Restricted Edge Connectivity of Edge Transitive Graphs *

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Abstract

Let $G = (V, E)$ be a connected graph and $S \subseteq E$. S is said to be an r-restricted edge cut if $G - S$ is disconnected and each component in $G - S$ contains at least r vertices. Define $\lambda^{(r)}(G)$ to be the minimum size of all r-restricted edge cuts and $\xi_r(G) = \min\{\omega(U)\colon U \subseteq V, |U| = r$ and the subgraph of G induced by U is connected$\}$, where $\omega(U)$ denotes the number of edges with one end in U and the other end in $V \backslash U$. A graph G with $\lambda^{(r)}(G) = \xi_r(G)$ $(r = 1, 2, 3)$ is called a $\lambda^{(3)}$-optimal graph. In this paper, we show that the only edge-transitive graphs which are not $\lambda^{(3)}$-optimal are the star graphs $K_{1,n-1}$, the cycles C_n and the cube Q_3. Based on this, we determine the expressions of $N_i(G)$ $(i = 0, 1, \cdots, \xi_3(G) - 1)$ for edge transitive graph G, where $N_i(G)$ denotes the number of edge cuts of size i in G.

Keywords: Restricted edge connectivity; Edge transitive graphs.

1 Introduction

We follow [1] and [2] for graph theoretical terminology not defined here.

A well known model for network reliability consists of an undirected graph $G = (V, E)$ in which the vertices are reliable while the edges may fail independently with the same probability $\rho \in (0, 1)$. The reliability of the network can be measured by the probability $P(G, \rho)$ of G being

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disconnected [3]

$$P(G,\rho) = \sum_{i=\lambda}^{m} N_i(G)\rho^i(1-\rho)^{m-i},$$

where $m = |E|$, $\lambda = \lambda(G)$ is the edge connectivity of G, and $N_i(G)$ is the number of edge cuts of size i in G. Generally, to determine $N_i(G)$ is difficult [3, 4, 5].

When ρ is sufficiently small, the problem of minimizing $P(G,\rho)$ becomes a matter of minimizing the first term

$$N_\lambda(G)\rho^\lambda(1-\rho)^{m-\lambda},$$

which can be obtained by maximizing λ first and then minimizing N_λ [6]. For a general graph G, $\lambda(G) \leq \delta(G)$, where $\delta(G)$ denotes the minimum degree of G. If $\lambda(G) = \delta(G)$, we call G a *λ-optimal* graph (or *maximally edge connected* graph in some literatures). To minimize N_λ, Bauer et al. [7] defined the so-called super-λ graphs. A connected graph G is said to be *super-λ* if every edge cut of size λ isolates a vertex.

For further study, Esfahanian and Hakimi [8] proposed the concept of restricted edge connectivity. *A restricted edge cut* of G is a set of edges whose removal disconnects G and every component of $G - S$ has at least two vertices. The *restricted edge connectivity* $\lambda'(G)$ is the minimum size of restricted edge cuts of G. Esfahanian and Hakimi also proved that a connected graph G with at least four vertices which is not a star graph $K_{1,n-1}$ has restricted edge cuts, and thus $\lambda'(G)$ is well-defined. Furthermore, $\lambda'(G) \leq \xi(G)$, where $\xi(G) = \min\{d(x) + d(y) - 2 \mid (x,y) \in E(G)\}$ is the minimum edge degree of G. A graph G with $\lambda(G) = \delta(G)$ and $\lambda'(G) = \xi(G)$ is called a *λ'-optimal* graph. M. Wang and Q. Li showed in [9] that when ρ is sufficiently small, a k-regular λ'-optimal graph is more reliable than those which are not λ'-optimal in the class of graphs with n vertices and $nk/2$ edges.

In recent years, there are many works studying a generalization of the above concept: r-restricted edge cuts [10, 11, 12, 13].

Definition 1 An edge subset S of a connected graph G is called a *r-restricted edge cut* if $G - S$ is disconnected and each component of $G - S$ contains at least r vertices. The *r-restricted edge connectivity* $\lambda^{(r)}(G)$ is the minimum size of all r-restricted edge cuts in G.

Let $G = (V, E)$ be a graph and $U_1, U_2 \subseteq V$. Denote by $[U_1, U_2]$ the set of edges with one end in U_1 and the other end in U_2. For $U \subseteq V$, denote by $\overline{U}$ the complement of U in V, and $G[U]$ the subgraph induced by U. Let $\omega(U) = |[U, \overline{U}]|$. The following inequality is well-known [14]:

$$\omega(U_1 \cup U_2) + \omega(U_1 \cap U_2) \leq \omega(U_1) + \omega(U_2). \quad (1)$$

Let $n = |V|$ and $1 \le r \le n$. Define

$$\xi_r(G) = \min\{\omega(U) : |U| = r,\ G[U] \text{ is connected}\}.$$

Clearly, $\lambda^{(1)}(G) = \lambda(G)$, $\lambda^{(2)}(G) = \lambda'(G)$ and $\xi_1(G) = \delta(G)$, $\xi_2(G) = \xi(G)$. So, the above upper bounds for $\lambda(G)$ and $\lambda'(G)$ can be rewritten as $\lambda^{(1)}(G) \le \xi_1(G)$ and $\lambda^{(2)}(G) \le \xi_2(G)$, and the graphs with the equalities are considered to be optimal with respect to these parameters.

In [10], Bonsma et al. studied the existence of 3-restricted edge cuts and showed that $\lambda^{(3)}(G) \le \xi_3(G)$ holds for any graph G containing 3-restricted edge cuts. Thus, it is natural to define a graph G to be $\lambda^{(3)}$*-optimal* if $\lambda^{(r)}(G) = \xi_r(G)$ for $r = 1, 2, 3$. It is shown in [12] and [13] that, when ρ is sufficiently small, a k-regular $\lambda^{(3)}$-optimal graph is more reliable than those which are not $\lambda^{(3)}$-optimal in the class of graphs with n vertices and $nk/2$ edges. The problem is which classes of graphs are $\lambda^{(3)}$-optimal.

In designing networks, graphs with some symmetry are often used as the topology for the reason that they have many desirable properties, such as simple routing algorithms and high (edge) connectivity. For edge transitive graphs, the following results are known:

Theorem A ([15]) Every connected edge transitive graph G is κ-optimal (that is, $\kappa(G) = \delta(G)$, where $\kappa(G)$ is the connectivity of G), and thus is $\lambda^{(1)}$-optimal.

Theorem B ([16, 17]) The only connected edge transitive graphs on n vertices which are not $\lambda^{(2)}$-optimal are the cycles C_n and the star graphs $K_{1,n-1}$.

Theorem C ([12]) The only connected edge and vertex transitive graphs which are not $\lambda^{(3)}$-optimal are the cycles C_n and the cube Q_3.

In this paper, we will generalize the characterization in Theorem C to that of all edge transitive graphs. Our main result is

Theorem 2 The only edge transitive graphs with order at least 6 which are not $\lambda^{(3)}$-optimal are the star graphs $K_{1,n-1}$, the cycles C_n and the cube Q_3.

The existence of 3-restricted edge cuts in an edge transitive graph which is not isomorphic to $K_{1,n-1}$ is fairly easy to be obtained just by the definition. Or in another way, the existence can also be seen from Theorem 2.2 in [10] which characterizes all graphs without 3-restricted edge cuts.

For edge transitive graphs, the following result is well known:

Theorem D ([1]) Let G be an edge transitive graph. If G is not vertex transitive, then G is a bipartite graph, and all vertices in a same bipartition have the same degree.

In light of Theorem D and Theorem C, we see that to complete the characterization of $\lambda^{(3)}$-optimal edge transitive graphs, we need only to consider bipartite graphs. The key to the proof lies in the disjointness of $\lambda^{(3)}$-atoms which will be defined and shown in section 2. Then in Section 3, Theorem 2 is proved, and based on it, the expressions of $N_i(G)$ ($i = 0, 1, \cdots, \xi_3(G) - 1$) for edge transitive graph G are determined.

2 Disjointness of $\lambda^{(3)}$-atoms

Let $G = (V, E)$ be a connected graph, and F a non-empty subset of V. F is called a $\lambda^{(r)}$-*fragment*, if $[F, \overline{F}]$ is an r-restricted edge cut with $\omega(F) = \lambda^{(r)}(G)$. Clearly, if F is a $\lambda^{(r)}$-fragment, so is $\overline{F}$, and both $G[F]$ and $G[\overline{F}]$ are connected. A $\lambda^{(r)}$-fragment with the least cardinality is called a $\lambda^{(r)}$-atom. Denote by $\alpha^{(r)}(G)$ the cardinality of a $\lambda^{(r)}$-atom in G. Clearly, $\alpha^{(r)}(G) \leq |V|/2$. Bonsma et al. [10] showed that a graph G which contains 3-restricted edge cuts is $\lambda^{(3)}$-optimal if and only if $\alpha^{(3)}(G) = 3$.

The concept of atoms, originated from M. Mader [15], is an important tool in analyzing higher order (edge) connectivity of graphs. The following theorem is the key to the proof of our main result.

Theorem 1 Let G be an edge transitive bipartite graph with $|V(G)| \geq 6$ and $\alpha^{(3)}(G) \geq 4$. U_1, U_2 are two distinct $\lambda^{(3)}$-atoms of G with $G[U_1] \cong G[U_2]$. If $G \ncong Q_3$, C_n and $K_{1,n-1}$, then $U_1 \cap U_2 = \emptyset$.

Proof. Write $G = (X, Y)$, where X and Y are the two bipartitions of G. Because of Theorem D, we may assume that each vertex in X has degree d_X and each vertex in Y has degree d_Y. Suppose, without loss of generality, that $d_X \leq d_Y$. Since $G \ncong K_{1,n-1}$ and C_n, we may assume that $d_X \geq 2$ and $d_Y \geq 3$.

By Theorem B, G is $\lambda^{(2)}$-optimal, so $\lambda^{(1)}(G) = \xi_1(G) = d_X$ and $\lambda^{(2)}(G) = \xi_2(G) = d_X + d_Y - 2$. Clearly, $\xi_3(G) = 2d_X + d_Y - 4$. Since $\alpha^{(3)}(G) \geq 4$, G is not $\lambda^{(3)}$-optimal, and so $\lambda^{(3)}(G) \leq \xi_3(G) - 1 = 2d_X + d_Y - 5$.

Set $A = U_1 \cap U_2$, $B = U_1 \cap \overline{U}_2$, $C = U_2 \cap \overline{U}_1$ and $D = \overline{U}_1 \cap \overline{U}_2 = \overline{U_1 \cup U_2}$. In the following, we assume $U_1 \cap U_2 \neq \emptyset$, and derive a contradiction by a series of claims.

	U_2	$\overline{U}_2$
U_1	A	B
$\overline{U}_1$	C	D

Figure 1. V is partitioned into four vertex sets A, B, C and D.

Clearly, one of the following two inequalities must hold:

$$|[B,A]| \quad \leq |[A,C]| + |[A,D]|, \qquad (2)$$
$$|[C,A]| \quad \leq |[A,B] + |[A,D]|. \qquad (3)$$

In the following, we always assume, without loss of generality, that inequality (2) holds.

Claim 1 Every component in $G[B]$ is isomorphic to either K_1 or K_2.
First, it follows from inequality (2) that

$$\omega(B) = |[B,D]| + |[B,A]| + |[B,C]| \leq |[U_1, \overline{U}_1]| = \lambda^{(3)}(G).$$

If this claim is not true, suppose $\tilde{G}$ is a component in $G[B]$ with $|V(\tilde{G})| \geq 3$. Set $E = V(\tilde{G})$. Since $G[U_2]$ and $G[\overline{U}_1]$ are both connected, and $U_2 \cap \overline{U}_1 \neq \emptyset$, we see that $G[\overline{B}]$ is connected. Furthermore, since G is connected, every component of $G[B]$ is joined to $G[\overline{B}]$, and thus $G[\overline{E}]$ is connected. So $[E, \overline{E}]$ is a 3-restricted edge cut with $\omega(E) \leq \omega(B) \leq \lambda^{(3)}(G)$. It follows that E is a smaller $\lambda^{(3)}$-fragment than U_1, a contradiction. □

Claim 2 For any vertex $v \in U_1$, $d_{G[U_1]}(v) \geq 2$; and for any vertex $u \in U_2$, $d_{G[U_2]}(u) \geq 2$.
By contradiction. Suppose there exists a vertex $v \in U_1$ with $d_{G[U_1]}(v) = 1$. Set $E = U_1 \backslash \{v\}$. Clearly $G[E]$ is connected. By the assumption $d_X \geq 2$, we have $|[v, \overline{U}_1]| \geq 1$. Then we deduce from the connectedness of $G[\overline{U}_1]$ that $G[\overline{E}]$ is connected. Since $|U_1| = \alpha^{(3)}(G) \geq 4$, we have $|E| \geq 3$. It follows from

$$\omega(E) = \omega(U_1) + |[v,E]| - |[v, \overline{U}_1]| \leq \omega(U_1) = \lambda^{(3)}(G)$$

that E is a smaller $\lambda^{(3)}$-fragment than U_1, a contradiction. Similarly, $d_{G[U_2]}(u) = 1$ for $u \in U_2$ is also impossible. □

Claim 3 $A \not\subseteq X$ and $A \not\subseteq Y$.
By contradiction. Suppose $A \subseteq X$. Then $G[A]$ is an independent set. Since we have assumed that $|[B,A]| \leq |[A,C]| + |[A,D]|$, there exists a vertex v in A such that

$$|[v,B]| \leq |[v,C]| + |[v,D]|. \qquad (4)$$

Set $E = U_1 \backslash \{v\}$. Then

$$\omega(E) = \omega(U_1) - |[v,C]| - |[v,D]| + |[v,B]| \leq \omega(U_1) = \lambda^{(3)}(G).$$

Since $G[U_1]$ is connected and $G[A]$ is independent, we have $|[v,B]| \geq 1$. It follows from inequality (4) that $|[v, \overline{U}_1]| \geq 1$. So, $G[\overline{E}]$ is connected.

We claim that every component in $G[E]$ has at least 3 vertices. In fact, if there is an isolated vertex u in $G[E]$, then v is the only vertex adjacent to u in $G[U_1]$, and thus $d_{G[U_1]}(u) = 1$, contradicting Claim 2. If there is an isolated edge uw in $G[E]$, suppose $u \in X$ and $w \in Y$. It can be seen that w is the only vertex adjacent to u in $G[U_1]$, and thus $d_{G[U_1]}(u) = 1$, also a contradiction. Now, similarly as in the proof of Claim 1, a contradiction arises, since E contains a smaller $\lambda^{(3)}$-fragment than U_1. $A \not\subseteq Y$ can be proved similarly. □

Claim 4 $|A| \geq 3$.

First, it can be seen from Claim 3 that $|A| \geq 2$. In the following, we will derive a contradiction under the assumption $|A| = 2$.

If $|A| = 2$, then also by Claim 3, A must contain a vertex v_1 in X and a vertex v_2 in Y. If $G[B]$ contains an isolated vertex v, then $d_{G[U_1]}(v) = 1$, contradicting Claim 2. So, it follows from Claim 1 that every component in $G[B]$ is an isolated edge. Furthermore, it can be seen from Claim 2 that each end of such an edge must be adjacent to either v_1 or v_2 (see Figure 2 (a)). Suppose there are t such components in $G[B]$. Then $d_X \geq t$. Since

$$2d_X + d_Y - 5 \geq \lambda^{(3)}(G) = \omega(U_1) = (t+1)d_X + (t+1)d_Y - 6t - 2|[v_1, v_2]|,$$

we have

$$(d_X + d_Y - 6)t \leq d_X - 5 + 2|[v_1, v_2]|. \tag{5}$$

If v_1 and v_2 are not adjacent, then

$$(d_X + d_Y - 6)t \leq d_X - 5. \tag{6}$$

If $d_X + d_Y \geq 6$, then since $t \geq 1$, we have $d_Y \leq 1$, a contradiction. So $d_X + d_Y \leq 5$. Since we have assumed $d_X \geq 2$ and $d_Y \geq 3$, we can see that $d_X = 2$ and $d_Y = 3$. But by inequality (6), we have $t \geq 3$, contradicting $t \leq d_X$. So, v_1 and v_2 are adjacent, and $t \leq d_X - 1$. In this case, inequality (5) becomes

$$(d_X + d_Y - 6)t \leq d_X - 3. \tag{7}$$

If $d_X + d_Y \leq 5$, we can similarly show that $d_X = 2$ and $d_Y = 3$. Since $t \leq d_X - 1 = 1$, we have $t = 1$, and thus $G[U_1]$ is a 4-cycle. Since $G[U_2] \cong G[U_1]$, $G[U_2]$ is also a 4-cycle. But then, $d_X \geq 3$, a contradiction. So $d_X + d_Y \geq 6$. Since $t \geq 1$, it can be derived from inequality (7) that $d_Y \leq 3$. It follows from $6 \leq d_X + d_Y \leq 2d_Y \leq 6$ that $d_X = d_Y = 3$. Since $t \leq d_X - 1$, we have $t \leq 2$. If $t = 2$, then $d_{G[U_1]}(v_1) = d_{G[U_1]}(v_2) = 3$, and thus $G[U_2]$ is not connected, a contradiction. So $t = 1$. In this case, $G[U_1] \cong G[U_2] \cong C_4$ (see Figure 2 (b)). Since v_1v_2 is the common edge of two 4-cycles (namely $v_1v_2v_4v_3$ and $v_1v_2v_6v_5$), by the edge transitivity of G, this property holds for every edge. Consider v_1v_5, it is contained in the

4-cycle $v_1v_2v_6v_5$. Since $d_G(v_1) = 3$, the other 4-cycle containing v_1v_5 must contain v_1v_3. Denote this 4-cycle by $v_1v_5v_7v_3$. Similarly, by considering edge v_2v_6, there is a 4-cycle $v_2v_6v_8v_4$ having v_2v_6 as its common edge with $v_1v_2v_6v_5$. We claim that $v_1, \cdots, v_8$ are the only vertices in G. Otherwise it can be seen from $d_X = d_Y = 3$ that $\{v_7, v_8\}$ is a vertex cut, and thus $\kappa(G) \leq 2$, but this contradicts Theorem A. Now, it is not difficult to see that $G \cong Q_3$, contradicting our assumption of the theorem. □

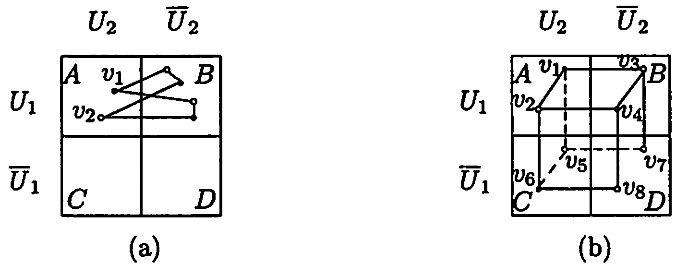

Figure 2. The solid dots represent vertices in X, and the small cycles represent vertices in Y.

Claim 5 $\omega(A) \geq \lambda^{(3)}(G)$, and equality holds only when $G[A]$ is connected.

First,

$$|\overline{A}| \geq |D| = |V| - |U_1| - |U_2| + |U_1 \cap U_2| = |V| - 2\alpha^{(3)}(G) + |A| \geq |A| \geq 3.$$

Clearly, $G[\overline{A}]$ is connected. If $G[A]$ contains a component of order at least 3, then similar to the proof of Claim 1, we can show that $[A, \overline{A}]$ contains a 3-restricted edge cut, and thus $\omega(A) \geq \lambda^{(3)}(G)$ with equality satisfied only when $G[A]$ is connected. Now, suppose every component in $G[A]$ is an isolated edge or an isolated vertex. If there is an isolated edge in $G[A]$, then by the fact $|A| \geq 3$, we have

$$\omega(A) \geq \xi_2(G) + \lambda^{(1)}(G) = 2d_X + d_Y - 2 > \lambda^{(3)}(G).$$

If all components in $G[A]$ are isolated vertices, since not all vertices in A are from the same bipartition, there must be at least one vertex from Y. For $|A| \geq 3$, we have

$$\omega(A) \geq d_Y + d_X + d_X > \lambda^{(3)}(G). \quad □$$

Claim 6 $\omega(D) < \lambda^{(3)}(D)$ and D is an independent set contained in X.

By Claim 5 and

$$\omega(A) + \omega(D) = \omega(U_1 \cap U_2) + \omega(U_1 \cup U_2) \leq \omega(U_1) + \omega(U_2) = 2\lambda^{(3)}(G), \quad (8)$$

we see that if $\omega(D) \geq \lambda^{(3)}(G)$, then $\omega(A) = \lambda^{(3)}(G)$ and thus $G[A]$ is connected. It follows that A is a smaller $\lambda^{(3)}$-fragment than U_1, a contradiction. So, $\omega(D) < \lambda^{(3)}(D)$, and then similar to the proof of Claim 5, we can show that D is an independent set contained in X. $\square$

Let $s = |D|$. Then $s \geq 3$ and

$$\omega(D) = sd_X. \tag{9}$$

Combining this with $\omega(D) < \lambda^{(3)}(G)$, we have

$$(s-2)d_X + 6 \leq d_Y. \tag{10}$$

Suppose there are t isolated edges and r isolated vertices in $G[B]$. Since $G[\overline{U}_2]$ is connected, every isolated vertex in $G[B]$ must belong to Y (see Figure 3). So $|B \cap X| = t$ and $|B \cap Y| = t + r$. Since $G(U_1) \cong G(U_2)$, we have $|C \cap X| = t$ and $|C \cap Y| = t + r$.

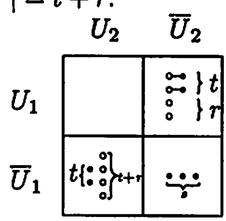

Figure 3.

Denote by e_1 the number of edges in $G[\overline{A}]$. Since G is a bipartite graph, so is $G[\overline{A}]$, and thus

$$e_1 \leq (s+2t)d_X. \tag{11}$$

On the other hand, vertices in $B \cap X$ can contribute at most $t + t(t+r)$ edges to $G[\overline{A}]$ (the first term accounts for edges in $G[B]$, and the second term accounts for edges between B and C with the end vertex of degree d_X belonging to B), vertices in D can contribute at most $2s(t+r)$ edges to $G[\overline{A}]$, and vertices in $C \cap X$ can contribute at most td_X edges to $G[\overline{A}]$. So,

$$e_1 \leq 2s(t+r) + t + t(t+r) + td_X. \tag{12}$$

Clearly,

$$\omega(A) = \omega(\overline{A}) = \sum_{v \in \overline{A}} d_G(v) - 2e_1 = (s+2t)d_X + 2(t+r)d_Y - 2e_1, \tag{13}$$

Combining this with (8) (9) and (11), we have

$$-(t+2)d_X + (t+r-1)d_Y + 5 \leq 0.$$

By (10) and the fact $s \geq 3$, we have

$$(r-3)d_X + 6(t+r) - 1 \leq 0, \qquad (14)$$

which may occur only when $r \leq 2$.

Similarly, combining inequality (8) (9) (10) (12) and (13), we have

$$(s-2)(t+r)d_X + (6-2s-t)(t+r) - t - 1 \leq 0. \qquad (15)$$

If $r = 2$, then it can be deduced from (14) and (15) that

$$s(6t^2 + 21t + 18) - (13t^2 + 43t + 33) \leq 0.$$

By the fact $s \geq 3$, we have

$$5t^2 + 20t + 21 \leq 0,$$

which is obviously impossible.

If $r = 1$, (14) becomes $d_X \geq 3t + \frac{5}{2}$. Since d_X is an integer, we have $d_X \geq 3(t+1)$. Combining this with inequality (15) and the fact $s \geq 3$, it can be deduced that $2t^2 + 4t + 2 \leq 0$, also impossible.

So $r = 0$, and (14) becomes $d_X \geq 2t - \frac{1}{3}$. Since d_X is an integer, we have $d_X \geq 2t$. Similarly as above, we have $t^2 - t - 1 \leq 0$, which may occur only when $t = 1$. In this case, (15) becomes

$$(s-2)d_X - 2s + 3 \leq 0.$$

It follows that

$$d_X \leq 2 + \frac{1}{s-2} \leq 3.$$

Now it can be seen that $e_1 \leq 2(s+2)$. Similarly as above, we have

$$(s-1)d_X - 2s + 1 \leq 0.$$

If $d_X = 3$, then $s \leq 2$, a contradiction. So, $d_X = 2$. Denote the vertex in $Y \cap B$ as v (see Figure 3 (b)). Since $d_{G[U_1]}(v) \geq 2$, there is a vertex $u \in A$ adjacent to v. Clearly $u \in X$, and thus $d_G(u) = 2$. It follows that $d_{G[U_2]}(u) \leq 1$, contradicting Claim 2. The theorem is proved. □

3 $\Lambda^{(3)}$-optimal edge transitive graphs

To characterize $\lambda^{(3)}$-optimal edge transitive graphs, we need the concept of imprimitive block [14]. Let A be a non-empty proper subset of $V(G)$. If for any automorphism ϕ of G, either $\phi(A) = A$ or $\phi(A) \cap A = \emptyset$, then A

is said to be *an imprimitive block* of G. The following theorem shows the importance of imprimitive blocks:

Theorem E ([14]) Let $G=(V,E)$ be a connected graph and A be an imprimitive block of G. If G is edge transitive, then A is an independent set of G.

We are ready to prove the main theorem.

Theorem 2 The only edge transitive graphs with order at least 6 which are not $\lambda^{(3)}$-optimal are the star graphs $K_{1,n-1}$, the cycles C_n and the cube Q_3.

Proof. In light of Theorem C and Theorem D, we only need to show the truth when G is an edge transitive bipartite graph. By contradiction. If $\lambda^{(3)}(G)<\xi_3(G)$, then $\alpha^{(3)}(G)\geq 4$, it follows from Theorem 1 that distinct $\lambda^{(3)}$-atoms which induce isomorphic subgraphs are disjoint. So, every $\lambda^{(3)}$-atom of G is an imprimitive block. Let U be a $\lambda^{(3)}$-atom of G. By Theorem E, U is an independent set, contradicting the fact that $G[U]$ is connected. □

As a corollary, in an edge transitive graph G which is not isomorphic to C_n, $K_{1,n-1}$ and Q_3, $N_i(G)$ can be determined for $i=0,1,\cdots,\xi_3(G)-1$. In fact, since G is $\lambda^{(3)}$-optimal, the deletion of any edge set S of size i ($i<\xi_3(G)$) may create either isolated vertices or isolated edges. So $N_i(G)$ can be counted as indicated in the following corollary:

Corollary 1 Let G be a connected edge transitive graph with $n\geq 6$ vertices and m edges, where G is not isomorphic to $K_{1,n-1}$, C_n and Q_3.

(1) If G is a k-regular vertex transitive graph, then

$$N_i(G)=\begin{cases} 0, & 0\leq i<k,\\ n\binom{m-k}{i-k}, & k\leq i<2k-2,\\ m\binom{m-2k+1}{i-2k+1}+\left(\frac{n(n-1)}{2}-m\right)\binom{m-2k}{i-2k}+n\left[\binom{m-k}{i-k}-k\binom{m-2k+1}{i-2k+1}-\right. & \\ \quad \left.-(n-k-1)\binom{m-2k}{i-2k}\right]+m\binom{m-2k+1}{i-2k+2}, & 2k-2\leq i\leq \xi_3(G)-1. \end{cases}$$

(2) If G is a bipartite graph with bipartition X and Y, where the degrees of vertices in X and Y are d_X and d_Y, respectively, with $d_X\leq d_Y$, then

$$N_i(G)=\begin{cases} 0, & 0\leq i<d_X,\\ \sum_{j=1}^{\lfloor \frac{i}{|X|}\rfloor}(-1)^{j-1}C_{|X|}^{j}\binom{m-jd_X}{i-jd_X}+|Y|\binom{m-d_Y}{i-d_Y}, & d_X\leq i<d_X+d_Y-2,\\ m\binom{m-(d_X+d_Y-1)}{i-(d_X+d_Y-2)}+|Y|\binom{m-d_Y}{i-d_Y}-C_{|Y|}^{2}\binom{m-2d_Y}{i-2d_Y} & \\ \quad +\sum_{j=1}^{\lfloor \frac{i}{|X|}\rfloor}(-1)^{j-1}C_{|X|}^{j}\binom{m-jd_X}{i-jd_X}-|X|(|Y|-d_X)\binom{m-(d_X+d_Y)}{i-(d_X+d_Y)} & \\ \quad -|X|d_X\binom{m-(d_X+d_Y-1)}{i-(d_X+d_Y-1)}, & d_X+d_Y-2\leq i<2d_X+d_Y-4. \end{cases}$$

Here $\binom{x}{y} = 0$ when $y < 0$. □

Remark The proof of disjointness of atoms is a key to the study of higher order (edge) connectivity. It is known that distinct $\lambda^{(1)}$-atoms are disjoint for any connected non-$\lambda^{(1)}$-optimal graph [15, 14]. In [17], J.M. Xu and K.L. Xu proved the disjointness of $\lambda^{(2)}$-atoms for any non-$\lambda^{(2)}$-optimal graphs containing 2-restricted edge cuts. But such a property no longer holds for $\lambda^{(3)}$-atoms. A counter example is given in [10]. In [12], Meng showed that for a non-$\lambda^{(3)}$-optimal vertex transitive graph which is not isomorphic to $C_m(K_2)$ (the lexicographic product of C_m with K_2) or $C_m \times K_2$ (the cartesian product of C_m and K_2), distinct $\lambda^{(3)}$-atoms are disjoint. In this paper, we show the disjointness of $\lambda^{(3)}$-atoms for edge transitive graphs. It remains a problem of characterizing general graphs in which $\lambda^{(3)}$-atoms are disjoint.

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