A NOTE ON EXCELLENT GRAPHS

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Abstract

A graph G is said to be excellent if given any vertex x of G, there is a γ - set of G containing x. It is known that any non - excellent graph can be imbedded in an excellent graph. For example for every graph G, its corona $G \circ K_1$ is excellent, but the difference $\gamma(G \circ K_1) - \gamma(G)$ may be high. In this paper we give a construction to imbed a non - excellent graph G in an excellent graph H such that $\gamma(H) \leq \gamma(G) + 2$. We also show that given a non - excellent graph G, there is subdivision of G which is excellent. The excellent subdivision number of a graph G, $ESdn(G)$ is the minimum number of edges of G to be subdivided to get an excellent subdivision graph H. We obtain upper bounds for $ESdn(G)$. If any one of these upper bounds for $ESdn(G)$ is attained, then the set of all vertices of G which are not in any γ - set of G is an independent set.

1. Introduction

The graphs considered here are finite, undirected, non - trivial without loops or multiple edges. Let G = (V, E) be a graph. A subset D of V is a dominating set of G if every vertex in V - D is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. A dominating set with minimum cardinality is said to be a $\gamma(G)$ - set.

In [1] G. H. Fricke et al. called a vertex of a graph G to be good if it is contained in some $\gamma(G)$ - set, and bad if it is not. They call a graph G to be excellent if every vertex of G is good.

G. H. Fricke et al. also proved that every graph (of order n) is an induced subgraph of an excellent graph (of order $2n$) i.e., the graph $G \circ K_1$ which is obtained from a copy of G, by adding to each vertex $v \in V(G)$ a new vertex v' and an pendant edge vv', is excellent. But in some cases it might happen that $\gamma(G \circ K_1) - \gamma(G)$ is large. For example, if $G = K_{1,n}, n \geq 2$, then $\gamma(G) = 1$, but $\gamma(G \circ K_1) =$

$n+1$. We provide a construction, where a non - excellent graph G is imbedded in an excellent graph H such that $\gamma(H) \leq \gamma(G)+2$.

2. Imbedding Into An Excellent Graph

Theorem 1 *Let G be a non - excellent graph. Then there exists a graph H such that*
1. H is excellent.
2. $\gamma(G) < \gamma(H) \leq \gamma(G)+2$.
3. G is an induced subgraph of H.

Proof. *Let G be a non - excellent graph. Let A be the set of all good vertices of G, and B be the set of all bad vertices of G. As G is non - excellent,* $B \neq \phi$. *Let* $B = \{b_1, b_2, ..., b_m\}$. *Let* B^* *be a nonempty subset of B. Then* $\gamma(G - B^*) \geq \gamma(G) - |B^*| + 1$. *[Otherwise for every set S of* $G - B^*, S \cup B^*$ *is a dominating set of G, and hence a* $\gamma(G)$ *set of G containing all the bad vertices of* B^* *which is a contradiction]. If* $\gamma(G - B^*) = \gamma(G) - |B^*| + 1$, *then we say that the set* B^* *is an optimal bad set. If* B^* *is an optimal bad set and* $G - B^*$ *is excellent, then we say that* B^* *is an extreme optimal bad set. If* $|B| = 1$, *we observe that B is an extreme optimal bad set. [If* $|B| = 1$, *then* $\gamma(G - B) \geq \gamma(G)$. *As every* γ *- set of G is a dominating set of* $G - B, \gamma(G - B) = \gamma(G)$. *As every vertex of* $G - B$ *is in some* γ *- set of G, (and hence of a* γ *- set of* $G - B), G - B$ *is excellent].*

Case 1 *: We assume that there is a nonempty subset* B^* *of B such that* B^* *is an extreme optimal bad set. Let* $B^* = \{b_1, b_2, ..., b_k\}$ *. In this case we construct H as follows.*
$V(H) = V(G) \cup \{u_1, u_2, ..., u_k\}$ *and* $E(H) = E(G) \cup \{u_i\ b_i |\ i = 1, 2, ..., k\}$.
Then clearly,
1. G is an induced subgraph of H.
2. $\gamma(G) < \gamma(H)$*.[For a given dominating set S of H, we can find a dominating set S′ for H such that* $B^* \subset S' \subset V(G), |S| = |S'|$ *and hence* $|S| = |S'| \geq \gamma(G) + 1$. *Thus* $\gamma(H) \geq \gamma(G) + 1$].
3. $\gamma(H) = \gamma(G) + 1$. *[For each* γ *- set S of* $G - B^*, S \cup B^*$ *is a dominating set for H].*
4. H is excellent. [As $G - B^*$ *is excellent, given a vertex x of* $V(G - B^*)$*, find a* γ *- set S of* $G - B^*$*, which contains x. Then* $S \cup B^*$ *and* $S \cup \{u_i |\ i = 1, 2, ..., k\}$ *are* γ *- sets for H containing x and* $\{b_i |\ i = 1, 2, ..., k\}$*, containig x and* $\{u_i |\ i = 1, 2, ..., k\}$ *respectively].*

Case 2: *Assume that no subset B^* of B is an optimal bad set. It follows that $|B| \geq 2$. We construct a graph H as follows. Let $\{u_1, u_2, ..., u_m, v_1, v_2, ..., v_m\}$ be a set disjoint with $V(G)$. Let $V(H) = V(G) \cup \{u_1, u_2, ..., u_m, v_1, v_2, ..., v_m\}$ and $E(H) = E(G) \cup \{b_i u_i | i = 1, 2, ..., m\} \cup \{u_i v_j, v_i v_j | i \neq j, i, j = 1, 2, ..., m\}$.*
Clearly G is an induced subgraph of H. Whenever S is a γ - set for G, $S \cup \{u_1, v_1\}$ is a dominating set for H. So $\gamma(H) \leq \gamma(G) + 2$. Let D be a minimum dominating set for H. The set D should contain at least one element from $V_1 \cup V_2$, where $V_1 = \{u_i | i = 1, 2, ..., m\}, V_2 = \{v_i | i = 1, 2, ..., m\}$. Let $V_0 = V(G)$.
Subcase 1 : *Let $D \cap V_1 = \phi$. Then either $|D \cap V_2| = 2$ or $|D \cap V_2| = 1$ and $D \cap B \neq \phi$. As $D \cap V_1 = \phi, D \cap V_0$ is a dominating set for G and hence*

$$|D \cap V_0| \geq \begin{cases} \gamma(G) & \text{if } D \cap B = \phi \\ \gamma(G) + 1 & \text{if } D \cap B \neq \phi \end{cases}$$

Thus in this case $|D| \geq \gamma(G) + 2$.
Subcase 2 : *Let $D \cap V_1 \neq \phi$. Then $|D \cap (V_1 \cup V_2)| \geq 2$. Let $B' = \{b_i | u_i \in D\}$. Then $D \cap V_0$ dominates $G - B'$. Hence $(D \cap V_0) \cup B'$ dominates G and contains at least one bad vertex of G. Then $|(D \cap V_0) \cup B'| \geq \gamma(G) + 1$ and $|D \cap V_0| \geq \gamma(G) + 1 - |B'|$. As $|B'| = |D \cap V_1|$ it follows that if $D \cap V_2 \neq \phi$, then $|D| \geq \gamma(G) + 2$. As $D \cap V_1$ does not dominate any vertex in $V_1 - D$, if $D \cap V_2 = \phi$, then $D \cap V_0$ must contain $B - B'$. In this case $(D \cap V_0) \cup B'$ is a dominating set for G, containing B. We claim that $|(D \cap V_0) \cup B'| \geq \gamma(G) + 2$.*

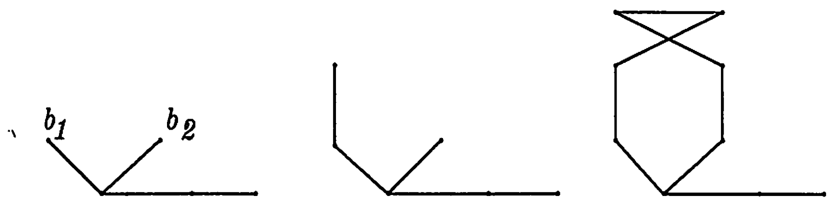
If possible assume that $|(D \cap V_0) \cup B'| = \gamma(G) + 1$. Fix any one vertex $b_{i_0} \in B'$. Then $(D \cap V_0) \cup (B' - b_{i_0})$ is a γ - set of $G - b_{i_0}$ containing $B - b_{i_0}$. It follows that $G - b_{i_0}$ is excellent and $\{b_{i_0}\}$ is an extreme optimal bad set, which is a contradiction to our assumption that no subset of B is an extreme optimal set. Then $|(D \cap V_0) \cup B'| \geq \gamma(G) + 2$. So $|D| = |(D \cap V_0)| + |(D \cap V_1)| = |D \cap V_0| + |B'| \geq \gamma(G) + 2$.

Given any vertex $a \in A$, let S be any $\gamma(G)$ - set for G containing a. Then $S \cup \{u_i, v_i\}$ is a γ - set for H containing u_i, v_i and a, for all $i = 1, 2, ..., m$ and $S \cup \{b_i, v_i\}$ is a γ - set for H containing b_i for $i = 1, 2, ..., , m$. So H is excellent.

Remark

If some subset B^* of B is an extreme optimal bad set, then the construction given in the case 2 may not yield an excellent graph. For example consider the graph given in Fig.1 (a). For this graph

$B = \{b_1, b_2\}$ and $B^* = \{b_1\}$ is an extreme optimal bad set. The graph given in Fig.1(b) constructed as in case 1 is excellent, while the graph given in Fig.1 (c) constructed as in case 2, is not excellent.



(a) (b) (c)

Fig.1

3. Excellent subdivision graphs

If G is a graph, then a graph obtained from G by subdividing each edge at most once is called a subdivision of G. The graph obtained from G by subdividing each edge of G exactly once is denoted by $S_1(G)$. The graph $S_1(G)$ need not be excellent, even for an excellent graph G. The corona $P_3 \circ K_1$ of P_3 is excellent, but $S_1(P_3 \circ K_1)$ is not excellent. If G is a star $K_{1,n}, (n \geq 2)$, then both G and $S_1(G)$ are not excellent. In the following theorem, we show that for each G, there is at least one subdivision of G which is excellent.

Theorem 2 *If a graph G is not excellent, then there is a subdivision graph H of G which is excellent.*

Proof. *Let G be a graph which is not excellent. Let A and B be the set of all good and bad vertices of G respectively. As G is not excellent, $B \neq \phi$. Fix one $x \in B$. Among the set of all γ - sets of G, select one γ - set S_1 such that $|N(x) \cap S_1|$ is maximum. Let $V_0 = N(x) \cap S_1$. Then $V_0 \subseteq A$. For each $y \in N(x) \cap S_1$, subdivide the edge xy. Let w_y be the vertex introduced while subdividing the edge xy. Let H_1 be the graph thus obtained. $V(H_1) = V(G) \cup \{w_y | y \in N(x) \cap S_1 \text{ in } G\}$.*

As $S_1 \cup \{x\}$ is a dominating set for $H_1, \gamma(H_1) \leq \gamma(G) + 1$. We claim that $\gamma(H_1) = \gamma(G) + 1$. Assume that $\gamma(H_1) = \gamma(G)$ and let D be a γ - set of H_1. If $x \notin D$ and $w_y \notin D$ for all $y \in V_0$, then $V_0 \subseteq D$ [otherwise for some y, w_y is not dominated by D], and D must contain at least one vertex of $N(x) \cap (V(G) - V_0)$. As

$|D| = \gamma(G)$ and $w_y \notin D$ for all $y \in D$, the set D is a γ - set for G also. Hence, $D \cap (N(x) \cap (V(G) - V_0)) \subseteq A$, and $|D \cap N(x)| > |S_1 \cap N(x)|$ which is a contradiction to the selection of S_1. Thus D must contain either x or at least one w_y. If $x \in D$, then take $D_1 = (D \cup \{y | w_y \in D\}) - \{w_y | w_y \in D\}$. Then D_1 is a γ - set for G and as $x \in D, x \in A$ which is a contradiciton. Hence $x \notin D$ and $w_y \in D$ for some y. Fix one y_0 such that $w_{y_0} \in D$. Then $D_2 = (D \cup \{y | y \neq y_0, w_y \in D\}) - \{w_y | y \neq y_0, w_y \in D\}$ is also a dominating set for H_1 [a γ - set for H_1]. Note that $x \notin D_2, w_y \notin D_2$ for every $y \neq y_0$ and $w_{y_0} \in D_2$. Then $D_2 \cup \{x\} - \{w_{y_0}\}$ is a γ - set for G which is a contradiciton as $x \notin A$. Thus $\gamma(H_1) \neq \gamma(G)$ and $\gamma(H_1) = \gamma(G) + 1$.

For each $y \in V_0, S_1 \cup \{w_y\}$ and $S_1 \cup \{x\}$ are γ - sets of H_1. Let $z \in A$ and S^* be a γ - set of G such that $z \in S^*$. Then $S^* \cup \{x\}$ is a γ - set of H_1 containing z. The set of all good vertices of H_1 contains $A \cup \{x, w_y | y \in V_0\}$, and hence the set of all bad vertices in H_1 is a proper subset of B. Note that if x_0 is a bad vertex of H_1, then $N(x_0)$ in H_1 is contained in $V(G)$ i.e., $w_y \notin N(x_0)$, for all $y \in V_0$.

If x_0 is a bad vertex of H_1 and S_2 is a γ - set of H_1 such that $|N(x_0) \cap S_2|$ is maximum, then obtain a subdivision of H_2 of H_1 by subdividing the edges x_0y, where $y \in N(x_0) \cap S_2$. As $N(x_0)$ of H_1 is contained in V (G), the subdivision H_2 of H_1 is a subdivision of G, i.e., the edges of H_1 which are subdivided to obtain H_2 are edges in G and they are not subdivided while obtaining H_1. Then the set of all bad vertices in H_2 is a proper subset of the set of all bad vertices of H_1.

Proceeding like this, we obtain a finite sequence $H_1, H_2, ..., H_k$ of subdivision of G such that, each H_{i+1} is a subdivision of H_i, and the number of bad vertices of H_{i+1} is less than the number of bad vertices of H_i. Hence for some $k, (\leq |B|)$, we obtain an excellent graph H_k. Denote this H_k by H.

Remark

An algorithm to obtain an excellent subdivision graph of a non excellent graph can be obtained using the proof of Theorem 2. We refer to this algorithm as Excellent Subdivision Algorithm (ESA) in the next section. The process of obtaining H_{i+1} from H_i is called one iteration of ESA.

4. Excellent Subdivision Number

For a given graph G, if S(G) is a subdivision of G, $|V(S(G))| - |V(G)|$ *is denoted by* $p(S(G))$*. The min* $\{p(S(G)) \mid S(G)$ *is a subdivision of G and S(G) is excellent* $\}$ *is called the excellent subdivision number of G and is denoted by ESdn(G). By Theorem 2, ESdn(G) exists. We note that*

1. If G itself is excellent, $ESdn(G) = 0$.

2. For $G = K_{1,n} (n \geq 2), ESdn(G) = n - 1$.

3. Let P_n *be a path on n vertices. Then* $ESdn(P_n) = 1, 0$, *or 2 according as* $n \equiv 0$, *1, or 2 (mod 3). If* $P_n = u_1, u_2, ..., u_n$, *then* $A = \{u_{2+3i} \mid 0 \leq i < \frac{n}{3}\}$ *if* $n \equiv 0$ *(mod 3),* $A = \{u_{1+3i}, u_{2+3i} \mid 0 \leq i < \frac{n}{3}\}$ *if* $n \equiv (2$ *mod 3).*

In the following theorem, we obtain an upper bound for ESdn(G).

Theorem 3 *Let G be a connected graph. The excellent subdivision number*
$ESdn(G) \leq q - \gamma(G)$, *where* $q = |E(G)|$.

Proof. *If G is excellent, then ESdn (G) = 0. So assume that G is not excellent. Let A and B be the set of all good and bad vertices of G respectively. Let* $\gamma(G) = m$. *Fix one* γ *- set* $S = \{x_1, x_2, ..., x_m\}$ *for G. Then* $S = X \cup Y_1 \cup Y_2$, *where*

$X = \{x \in S \mid N(x) \cap (A - S) \neq \phi\}$

$Y_1 = \{x \in S \mid N(x) \cap (A - S) = \phi$, *but* $N(x) \cap S \neq \phi\}$ *and*

$Y_2 = \{x \in S \mid N(x) \cap A = \phi\}$. *Let* $Y = Y_1 \cup Y_2$.

If $u \in S$, *let* $PN(u) = \{v \in V - S \mid N(v) \cap S = \{u\}\}$. *We claim that* $|PN(u)| \geq 2$, *for all* $u \in Y$. *If* $y \in Y_1$, *as* $N(y) \cap S \neq \phi, PN(y) \neq \phi$. *If* $|PN(y)| = 1$ *and* $|PN(y)| = \{v\}$, *then* $v \in B$, *(as* $N(y) \cap (A - S) = \phi$), *and* $(S - y) \cup \{v\}$ *is a* γ *- set for G, which is a contradiction to* $v \in B$. *So* $|PN(y)| \geq 2$ *for all* $y \in Y_1$. *If* $z \in Y_2$ *,(and as G is connected),* $N(z)$ *is a nonempty subset of B. As* $N(z) \neq \phi$ *and* $N(z) \subseteq B, PN(z) \neq \phi$. *If* $PN(z) = \{w\}$, *then* $(S - \{z\}) \cup \{w\}$ *is a* γ *- set for G, which is a contradiction as* $w \in N(z) \subseteq B$. *So* $|PN(z)| \geq 2$, *for all* $z \in Y_2$.

Now we show that there is a subdivision graph H of G which is excellent and to each $u \in S$, *there is one* $g(u) \in (V - S) \cap (N(u))$ *such that the edge* $ug(u)$ *is not subdivided in the process of obtaining the graph H.*

To each $x \in X$, *select one vertex* $g(x) \in (A - S) \cap N(x)$. *(It may happen that* $g(x_1) = g(x_2), x_1 \neq x_2$ *in X).*

If $Y = \phi$, *apply ESA to obtain a subdivision graph H of G* *which is excellent. In any iteration of the ESA the edges $xg(x), x \in X$ are not subdivided . Thus in this case, $ESdn(G) \leq q - \gamma(G), as |S| = |X| = \gamma(G)$.*

So assume that $Y \neq \phi$. Let $Y = \{y_1, y_2, ..., y_k\}$. Then as $Y \subseteq S$, $k \leq m$. To each $i, 1 \leq i \leq k$, let $PN(y_i) = \{w_{i1}, w_{i2}, ..., w_{is_i}\}$, where $s_i = |PN(y_i)|$.

Start the first iteration of the ESA by selecting the vertex w_{11}. At the end of this iteration, w_{11} has become a good vertex. Possibly some other $w_{1j} (j > 1)$ have also become good in the process. If w_{1j} has become a good vertex, for some $j > 1$, select one such vertex and call it $g(y_1)$. In future iterations of ESA, the edge $y_1 g(y_1)$ remains unsubdivided. If all $w_{1j} (j > 1)$ remain bad at the end of the first iteration, start the next iteration of ESA by selecting the vertex w_{12}. Proceed similarly till one of the vertices w_{1j} has become a good vertex for some $j > t$ at the end of the t^{th} iteration for some $t < s_1$. Once we get a good vertex w_{1j} for some $j > t$ at the end of the t^{th} iteration, for some $t < s_1$, select one such good vertex and call it $g(y_1)$. We claim that there is one $t < s_1$, such that at the end of the t^{th} iteration, $PN(y_1)$ contains atleast t + 1 good vertices. Assume that for every $t < s_1 - 1$, at the end of the t^{th}iteration $PN(y_1)$ contains exactly t good vertices. Then at the end of the $(s_1 - 1)^{th}$ iteration, $S \cup \{w_{11}, w_{12}, ..., w_{1(s_1 - 1)}\}$ is a γ - set of the resulting graph and $S \cup PN(y_1) - \{y_1\}$ is also a γ - set and hence, in this case $PN(y_1)$ contains s_1 good vertices.

Thus starting the first iteration of ESA by selecting the bad vertex w_{11}, continue the iterations until $PN(y_1)$ contains more number of good vertices than the number of iterations completed (i.e., until we get a vertex $g(y_1)$ in $PN(y_1)$). We call this process one cycle of iterations of ESA at $PN(y_1)$. Thus at the end of the first cycle of iterations of ESA we get a vertex $g(y_1)$ in $PN(y_1)$ such that the edge $y_1 g(y_1)$ (of G) is not subdivided in all the iterations of this cycle and also in future iterations.

At the end of this cycle of iterations at $PN(y_1)$, there may be some $i > 1$ such that some of the vertices of $PN(y_i)$ have become good. For each such i, select one such vertex in $PN(y_i)$ and call it $g(y_i)$. Select a least $j > 1$, if it exists, such that $PN(y_j)$ contains only bad vertices even at the end of the previous cycle of iteration, and do the cycle of iterations of ESA at $PN(y_j)$ till we get $g(y_j)$. Continue the cycle of iteration process till we get $g(y_1), g(y_2), ..., g(y_m)$.

Let H_1 be the graph thus obtained from G after performing *these cycle of iterations. Untill now the edges $y_ig(y_i)$ of G are not subdivided. H_1 need not be an excellent graph. So apply ESA algorithm to H_1 to get an excellent graph H_2 which is the subdivision graph of H_1 (and hence of G). In H_2, the edges $y_ig(y_i)(1 \leq i \leq m)$ remain unsubdivided.So $ESdn(G) \leq q - \gamma(G)$.*

Remarks

1. We observe that in addition to the $\gamma(G)$ edges obtained above, every edge in the induced graph$\prec S \succ$ of G remains unsubdivided. By considering a γ - set S of G, for which $|E \prec S \succ|$ is maximum, we obtain the following bound.
$ESdn(G) \leq q - \gamma(G) - max\{|E \prec S \succ| : S$ *is a*γ - *set of* $G\}$*.....(1)*

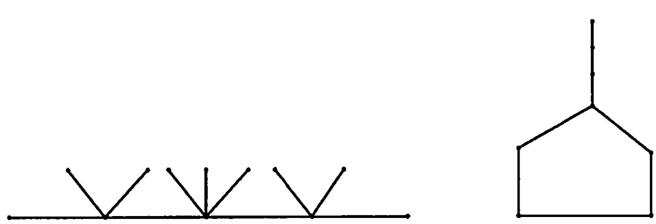
2. As in no iteration of ESA algorithm, the edges in $\prec A \succ$, the subgraph in G induced by the set of good vertices A, is subdivided, we have
$ESdn(G) \leq q - |E \prec A \succ|$*.....(2)*

3. The upper bound given in (1) and (2) are the best , as there are many graphs for which $ESdn(G)$ attains these upper bounds.

(a) *(b)*

Fig.2

- *For example the graph given Fig.2(a), $|A| = 10, |B| = 1$, $|E \prec A \succ| = 8$, max $|E \prec S \succ| = 3, \gamma(G) = 5$ and $ESdn(G) = 2$. For the graph given in Fig.2(b), $|B| = 1, |E \prec S \succ| = 3$, $q = 5, \gamma = 2$, max $|E \prec S \succ| = 1$.*



(a) *(b)*

Fig.3

- *The graph in Fig.3(a) is an example for which the upper bound (1) is attained but (2) is not attained. Here* $|A| = 3$, $|B| = 9, q = 11,$ $max\,|E \prec S \succ| = 2, \gamma(G) = 3, ESdn(G) = 6$.

- *The graph in Fig.3(b) is an example for which the upper bound (2) is attained but (1) is not attained. Here* $\gamma = 3, |E \prec S \succ|$ $= 0, q = 8, ESdn = 2, E \prec A \succ = 6$.

Theorem 4 *If* $ESdn(G) = q - |E \prec A \succ|$ *or* $q - \gamma(G) - max\{|E \prec S \succ| : S$ *is a* γ *- set of* $G\}$*, then in G, the set B of bad vertices in G is an independent set.*

Proof. Assume that there exist $b_1, b_2 \in B$ such that $b_1b_2 \in E(G)$. It is enough to prove that $ESdn(G)$ does not attain any of these two upper bounds. Let S_0 be a γ - set of G such that $|E \prec S_0 \succ| = max\{|E \prec S \succ| : S$ is a γ set of G$\}$. Now subdivide the edge b_1b_2 by introducing a new vertex w. Let the resulting graph be H_1.

As S is a γ - set for G, each $b_i (i = 1, 2)$ is adjacent to some vertex in S. Let $|N(b_2) \cap S| \leq |N(b_1) \cap S|$ and $\lambda = |N(b_1) \cap S|$. Clearly $\lambda \geq 1$. Now let $S' = S_0 \cup \{b_1\}$. Then S' is a γ - set for H_1 and $|E \prec S' \succ| = |E \prec S_0 \succ| + \lambda > |E \prec S_0 \succ|$. The vertices b_1, b_2, w are good in H_1. So the edges b_1w, wb_2 remain unsubdivided under ESA algorithm.

Now apply ESA to H_1 as given in the proof of the Theorem 3, using the γ - set S'. At the end we get an excellent graph H_2 which is a subdivision of H_1, and hence of G (as the edges b_1w, b_2w are not subdivided in the process). As in the proof of the Theorem 3, for each $u \in S'$, $\exists\, g(u) \in V(H_1) - S'$ such that the edges $ug(u)$ are not subdivided. Out of these edges the edges $ug(u), u \in S_0$, are the edges in G also. So these $\gamma(G)$ edges together with $E \prec S' \succ$, also remain as edges in H_2. So $ESdn(G) \leq q - \gamma(G) - |E \prec S_0 \succ| - \lambda$. Thus $ESdn(G)$ does not attain the upper bound given in (1).

Let A' be the set of edges of G whose one end is in $\{b_1, b_2\}$ and other end in A. As S is a γ - set for G, A' contains atleast two edges. These edges along with $E \prec A \succ$ are not subdivided in the process of obtaining the excellent graph H_2. So $ESdn(G) \leq q - |E \prec A \succ| - |A'| < q - |E \prec A \succ|$ and $ESdn$ does not attain the upper bound given in (2).

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Reference

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