RECONSTRUCTION NUMBER FOR ULAM'S CONJECTURE

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Abstract

A vertex-deleted subgraph (subdigraph) of a graph (digraph) G is called a **card** of G. A card of G with which the degree (degree triple) of the deleted vertex is also given is called a degree associated card or **dacard** of G. To investigate the failure of digraph reconstruction conjecture and its effect on Ulam's conjecture, we study the parameter **degree associated reconstruction number** drn(G) of a graph (digraph) G defined as the minimum number of dacards required in order to uniquely identify G. We find drn for some classes of graphs and prove that for $t \geq 2$, drn(tG)$\leq$ 1+drn(G) when G is connected nonregular and drn(tG) $\leq$ m+2-r when G is connected r-regular of order m > 2 and these bounds are tight. drn$\leq$3 for other disconnected graphs. Corresponding results for digraphs are also proved.

1. INTRODUCTION AND DEFINITIONS

A digraph consists of a finite set V of vertices and a set of ordered pairs of distinct vertices. Any such pair is called an **arc**. If uv and vu are both arcs, then they together are called a **symmetric pair** of arcs. If uv is an arc and vu is not an arc, then uv is called an **unpaired outarc** incident with u and an **unpaired inarc** incident with v. For a vertex v of a digraph, the ordered triple (r,s,t) is called the **degree triple** of v where r,s, and t are respectively the number of unpaired outarcs, unpaired inarcs and symmetric pairs of arcs incident with v.

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A vertex deleted subgraph or card G-v of a graph (digraph) G is the unlabeled graph(digraph) obtained from G by deleting the vertex v and all edges (arcs) incident with v. The **deck** of a graph (digraph) G is its collection of cards. The ordered pair (d(v),G-v) is called a **degree associated card** or **dacard** of the graph (digraph) G where d(v) is the degree (degree triple) of v in G. The **degree associated deck** or **dadeck** of a graph (digraph) G is its collection of dacards.

Ulam's Conjecture (UC), also called **reconstruction conjecture (RC)** asserts that a graph on at least three vertices is determined uniquely by its deck. An extension of UC to digraphs—the **digraph reconstruction conjecture (DRC)** proposed by Harary [1] was disproved when Stockmeyer exhibited [10,11] several infinite families of counter examples. Ever since, a search is on for reconstruction related properties of graphs whose analogues do not hold for digraphs , in the hope that such properties will play a key role in proving UC if at all it is true. Also by imposing their analogues on digraphs as additional conditions,we can try to reconstruct digraphs. One such is the following:
Porperty P : When the deck of a graph is known, the degree of the deleted vertex and the sequence of degrees of the neighbours (neighbourhood degree sequence) of the deleted vertex can be determined for each card.

P is not true in the case of digraphs even though the collection of degree triples of the vertices can be determined from the deck [4]. This led to the concept of **reconstruction from dacards** (N- reconstruction) and the proof that all digraphs in Stockmeyer's counter examples to DRC fall within classes of digraphs that are reconstructible from their dadecks [7,8]. No pair of nonisomorphic digraphs with the same dadeck is so far known.

For a reconstructible graph G, Harary and Plantholt [3] have defined the **reconstruction number** rn(G) to be the size of the smallest subcollection of the deck of G which is not contained in the deck of any other graph H, H $\not\cong$ G. Myrvold [6] referred to this parameter as **ally-reconstruction number** of G. She has also studied **adversary reconstruction number** of G which is the smallest k such that no subcollection of the deck of G of size k is contained in the deck of any other graph H, H $\not\cong$ G. However, if G and H are graphs having the same deck, then they have the same dadeck, whereas a subcollection of the deck of a graph need not always give the corresponding dacards. So, rn(G), which uses partial deck can not be expected to contribute much to the ultimate solution of UC as it ignores an important aspect (degree association) which differentiates UC from its disproved extension DRC. To make up this deficiency and to address UC more effectively, drn(G) covering graphs and digraphs alike was defined in [9] as follows.

The **degree (degree triple) associated reconstruction number** drn(G) of a graph (digraph) G which is reconstructible from its dadeck is the size of the smallest subcollection of the dadeck of G which is not contained in the dadeck of any other H, H $\not\cong$ G. An **s-blocking set** of G is a family F of graphs(digraphs) such that G $\notin$ F and each collection of s dacards of G will also appear in the dadeck of some graph (digraph) of F.

There are graphs for which rn and drn have different values. As an example, $rn(2K_3)=5$ whereas $drn(2K_3)=3$. We find upper bounds for the value of drn for some classes of graphs and digraphs and evaluate drn for some classes. We use the terminology in [2].

2. drn(G) FOR GRAPHS G

The following observations are obvious.

1. If G and H have a common dacard, then the number of vertices in G and H are equal as also the number of edges in G and H.
2. $drn(G) = drn(\overline{G}) \leq rn(G)$
3. $drn(K_n) = 1$ and $drn(nK_2) = 3$ for $n \geq 2$.

Result 2. 1[9]: $drn(C_n) = 3$ for $n \geq 5$.

Result 2.2: $drn(K_{m,m}) = 3$ for $m \geq 2$.

Proof: All dacards of $K_{m,m}$ are $(m, K_{m-1,m})$. Let A and B denote the parts of $K_{m-1,m}$ containing m-1 and m vertices respectively. All graphs having a dacard $(m, K_{m-1,m})$ can be obtained by adding a vertex w to $K_{m-1,m}$ and joining it with m vertices in $A \cup B$. In this process, w is joined to at least one vertex v_1 of B. If w is joined to no vertex of A, the graph obtained is $K_{m,m}$. If w is joined to at least one vertex v_2 of A, then the resulting graph G^* has a triangle v_1v_2w and hence at most three dacards of G^* are triangle-free. Of these, deg $v_2 = m+1$ and hence G^* has at most two dacards in common with $K_{m,m}$. Hence $drn(K_{m,m}) = 3$. ($\{K_{m-1,m+1}+e\}$ where e joins a pair of vertices each of degree m-1 is a 2-blocking set).

Similarly we can prove

Result 2.3: $drn(K_{m,n}) = 2$ for $2 \leq m < n$.

Result 2.4: For $t \geq 2$, $drn(tC_n) = 3$, $n \geq 3$, $n \neq 4$ and $drn(tC_4) = 4$.

For disconnected graphs, Myrvold and Molina have proved the following.

Theorem 2.5[5,6]: If G is a disconnected graph and the components are not all isomorphic, then $rn(G) \leq 3$.

Theorem 2.6[5,6]: $rn(tG) \leq m+2$ for $t \geq 2$ where G is a connected graph of order m.

For disconnected graphs all whose components are isomorphic, we prove the following results.

Theorem 2.7: For $t \geq 2$, $drn(tG) \leq 1+drn(G)$ where G is a connected nonregular graph which is reconstructible from its dadeck and has at least 3 vertices.

Proof: Let c be the order of G. Each dacard of tG has at least t components, t-1 of which are isomorphic to G. Let F be a collection (d_i, G_i), $i=1,2,\ldots,m$ of $m = drn(G)$ dacards that identify G uniquely. From the dadeck of tG, we select a subdeck S of size at most m+1 as follows.

Case 1: At least one of the G_i in F is connected.

If $d_1=d_2=\ldots=d_m$, let (d_{m+1},G_{m+1}) be a dacard of G with $d_{m+1} \neq d_1$. Let S be the collection $(d_i,G_i\cup(t-1)G)$, $i=1,2,\ldots,k$ where $k=m+1$ or m according as $d_1=d_2=\ldots=d_m$ or not.

Case 2: G_i is disconnected for all i.

Let (d_{m+1},G_{m+1}) be a dacard of G which is connected. Let S be the collection $(d_i,G_i\cup(t-1)G)$, $i=1,2,\ldots,m+1$.

Now let A be a dacard in S having exactly t components and associated degree d_i. S has either a dacard B_1 with associated degree d_j different from d_i or a dacard B_2 with at least t+1 components. We will prove that every graph whose dadeck contains $\{A,B_1\}$ or $\{A,B_2\}$ has exactly t components, each of order c of which t-1 are isomorphic to G ...(1)

A graph R having dacard A can be obtained by annexing a vertex v to A and joining v to suitable vertices of A.

(i) If v is joined to at least two components of order c, all dacards of R other than A have a component of order at least c+1. So neither $\{A,B_1\}$ nor $\{A,B_2\}$ is contained in the dadeck of R.

(ii) If v is joined to a component of order c and the component H of order c-1 then in R, $\langle V(H)\cup\{v\}\rangle \not\cong G$ as it has less number of edges than G. Hence an arbitrary dacard of R other than A either has a component of order at least c+1 or does not have t-1 components each isomorphic to G. Hence neither $\{A,B_1\}$ nor $\{A,B_2\}$ is contained in the dadeck of R.

(iii) If v is joined to only one component of A and it is a component of order c, then dacard B_1 or B_2 is obtained from R by deleting a vertex w of the component M of R with c+1 vertices. If $M-w \cong G$, the dacard is isomorphic to A and is different from B_1 and B_2. If $M-w \not\cong G$, the dacard does not have t-1 components isomorphic to G and hence neither $\{A,B_1\}$ nor $\{A,B_2\}$ is contained in the dadeck of R.

In the only other option, v is joined only to the component of order c-1 of A and this proves (1).

We now prove that the collection S identify tG uniquely among graphs of the type given in (1).

Let $J=H\cup(t-1)G$ have the collection S in its dadeck.

(i) If S is contained in the subdeck of J obtained by deleting vertices of H, then (d_i,G_i), i=1 to m are in the dadeck of H also and hence $H \cong G$.

(ii) If some dacard $(d_i, G_i \cup (t-1)G)$ in S is obtained by deletion of a vertex w of J which is not in H, then $G_i \cup (t-1)G \cong H \cup (t-2)G \cup (G-w)$ and hence $H \cong G$ since G and H have the same order.
Thus $J = tG$ and hence $drn(tG) \leq |S| \leq 1+drn(G)$.

Note: The inequality in the above theorem can not be improved $(drn(K_{1,2})=1$ and $drn(2K_{1,2})=2)$.

Theorem 2.8: For $t>1$, $drn(tG) \leq m-r+2$ where G is a connected r-regular $(r \geq 2)$ graph of order $m>2$.

Proof: Each dacard of tG has exactly t-1 components of order m, exactly r vertices of degree r-1 and all other vertices of degree r . Let S be a chosen collection of m-r+2 dacards of tG . Let C_1, C_2, and C_3 be any three members of S. As in the proof of Lemma 4 in [6], a graph having the dacards C_1, C_2, and C_3 in its dadeck must have either (i) all components of order m or (ii) t-2 components of order m, one of order m+1 and one of order m-1.

A graph of type (i) having dacard C_1 is obtained from C_1 by adding a new vertex w and joining it only with components of C_1 of order less than m. If the resulting graph R is regular, then it must be tG. If it is not regular, then the component H of R containing w has order m and has a vertex v of degree r-l. So $H \ncong G$ and the dacards of R that are in S must be obtained by deletion of vertices of H (since no member of S has H as a component). However, the dacards of R obtained by deleting v and its neighbours can not be in S. So at most m-r dacards of R are in S. So S is not contained in the dadeck of R.

A graph of type (ii) having a dacard C_1 is obtained from C_1 by adding a new vertex w and joining it with vertices of a single component G of C_1. The resulting graph R has a component H of order m+1. So dacards of R that are in S must be obtained by deletion of vertices of H. However, H has r vertices each of degree r+1 and the dacards of R obtained by deleting these vertices are not in S. So at most m+1-r dacards of R are in S. So S is not contained in the dadeck of R.

Thus a graph containing S in its dadeck is tG and hence $drn(tG) \leq |S| = m-r+2$.

The inequality in the above theorem can not be improved as seen from the following.

Corollary 2.9: For $t > 1$, $drn(tK_n) = 3$ for $n \geq 2$.
(The graph $K_{n-1} \cup (t-2)K_n \cup K_{n+1}-e$ constitute a 2-blocking set).

Corollary 2.10: For $t > 1$ and $m>1$, $drn(tK_{m,m}) = m+2$.

Proof: By Theorem 2.8, $drn(tK_{m,m}) \leq m+2$. Also $tK_{m,m}$ has $\{(t-2)K_{m,m} \cup K_{m,m+1} \cup K_{m-1,m}\}$ as an (m+1)-blocking set.

3.drn(D) FOR DIGRAPHS D

If two digraphs have a dacard in common, then they have the same number of vertices, arcs and symmetric pairs of arcs.
A digraph is called connected or disconnected according as its underlying graph is connected or not.

The proofs in [5,6] can be easily extended to prove the following for digraphs.

Theorem 3.1: If D is a disconnected digraph and the components are not all isomorphic, then $drn(D) \leq 3$.
Theorem 3.2: For $t > 1$, $drn(tD) \leq m+2$ where D is a connected digraph of order m, $m \geq 3$.

The proof of Theorem 2.7 above, which is based mainly on the number of edges incident with vertices can be easily extended to prove the following.

Theorem 3.3: For $t > 1$, $drn(tD) \leq 1 + drn(D)$ where D is a digraph whose underlying graph (multigraph) is connected and nonregular.

Theorem 3.4: For $t > 1$ and $r > 0$, $drn(tD) \leq 1 + \max\{|D|-2r+1, r+1\}$ where D is a connected digraph all whose vertices have degree triple (r,r,0).
Proof: All the dacards of tD will be of the form $((r,r,0),(D-v)\cup(t-1)D)$ where v is a suitable vertex of D. Let A be a dacard of tD containing exactly t components. Let S be a subdeck of the dadeck of tD consisting of $1+\max\{|D|-2r+1, r+1\}$ dacards including A.

All digraphs E having the dacard A can be obtained by adding a vertex w to A and suitable arcs. If w is joined to a component of A isomorphic to D and another component, then the resulting digraph will have only the dacard A common with tD (as all others have either a component with more than |D| vertices or the degree triple associated is other than (r,r,0)) and hence the dadeck of the resulting digraph do not contain S.

If w is joined only to vertices of a component D of A, then in E, the component H containing w has r vertices of degree triple (r+1,r,0) and r vertices of degree triple (r,r+1,0). As H has |D|+1 vertices, a dacard of E will be isomorphic to a dacard of tD only when the deleted vertex belongs to H . As H has at most |D|+1-2r vertices with degree triple (r,r,o), the dadeck of E cannot contain S.

If w is joined to no component of A isomorphic to D, then w must be joined only to the component of A corresponding to D-v, resulting in a component H with |D| vertices. If all the vertices of H have the same degree triple (r,r,o), then $H \cong D$ and $E \cong tD$. Otherwise, H is not isomorphic to D and hence dacards of E isomorphic to dacards of tD must come only by deletion of vertices of H.
Case 1. E has a vertex u with degree triple (r+1,r-1,0).

A dacard of tD can be got from E only by deleting a vertex of E lying on an outarc incident with u. Hence E has at most r+1 dacards common with tD. Hence the dadeck of E cannot contain S.

Case 2. E has a vertex u with degree triple (r-1,r+1,0).

As in case 1 above, the dadeck of E cannot contain S.

Case 3. Some vertex u of E has degree triple (r-l,r,0) or (r,r-l,0).

By deleting u or any of the 2r-1 vertices joined to u, we cannot get a dacard of tD. Hence E has at most |D|-2r dacards common with tD and the dadeck of E does not contain S.

So the dadeck of a digraph E containing dacard A does not contain S whenever E $\not\cong$ tD. Hence drn(tD)≤|S|.

The inequality in the above theorem cannot be improved as seen from the following result.

Result 3.5: For t ≥ 2 and m > 0, drn(tD) = m+2 where D is a digraph with 3m vertices and $3m^2$ arcs such that the vertex set of D can be partitioned into sets A, B and C of equal size and arcs of D are precisely those directed from A to B, B to C and C to A.

Proof: By the above theorem, drn(tD) ≤ m+2.

Let E be a digraph with 3m+1 vertices and $3m^2+2m$ arcs such that its vertex set can be partitioned into sets X,Y and Z of cardinality m+1,m and m respectively and arcs are precisely those directed from X to Y, Y to Z and Z to X. All dacards of tD are identical and the digraph E∪(D-v)∪(t-2)D has m+1 dacards in common with tD so that drn(tD) ≥ m+2. This completes the proof.

4.CONCLUSION

In our attempt to investigate the failure of DRC and the possible truth of Ulam's conjecture, we have defined reconstruction from dacards and drn(G) of graphs and digraphs. There are graphs and digraphs G with given value for drn(G). However, all those with high drn(G) so far known are regular with all dacards identical and either they or their complements are disconnected with isomorphic components and hence are easily reconstructible. This suggests the following:

(1) The conjecture in [3] that if G has odd prime order, then rn(G)≤3 can be first tried for drn(G) of graphs and digraphs.

(2) Some more information that is derivable from the full deck of graphs may be attached with the cards while trying reconstruction from partial decks.

(3) The pre 1964 approach of establishing G≅H to settle Ulam's conjecture deserves more attention. Unless sufficient conditions for legitimate decks are found and used. `reconstruction from deck` approach is not likely to settle Ulam's conjecture in full.

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