SYMMETRIC COLORINGS OF REGULAR POLYGONS

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ABSTRACT. It is calculated the number of symmetric r-colorings of vertices of a regular n-gon and the number of equivalence classes of symmetric r-colorings. A coloring is symmetric if it is invariant in respect to some mirror symmetry with an axis crossing the center of polygon and one of its vertices. Colorings are equivalent if we can get one from another by rotating about the polygon center.

Let G be a finite Abelian group and let $r \in \mathbb{N}$. An *r-coloring* of G is any mapping $\chi : G \to \{1, \dots, r\}$. The group G naturally acts on the set of colorings. Given any coloring χ and element $g \in G$, the coloring χg is defined by

$$\chi g(x) = \chi(x - g)$$

for every $x \in G$. Colorings χ and φ are *equivalent* if they belong to the same orbit (i.e. there exists an element $g \in G$ such that $\chi(x - g) = \varphi(x)$ for every $x \in G$). Obviously, the number of all r-colorings of G equals $r^{|G|}$. The number of equivalence classes can be easily calculated by using Burnside's Lemma [1, I.§3]. It equals $\frac{1}{|G|} \sum_{g \in G} r^{|G/\langle g \rangle|}$, where $\langle g \rangle$ is the subgroup generated by g.

A coloring χ of G is *symmetric* with respect to an element $g \in G$ if

$$\chi(2g - x) = \chi(x)$$

for every $x \in G$. If χ is symmetric with respect to $g \in G$, then χh is symmetric with respect to $g+h \in G$. Indeed, $\chi h(2(g+h)-x) = \chi(2(g+h)-x-h) = \chi(2g-(x-h)) = \chi(x - h) = \chi h(x)$. In particular, if a coloring is symmetric, then every coloring equivalent to it will be also symmetric. The number of all symmetric r-colorings of G is denoted by $S_r(G)$. The number of equivalence classes of symmetric r-colorings of G is denoted by $s_r(G)$. In this note we deduce general formulae for calculating $S_r(G)$ and $s_r(G)$ (Theorem 1) and simplify them to elementary ones in the case of a finite cyclic group $G = \mathbb{Z}_n$ (Theorem 2). A result close to Theorem 1 was announced in [3]. Ukrainian version of Theorem 2 was published in [4].

Theorem 2 has a special interest since $S_r(\mathbb{Z}_n)$ and $s_r(\mathbb{Z}_n)$ can be interpreted in the following geometric sense: $S_r(\mathbb{Z}_n)$ is the number of symmetric r-colorings of vertices of a regular n-gon and $s_r(\mathbb{Z}_n)$ is the number of equivalence classes of symmetric r-colorings. A coloring is symmetric if it is invariant in respect to some mirror symmetry with an axis crossing the center of polygon and one of its vertices.

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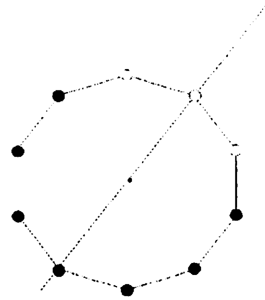
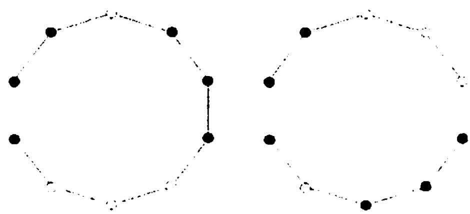


Colorings are equivalent if we can get one from another by rotating about the polygon center.



Note also that $s_r(\mathbb{Z}_n)$ is the number of symmetric necklaces of length n consisting of beads of r colors. It is well known that the number of all necklaces of length n consisting of beads of r colors equals $\frac{1}{n}\sum\limits_{d|n}\varphi(d)r^{\frac{n}{d}}$, where φ is the Euler function (see, for example, [2, §3]).

Theorem 1.

$$s_r(G) = \sum_{X \leq G}\sum_{Y \leq X} \frac{\mu(Y,X)}{|B(G/Y)|} r^{\frac{|G/X|+|B(G/X)|}{2}},$$

$$S_r(G) = \sum_{X \leq G}\sum_{Y \leq X} \frac{\mu(Y,X)|G/Y|}{|B(G/Y)|} r^{\frac{|G/X|+|B(G/X)|}{2}},$$

where $\mu(Y,X)$ is the Möbius function on a lattice of subgroups of G, $B(H) = \{x \in H : 2x = 0\}$.

Proof. Let Y be a subgroup of G, let $C(Y)$ be the set of r-colorings of G symmetric in respect to 0 with the stabilizer Y, and let $\chi \in C(Y)$. Obviously, the number of r-colorings of G equivalent to χ equals $|G/Y|$. We claim that the number of r-colorings of G equivalent to χ and symmetric with respect to 0 equals $|B(G/Y)|$.

Since $\chi g = \chi h$ if and only if $g - h \in Y$, it suffices to check that χg is symmetric with respect to 0 if and only if $2g \in Y$. Indeed,

$$\begin{aligned}\chi g(-x) = \chi g(x) &\Leftrightarrow \chi g(-(x-g)) = \chi g(x-g)\\ &\Leftrightarrow \chi g(-x+g) = \chi g(x-g)\\ &\Leftrightarrow \chi(-x) = \chi(x-2g)\\ &\Leftrightarrow \chi(x) = \chi 2g(x)\\ &\Leftrightarrow 2g \in Y.\end{aligned}$$

So, the number of equivalence classes of symmetric colorings with the stabilizer Y equals $\frac{|C(Y)|}{|B(G/Y)|}$. Consequently,

$$s_r(G) = \sum_{Y \le G} \frac{|C(Y)|}{|B(G/Y)|},$$

$$S_r(G) = \sum_{Y \le G} \frac{|G/Y| \cdot |C(Y)|}{|B(G/Y)|}.$$

On the other side

$$\sum_{Y \le X \le G} |C(X)| = r^{\frac{|G/Y|-|B(G/Y)|}{2}+|B(G/Y)|} = r^{\frac{|G/Y|+|B(G/Y)|}{2}}.$$

Applying Möbius inversion (see [1, IV.§2]), we obtain

$$|C(Y)| = \sum_{Y \le X \le G} \mu(Y,X) r^{\frac{|G/X|+|B(G/X)|}{2}},$$

where $\mu(Y,X)$ is the Möbius function on a lattice of subgroups of G. Finally,

$$\begin{aligned} s_r(G) &= \sum_{Y \le G} \frac{1}{|B(G/Y)|} \sum_{Y \le X \le G} \mu(Y,X) r^{\frac{|G/X|+|B(G/X)|}{2}} \\ &= \sum_{X \le G} \sum_{Y \le X} \frac{\mu(Y,X)}{|B(G/Y)|} r^{\frac{|G/X|+|B(G/X)|}{2}}, \\ S_r(G) &= \sum_{Y \le G} \frac{|G/Y|}{|B(G/Y)|} \sum_{Y \le X \le G} \mu(Y,X) r^{\frac{|G/X|+|B(G/X)|}{2}} \\ &= \sum_{X \le G} \sum_{Y \le X} \frac{\mu(Y,X)|G/Y|}{|B(G/Y)|} r^{\frac{|G/X|+|B(G/X)|}{2}}. \end{aligned}$$

□

From this point p is a variable of prime value. For instance, $\prod_{p|a} f(p)$ means a product where p takes on values of all prime divisors of a, in contrast to $\sum_{d|a} f(d)$, where d takes on values of all divisors of a.

Theorem 2. *If n is odd then*

$$s_r(\mathbb{Z}_n) = r^{\frac{n+1}{2}},$$
$$S_r(\mathbb{Z}_n) = \sum_{d|n} d \prod_{p|\frac{n}{d}} (1-p) r^{\frac{d+1}{2}}.$$

If $n = 2^l m$, where $l \ge 1$ and m is odd then

$$s_r(\mathbb{Z}_n) = \frac{r^{\frac{n}{2}+1} + r^{\frac{m+1}{2}}}{2},$$
$$S_r(\mathbb{Z}_n) = \sum_{d|\frac{n}{2}} d \prod_{p|\frac{n}{2d}} (1-p) r^{d+1}.$$

Proof. Applying formulae from Theorem 1 to $\mathbb{Z}_n$ we obtain

$$s_r(\mathbb{Z}_n) = \sum_{d|n}\sum_{k|d} \frac{\mu(\frac{d}{k})}{2-\delta(\frac{n}{k})} r^{\frac{\frac{n}{d}+2-\delta(\frac{n}{d})}{2}},$$
$$S_r(\mathbb{Z}_n) = \sum_{d|n} \frac{n}{d} \sum_{k|d} \frac{\mu(\frac{d}{k})\cdot\frac{d}{k}}{2-\delta(\frac{n}{k})} r^{\frac{\frac{n}{d}+2-\delta(\frac{n}{d})}{2}},$$

where $\mu(n)$ is the numeric Möbius function (see [5, II.§3]),

$$\delta(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

If $\frac{n}{d}$ is even, then $\delta(\frac{n}{k}) = 0$ and we have

$$\sum_{k|d} \frac{\mu(\frac{d}{k})}{2-\delta(\frac{n}{k})} = \frac{1}{2}\sum_{k|d}\mu(\tfrac{d}{k}) = \frac{1}{2}\sum_{k|d}\mu(k) = \begin{cases} \frac{1}{2} & \text{if } d = 1 \\ 0 & \text{otherwise,} \end{cases}$$
$$\sum_{k|d} \frac{\mu(\frac{d}{k})\cdot\frac{d}{k}}{2-\delta(\frac{n}{k})} = \frac{1}{2}\sum_{k|d}\mu(k)\cdot k = \frac{1}{2}\prod_{p|d}(1-p).$$

Let $\frac{n}{d}$ be odd. Then $\delta(\frac{n}{k}) = \delta(\frac{d}{k})$ and we have

$$\sum_{k|d} \frac{\mu(\frac{d}{k})}{2-\delta(\frac{n}{k})} = \sum_{k|d}\mu(\tfrac{d}{k})\frac{1}{2-\delta(\frac{n}{k})} = \sum_{k|d}\mu(k)\frac{1}{2-\delta(k)},$$
$$\sum_{k|d} \frac{\mu(\frac{d}{k})\cdot\frac{d}{k}}{2-\delta(\frac{n}{k})} = \sum_{k|d}\mu(k)\frac{k}{2-\delta(k)}.$$

Recall that a function $f : \mathbb{N} \to \mathbb{C}$ is *multiplicative* if $f(1) = 1$ and $f(nm) = f(n)f(m)$ for all mutually prime n, m.

Lemma. *$\frac{1}{2-\delta(k)}$ and $\frac{k}{2-\delta(k)}$ are multiplicative functions.*

Proof. Since the function k is multiplicative and a product of multiplicative functions is a multiplicative function, it is sufficient to prove that the function $f(k) = \frac{1}{2-\delta(k)}$ is multiplicative.

Obviously, $f(1) = 1$.

Now let n and m be natural numbers such that $GCD(n, m) = 1$.

Case 1: n and m are odd. Then $\delta(n) = \delta(m) = \delta(nm) = 1$. Thus, $f(n) = f(m) = f(nm) = 1$, so $f(nm) = f(n)f(m)$.

Case 2: n is even and m is odd. Then $\delta(n) = \delta(nm) = 0$, $\delta(m) = 1$. Thus, $f(n) = f(nm) = \frac{1}{2}$, $f(m) = 1$, so $f(nm) = f(n)f(m)$.

□

Now we can apply a formula

$$\sum_{k|d}\mu(k)f(k) = \prod_{p|d}(1 - f(p))$$

that holds for every multiplicative function $f(k)$ [5, II.§3.b]. We obtain

$$\sum_{k|d} \frac{\mu(\frac{d}{k})}{2-\delta(\frac{n}{k})} = \prod_{p|d}\left(1 - \frac{1}{2-\delta(p)}\right) = \begin{cases} 1 & \text{if } d = 1 \\ \frac{1}{2} & \text{if } d = 2^i \\ 0 & \text{otherwise,} \end{cases}$$
$$\sum_{k|d} \frac{\mu(\frac{d}{k})\cdot\frac{d}{k}}{2-\delta(\frac{n}{k})} = \prod_{p|d}\left(1 - \frac{p}{2-\delta(p)}\right) = \begin{cases} \prod_{p|d}(1-p) & \text{if } d \text{ is odd} \\ 0 & \text{if } d \text{ is even.} \end{cases}$$

Hence, if n is odd, then for each $d|n$ one has

$$\sum_{k|d} \frac{\mu(\frac{d}{k})}{2-\delta(\frac{n}{k})} = \begin{cases} 1 & \text{if } d=1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\sum_{k|d} \frac{\mu(\frac{d}{k})\cdot\frac{d}{k}}{2-\delta(\frac{n}{k})} = \prod_{p|d}(1-p),$$

and consequently,

$$s_r(\mathbb{Z}_n) = r^{\frac{n+1}{2}},$$

$$S_r(\mathbb{Z}_n) = \sum_{d|n} \frac{n}{d} \prod_{p|d}(1-p) r^{\frac{\frac{n}{d}+1}{2}} = \sum_{d|n} d \prod_{p|\frac{n}{d}} (1-p) r^{\frac{d+1}{2}}.$$

If n is even, then

$$\sum_{k|d} \frac{\mu(\frac{d}{k})}{2-\delta(\frac{n}{k})} = \begin{cases} \frac{1}{2} & \text{if } d \in \{1, 2^i\} \\ 0 & \text{otherwise,} \end{cases}$$

$$\sum_{k|d} \frac{\mu(\frac{d}{k})\cdot\frac{d}{k}}{2-\delta(\frac{n}{k})} = \begin{cases} \frac{1}{2}\prod_{p|d}(1-p) & \text{if } d|\frac{n}{2} \\ 0 & \text{otherwise,} \end{cases}$$

and consequently,

$$s_r(\mathbb{Z}_n) = \frac{1}{2} r^{\frac{n}{2}+1} + \frac{1}{2} r^{\frac{m+1}{2}},$$

$$S_r(\mathbb{Z}_n) = \sum_{d|\frac{n}{2}} \frac{n}{d} \cdot \frac{1}{2} \prod_{p|d}(1-p) r^{\frac{\frac{n}{d}+2}{2}} = \sum_{d|\frac{n}{2}} d \prod_{p|\frac{n}{2d}} (1-p) r^{d+1}.$$

□

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