$D_q\{f(a)\} = \frac{1}{a}(f(a) - f(aq))$$

and

$$\eta\{f(a)\} = f(aq),$$

respectively. In [2] Chen and Liu construct operator

$$\theta = \eta^{-1}D_q.$$

Based on these, they introduce two operators:

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q;q)_n}$$

and

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{\binom{n}{2}}}{(q;q)_n}.$$

Then the following operator identities are obtained.

Theorem 1.1 *(Chen and Liu, [2] and [3]) Let $T(bD_q)$ and $E(b\theta)$ are defined as above respectively. Then*

$$T(bD_q)\left\{\frac{1}{(at;q)_\infty}\right\} = \frac{1}{(at,bt;q)_\infty}, \tag{8}$$

$$T(bD_q)\left\{\frac{1}{(as,at;q)_\infty}\right\} = \frac{(abst;q)_\infty}{(as,at,bs,bt;q)_\infty}, \tag{9}$$

$$E(b\theta)\left\{(at;q)_\infty\right\} = (at,bt;q)_\infty, \tag{10}$$

$$E(b\theta)\left\{(as,at;q)_\infty\right\} = \frac{(as,at,bs,bt;q)_\infty}{(abst/q;q)_\infty}. \tag{11}$$

2 The results and their proofs

Theorem 2.1

$$
{}_3\psi_3\left(\begin{matrix} a, & b, & \frac{cde}{qabz} \\ c, & d, & e \end{matrix}; q, z\right)
= \frac{(az, \frac{d}{a}, \frac{c}{b}, \frac{e}{b}, \frac{dq}{abz}, \frac{cde}{qabz}; q)_\infty}{(z, d, \frac{q}{b}, \frac{cd}{abz}, \frac{de}{abz}, \frac{ce}{qb}; q)_\infty}\, {}_3\psi_3\left(\begin{matrix} a, & \frac{abz}{d}, & \frac{ce}{qb} \\ az, & c, & e \end{matrix}; q, \frac{d}{a}\right) \quad (12)
$$

where $\max(|z|, |d/a|) < 1$.

Proof. By (1), (5) can be rewritten as

$$
\sum_{k=-\infty}^{\infty} \frac{(a, b; q)_k}{(d; q)_k} z^k \cdot \left\{(cq^k, cd/abz; q)_\infty\right\} = \frac{(az, d/a, qd/abz; q)_\infty}{(z, d, q/b; q)_\infty} \times
$$

$$
\times \sum_{k=-\infty}^{\infty} \frac{(a, abz/d; q)_k}{(az; q)_k} \left(\frac{d}{a}\right)^k \cdot \left\{(cq^k, c/b; q)_\infty\right\}. \quad (13)
$$

Applying $E(e\theta)$ to both sides of the equation with respect to the variable c gives

$$
\sum_{k=-\infty}^{\infty} \frac{(a, b; q)_k}{(d; q)_k} z^k \cdot E(e\theta) \left\{(cq^k, cd/abz; q)_\infty\right\}
$$

$$
= \frac{(az, \frac{d}{a}, \frac{qd}{abz}; q)_\infty}{(z, d, \frac{q}{b}; q)_\infty} \sum_{k=-\infty}^{\infty} \frac{(a, \frac{abz}{d}; q)_k}{(az; q)_k} \left(\frac{d}{a}\right)^k E(e\theta) \left\{(cq^k, \frac{c}{b}; q)_\infty\right\}. \quad (14)
$$

By (8), we have

$$
E(e\theta) \left\{(cq^k, cd/abz; q)_\infty\right\} = \frac{(cq^k, cd/abz, eq^k, ed/abz; q)_\infty)}{(cdeq^k/qabz; q)_\infty}, \quad (15)
$$

and

$$
E(e\theta) \left\{(cq^k, c/b; q)_\infty\right\} = \frac{(cq^k, c/b, eq^k, e/b; q)_\infty}{(ceq^k/qb; q)_\infty}. \quad (16)
$$

Substituting these two identities to (14) and then using (2), we obtain the proof of the theorem. □

Theorem 2.2

$$
{}_3\psi_3\left(\begin{matrix} a, & b, & \frac{cde}{qabz} \\ c, & d, & e \end{matrix}; q, z\right)
$$

$$
= \frac{(az, \frac{d}{a}, \frac{c}{b}, \frac{e}{b}, \frac{dq}{abz}, \frac{qe}{abz}, \frac{cde}{qabz}; q)_\infty}{(z, d, e, \frac{q}{b}, \frac{cd}{abz}, \frac{ce}{qb}, \frac{de}{a^2bz}; q)_\infty} \; {}_3\psi_3\left(\begin{matrix} a, & \frac{abz}{d}, & \frac{abz}{e} \\ az, & c, & qa^2bz/de \end{matrix}; q, q\right) \tag{17}
$$

where $\max(|z|, |c/b|) < 1$.

Proof. By (1), (5) can be rewritten as

$$
\sum_{k=-\infty}^{\infty} \frac{(a, b; q)_k}{(c; q)_k} z^k \cdot \left\{(dq^k, cd/abz; q)_\infty\right\}
$$

$$
= \frac{(az, c/b; q)_\infty}{(z, q/b; q)_\infty} \sum_{k=-\infty}^{\infty} \frac{(a; q)_k}{(az, c; q)_k} q^{\binom{k}{2}} (-bz)^k \cdot \left\{(q^{-k}qd/abz, d/a; q)_\infty\right\}. \tag{18}
$$

Applying $E(e\theta)$ to both sides of the equation with respect to the variable d gives

$$
\sum_{k=-\infty}^{\infty} \frac{(a, b; q)_k}{(c; q)_k} z^k \cdot E(e\theta) \left\{(dq^k, cd/abz; q)_\infty\right\}
$$

$$
= \frac{(az, \frac{c}{b}; q)_\infty}{(z, \frac{q}{b}; q)_\infty} \sum_{k=-\infty}^{\infty} \frac{(a; q)_k}{(az, c; q)_k} q^{\binom{k}{2}} (-bz)^k E(e\theta) \left\{(\frac{q^{1-k}d}{abz}, \frac{d}{a}; q)_\infty\right\}. \tag{19}
$$

By (8), we have

$$
E(e\theta) \left\{(dq^k, cd/abz; q)_\infty\right\} = \frac{(dq^k, cd/abz, eq^k, \frac{ce}{abz}; q)_\infty}{(\frac{cdeq^k}{qabz}; q)_\infty}, \tag{20}
$$

and

$$E(e\theta)\left\{(q^{-k}qd/abz,d/a;q)_\infty\right\}$$
$$= \frac{(q^{-k}qd/abz,d/a,q^{-k}qe/abz,e/a;q)_\infty}{(\frac{q^{-k}de}{a^2bz};q)_\infty)}. \tag{21}$$

Substituting these two identities to (19) and then using (2), we obtain the proof of the theorem. □

Corollary 2.3

$$ {}_3\psi_3\left(\begin{matrix} a, & \frac{abz}{d}, & \frac{ce}{qb} \\ az, & c, & e \end{matrix};q,\frac{d}{a}\right)$$
$$= \frac{(\frac{qe}{abz},\frac{de}{abz};q)_\infty}{(e,\frac{de}{a^2bz};q)_\infty}\ {}_3\psi_3\left(\begin{matrix} a, & \frac{abz}{d}, & \frac{abz}{e} \\ az, & c, & qa^2bz/de \end{matrix};q,q\right) \tag{22}$$

where $\max(|d/a|,|c/b|)<1$.

Proof. Compare (12) and (17). □

Theorem 2.4

$$ {}_3\psi_3\left(\begin{matrix} a, & b, & \frac{cde}{qabz} \\ c, & d, & e \end{matrix};q,z\right)$$
$$= \frac{(az,bz,\frac{e}{a},\frac{e}{b},\frac{qc}{abz},\frac{qd}{abz},\frac{cde}{qabz};q)_\infty}{(\frac{q}{a},\frac{q}{b},c,d,\frac{de}{abz},\frac{ce}{abz},\frac{aez}{qa};q)_\infty}\ {}_3\psi_3\left(\begin{matrix} \frac{abz}{c}, & \frac{abz}{d}, & \frac{aez}{qa} \\ az, & bz, & e \end{matrix};q,\frac{cd}{abz}\right) \tag{23}$$

where $\max(|z|,|cd/abz|)<1$.

Proof. Iterate Theorem 2.1. □

Theorem 2.4 is a generalization of Bailey's ${}_2\psi_2$ transformation (6). Next we give another generalization.

Theorem 2.5

$$
{}_3\psi_3\left(\begin{matrix} a, & b, & \frac{ce}{q} \\ c, & d, & e \end{matrix}; q, z\right)
$$
$$
= \frac{(az, bz, \frac{ce}{q}, \frac{qc}{abz}, \frac{qd}{abz}, \frac{qe}{abz}; q)_\infty}{(\frac{q}{a}, \frac{q}{b}, c, d, e, \frac{ce}{abz}; q)_\infty} \; {}_3\psi_3\left(\begin{matrix} \frac{abz}{c}, & \frac{abz}{d}, & \frac{abz}{e} \\ az, & bz, & \frac{qabz}{ce} \end{matrix}; q, \frac{qd}{abz}\right) \tag{24}
$$

where $\max(|z|, |qd/abz|) < 1$.

Proof. By (1), (6) can be rewritten as

$$
\sum_{k=-\infty}^{\infty} \frac{(a, b; q)_k}{(d; q)_k} z^k \cdot \left\{(cq^k, c; q)_\infty\right\}
$$
$$
= \frac{(az, bz, \frac{qd}{abz}; q)_\infty}{(\frac{q}{a}, \frac{q}{b}, d; q)_\infty} \sum_{k=-\infty}^{\infty} \frac{(\frac{abz}{d}; q)_k}{(az, bz; q)_k} \left(-\frac{d}{q}\right)^k q^{\binom{k+1}{2}} \cdot \left\{(\frac{q^{1-k}c}{abz}, c; q)_\infty\right\}. \tag{25}
$$

Applying $E(e\theta)$ to both sides of the equation with respect to the variable c gives

$$
\sum_{k=-\infty}^{\infty} \frac{(a, b; q)_k}{(d; q)_k} z^k \cdot E(e\theta)\left\{(cq^k, c; q)_\infty\right\} = \frac{(az, bz, qd/abz; q)_\infty}{(q/a, q/b, d; q)_\infty} \times
$$
$$
\times \sum_{k=-\infty}^{\infty} \frac{(abz/d; q)_k}{(az, bz; q)_k} \left(-\frac{d}{q}\right)^k q^{\binom{k+1}{2}} E(e\theta)\left\{(q^{1-k}c/abz, c; q)_\infty\right\}. \tag{26}
$$

By (8), we have

$$
E(e\theta)\left\{(cq^k, c; q)_\infty\right\} = \frac{(cq^k, c, eq^k, e; q)_\infty}{(ceq^k/q; q)_\infty}, \tag{27}
$$

and

$$
E(e\theta)\left\{(q^{1-k}c/abz, c; q)_\infty\right\} = \frac{(q^{1-k}c/abz, c, q^{1-k}e/abz, e; q)_\infty}{(q^{-k}ce/abz; q)_\infty}. \tag{28}
$$

Substituting these two identities to (26) and then using (4), we obtain the proof of the theorem. □

Theorem 2.6

$$
{}_3\psi_3\left(\begin{matrix} a, & b, & \frac{cd}{q} \\ c, & d, & qb \end{matrix} ;q,\frac{q}{a}\right) = \frac{(q,q,qb/a,c/b,d/b,cd/q;q)_\infty}{(q/a,q/b,qb,cd/qb,c,d;q)_\infty} \tag{29}
$$

where $|q/a| < 1$.

Proof. By (1), (7) can be rewritten as

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty} \frac{(a,b;q)_k}{(qb;q)_k}\left(\frac{q}{a}\right)^k \cdot \left\{(cq^k,c;q)_\infty\right\} \\
= \; & \frac{(q,q,qb/a;q)_\infty}{(q/a,q/b,qb;q)_\infty}\cdot\{(c/b,c;q)_\infty\}.
\end{aligned} \tag{30}
$$

Applying $E(d\theta)$ to both sides of the equation with respect to the variable c gives

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty} \frac{(a,b;q)_k}{(qb;q)_k}\left(\frac{q}{a}\right)^k \cdot E(d\theta)\left\{(cq^k,c;q)_\infty\right\} \\
= \; & \frac{(q,q,qb/a;q)_\infty}{(q/a,q/b,qb;q)_\infty}\cdot E(d\theta)\{(c/b,c;q)_\infty\}.
\end{aligned} \tag{31}
$$

By (8), we have

$$
E(d\theta)\left\{(cq^k,c;q)_\infty\right\} = \frac{(cq^k,c,dq^k,d;q)_\infty}{(cdq^k/q;q)_\infty}, \tag{32}
$$

and

$$
E(d\theta)\{(c/b,c;q)_\infty\} = \frac{(c/b,c.d/b,d;q)_\infty}{(cd/qb;q)_\infty}. \tag{33}
$$

Substituting these two identities to (31) and then using (2), we obtain the proof of the theorem. □

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