COMPLEXITY OF COMPUTING OF THE DOMINATION NUMBER IN HEREDITARY CLASSES OF GRAPHS

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ABSTRACT. Let $\gamma(G)$ be the domination number of a graph G. A class $\mathcal{P}$ of graphs is called *γ-complete* if the problem of determining $\gamma(G)$, $G \in \mathcal{P}$, is NP-complete. A class $\mathcal{P}$ of graphs is called *γ-polynomial* if there is a polynomial-time algorithm for calculating $\gamma(G)$ for all graphs $G \in \mathcal{P}$.

We denote $\Gamma = \{P_k \cup nK_1 : k \leq 4 \text{ and } n \geq 0\}$. Korobitsin [4] proved that if $\mathcal{P}$ is a hereditary class defined by a unique forbidden induced subgraph H, then

(i) when $H \in \Gamma$, $\mathcal{P}$ is a γ-polynomial class,

(ii) when $H \notin \Gamma$, $\mathcal{P}$ is a γ-complete class.

We extend a positive result (i) in the following way. The class Γ is hereditary, and it is characterized by the set

$$Z(\Gamma) = \{2K_2, K_{1,3}, C_3, C_4, C_5\}$$

of minimal forbidden induced subgraphs.

For each $Z \subseteq Z(\Gamma)$ we consider a hereditary class FIS(Z) defined by the set Z of minimal forbidden induced subgraphs. We prove that FIS(Z) is γ-complete in 16 cases, and it is γ-polynomial in the other 16 cases. We also prove that $2K_2$-free graphs with bounded clique number constitute a γ-polynomial class.

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1. Intoduction

We use standard graph-theoretic terminology, see, for example, [6]. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The notation $x \sim y$ (respectively, $x \not\sim y$) means that vertices $x, y \in V(G)$ are adjacent (respectively, non-adjacent). The *neighborhood* of a vertex $x \in V(G)$ is the set $N(x) = N_G(x) = \{y \in V(G) : x \sim y\}$. The *degree* of a vertex $x \in V(G)$ is $\deg x = |N(x)|$.

We use the notation P_n for a path with $n \geq 1$ vertices, C_n for a cycle with $n \geq 3$ vertices, $K_{1,n}$, $n \geq 1$, for a star, and K_n for a complete graph of order $n \geq 1$. The union $G \cup H$ of graphs G and H is assumed to be disjoint. The union of n disjoint copies of a graph G is denoted by nG.

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A set $D \subseteq V(G)$ is a *domination* set in a graph G if every vertex of $V(G) \setminus D$ is adjacent to a vertex of D. In other words, D *dominates* $V(G)$. The *domination number* $\gamma(G)$ of a graph G is cardinality of a minimum domination set in G.

Definition 1. *A class $\mathcal{P}$ of graphs is called* γ-complete *if the problem of determining $\gamma(G)$, $G \in \mathcal{P}$, is NP-complete. A class $\mathcal{P}$ of graphs is called* γ-polynomial *if there is a polynomial-time algorithm for calculating $\gamma(G)$ for all graphs $G \in \mathcal{P}$.*

An *induced subgraph* H of a graph G is obtained from G by deleting a vertex set [possibly, empty]. Let $\mathrm{ISub}(G)$ be the set of all induced subgraphs of a graph G. A class $\mathcal{P}$ is *hereditary* if $\mathrm{ISub}(G) \subseteq \mathcal{P}$ for every graph $G \in \mathcal{P}$.

Let Z be a set of graphs. We put $\mathrm{FIS}(Z) = \{G : \mathrm{ISub}(G) \cap Z = \emptyset\}$; Z is called a set of *forbidden induced subgraphs* for $\mathrm{FIS}(Z)$. A forbidden induced subgraph G for a hereditary class $\mathcal{P}$ is *minimal* if $\mathrm{ISub}(G) \setminus \mathcal{P} = \{G\}$.

We denote by $Z(\mathcal{P})$ the set of all minimal forbidden induced subgraphs for $\mathcal{P}$. It is well-known that $\mathcal{P} = \mathrm{FIS}(Z(\mathcal{P}))$ for any hereditary class $\mathcal{P}$. If $|Z(\mathcal{P})| = 1$ then $\mathcal{P}$ is called a *monogenic* class. We denote

$$\Gamma = \{P_k \cup nK_1 : k \leq 4 \text{ and } n \geq 0\}.$$

Theorem 1 (Korobitsin [4]). *If $\mathcal{P}$ is a monogenic hereditary class defined by a unique forbidden induced subgraph H, then*

(i) *when $H \in \Gamma$, $\mathcal{P}$ is a γ-polynomial class,*

(ii) *when $H \notin \Gamma$, $\mathcal{P}$ is a γ-complete class.*

Proposition 1. *Γ is a hereditary class and*

$$Z(\Gamma) = \{2K_2, K_{1,3}, C_3, C_4, C_5\}.$$

Proof. It is easy to see that Γ is a hereditary class, and each of $2K_2$, $K_{1,3}$, C_3, C_4, C_5 is a minimal forbidden induced subgraph for Γ.

Let $G \in Z(\Gamma)$. If G contains a vertex u with $\deg u \geq 3$, then either $K_{1,3} \in \mathrm{ISub}(G)$, or $C_3 \in \mathrm{ISub}(G)$. By minimality, $G \in \{K_{1,3}, C_3\}$.

Suppose now that all vertex degrees in G do not exceed 2. Then G is a disjoint union of paths and cycles. If G has at least two nontrivial components, then $2K_2 \in \mathrm{ISub}(G)$ and $G = 2K_2$. Therefore we may assume that G has at most one nontrivial component. By the minimality, G does not contain isolated vertices. It follows that either $G = C_n$, $n \geq 3$, or $G = P_m$, $m \geq 5$. Indeed, if $G = P_m$ and $m \leq 4$ then $G \in \Gamma$, a contradiction to $G \in Z(\Gamma)$.

If either $G = C_n$ $(n \geq 6)$ or $G = P_m$ $(m \geq 5)$, then $2K_2 \in \mathrm{ISub}(G)$ which contradicts to the minimality of G. Hence $G \in \{C_3, C_4, C_5\}$. □

By Theorem 1, the class FIS(H) is γ-complete for any graph

$$H \in Z(\Gamma) = \{2K_2, K_{1,3}, C_3, C_4, C_5\}$$

and $Z(\Gamma)$ is the set of all minimal graphs with this property. We find conditions for FIS(Z) to be γ-polynomial/ γ-complete for each subset $Z \subseteq Z(\Gamma)$.

2. Main result

Here is our main result.

Theorem 2. *For $Z \subseteq \{2K_2, K_{1,3}, C_3, C_4, C_5\}$ and $\mathcal{P} = \text{FIS}(Z)$,*

(i) *if Z contains at least two graphs of $2K_2, K_{1,3}, C_3$, then $\mathcal{P}$ is a γ-polynomial class,*

(ii) *otherwise $\mathcal{P}$ is a γ-complete class.*

Proof. (i) Let G be an arbitrary graph in $\mathcal{P} = \text{FIS}(Z)$.

(i1) If $K_{1,3}, C_3 \in Z$ then vertex degrees in G are at most two. Clearly, the class $\mathcal{P}$ is γ-polynomial in this case.

(i2) Let $2K_2, C_3 \in Z$. Without loss of generality we may assume that G has no isolated vertices. If G does not contain P_4 as an induced subgraph, then by Theorem 1(i) there is a polynomial-time algorithm for computing $\gamma(G)$. Otherwise we choose an induced path $H = P_4 = (u_1, u_2, u_3, u_4)$ such that $V(H)$ dominates a maximum number of vertices [among all induced P_4's in G]. If $V(H)$ dominates all vertices of G, then $\gamma(G) \leq 4$ and the proof is complete. Otherwise there exists a vertex w which is not dominated by $V(H)$. We choose a vertex $x \in N(w)$.

Since $2K_2$ and C_3 are not induced subgraphs of G, x is adjacent to exactly two vertices of H, namely, to either u_1 and u_3, or u_2 and u_4. By symmetry, let $x \sim u_2$ and $x \sim u_4$. The path $F = (u_1, u_2, x, u_4)$ dominates w. Hence there is a vertex y which is dominated by $V(H)$ and is not dominated by $V(F)$. Clearly, y is adjacent to u_3 and y is not adjacent to u_1, u_2, x and u_4. Since the set $\{u_3, w, x, y\}$ does not induces $2K_2$, $y \sim w$. Then $\{u_1, u_2, w, y\}$ induces $2K_2$, a contradiction.

Thus, we have proved that $\gamma(G) \leq 4$. Hence $\mathcal{P}$ is a γ-polynomial class. Note that $\max\{\gamma(G) : G \in \text{FIS}(2K_2, C_3) \text{ and } G \text{ is connected }\} = 3$.

(i3) Now let $2K_2, K_{1,3} \in Z$. As before, we assume that G has no isolated vertices. First we show that if $C_5 \in \text{ISub}(G)$ then $\gamma(G) \leq 5$. Let $H = C_5 = (u_1, u_2, u_3, u_4, u_5)$. Suppose that $V(H)$ does not dominate a vertex w and $w \sim x$. Since $\{u_1, u_2, w, x\}$ does not induce $2K_2$, we may assume that $x \sim u_1$. Since $\{u_3, u_4, w, x\}$ does not induce $2K_2$, either $x \sim u_3$ or $x \sim u_4$.

Let $x \sim u_4$. Then $\{u_1, u_4, w, x\}$ induces $K_{1,3}$, a contradiction.

Now we show that if $C_4 \in \text{ISub}(G)$ then $\gamma(G) \leq 4$. Let $H = C_4 = (u_1, u_2, u_3, u_4)$. Suppose that $V(H)$ does not dominate a vertex w and $w \sim x$. It follows from $2K_2 \notin \text{ISub}(G)$ that x is adjacent to either u_1 and u_3 or u_2 and u_4. Then either $\{u_1, u_3, w, x\}$ or $\{u_1, u_3, w, x\}$ induces $K_{1,3}$, a contradiction. Thus, we may assume that G does not contain $2K_2, C_4$ and C_5 as induced subgraphs.

A graph G is called *split* if there is a partition $A \cup B = V(G)$ [a *split partition* of G] such that A induces a complete graph, and B is a stable set.

Fact 1 (Földes and Hammer [2]). *A graph is split if and only if it does not contain each of $2K_2, C_4, C_5$ as an induced subgraph.*

By Fact 1, G is a split graph. We choose a split partition $A \cup B$ of G. We construct a graph H with the vertex set B in the following way: vertices $u, v \in B$ (possibly, $u = v$) are adjacent in H if and only if $\{u, v\} = N_G(w) \cap B$ for some vertex $w \in A$.

Note that $G \in \text{FIS}(K_{1,3})$ that $|N_G(w) \cap B| \leq 2$ for every vertex $w \in A$ [since $K_{1,3}$ is a forbidden induced subgraph]. In case of $u = v$, H has a loop at u. Therefore H is a graph with possible loops. Since G has no isolated vertices, H has no isolated vertices, i.e., every vertex in H is incident to some edge [possibly, a loop].

It is evident that there exists a minimum dominating set $D \subseteq A$ in G. It can be easily seen that there is a natural bijection between domination sets $D \subseteq A$ of G and edge coverings of H, with corresponding sets being equal cardinality. Denote by H' a subgraph of H obtaining by deleting all loops of H and arising isolated vertices. Let $\rho(H)$ be the cardinality of a minimum edge covering of H [the *edge covering number*]. If all edges in H are loops then $\rho(H) = |V(H)|$ and $\gamma(G) = |V(H)|$.

Suppose that G has an edge uv with $u \neq v$. There is a vertex $w \in A$ which is adjacent to both u and v. For every vertex $x \in A$, the set $\{u, v, w, x\}$ does not induce $K_{1,3}$. Since $w, x \in A$, $w \sim x$. Therefore either $w \sim u$ or $w \sim v$. It follows that each edge in H [including loops] is adjacent to uv. Any loop in a minimum edge covering may be changed by uv. So we may consider that H does not contain loops.

Since H has no independent edges, by Gallai's identity, see Lovász and Plummer [5], $\rho(H) = |V(H)| - 1$ and $\gamma(H) = |V(H)| - 1$.

(ii) (iil) Let $K_{1,3}, C_3 \notin Z$. Then $Z \subseteq \{2K_2, C_4, C_5\}$. By Fact 1, $\mathcal{P}$ contains all split graphs.

Fact 2 (see Bertossi [1] and Johnson [3]). *Split graphs constitute a γ-complete class.*

By Fact 2, $\mathcal{P}$ is a γ-complete class.

(ii2) Let $2K_2, K_{1,3} \notin Z$. Then $Z \subseteq \{C_3, C_4, C_5\}$. We denote by T_1 the class of all graphs in which each component is either homeomorphic $K_{1,3}$ or a path.

Fact 3 (Korobitsin [4]). *If $\mathcal{P}$ is a hereditary class, $Z(\mathcal{P})$ is finite and $Z(\mathcal{P}) \cap T_1 = \emptyset$, then $\mathcal{P}$ is a γ-complete class.*

In our case, $|Z| \leq 3$ and $Z \cap T_1 = \emptyset$. By Fact 3, $P = \text{FIS}(Z)$ is γ-complete.

(ii3) Finally, let $2K_2, C_3 \notin Z$. Then $Z \subseteq \{K_{1,3}, C_4, C_5\}$.

Let P_{n_1}, P_{n_2} and P_{n_3} be three disjoint paths, $n_i \geq 1$, and u_i be a pendant vertex of P_{n_i}, $i = 1, 2, 3$. Add edges u_1u_2, u_1u_3 and u_2u_3 to produce C_3-*extension*. We denote by T_2 the class of all graphs in which each component is either C_3-extension or a path.

Fact 4 (Korobitsin [4]). *If $\mathcal{P}$ is a hereditary class, $Z(\mathcal{P})$ is finite and $Z(\mathcal{P}) \cap T_2 = \emptyset$, then $\mathcal{P}$ is an γ-complete class.*

In our case, $|Z| \leq 3$ and $Z \cap T_2 = \emptyset$. By Fact 4, $\mathcal{P} = \text{FIS}(Z)$ is γ-complete. □

Problem 1. *Let $\mathcal{P}$ be a hereditary class with the set $Z(\mathcal{P})$ of minimal forbidden induced subgraphs. Is $\mathcal{P}$ a γ-polynomial class or not?*

Here and below we assume that $P \neq NP$.

3. α-Complete and ω-bounded classes

A set $I \subseteq V(G)$ is *stable* if there are no adjacent vertices in I. The *stability number* $\alpha(G)$ of a graph G is the maximum cardinality of a stable set in G.

Definition 2. *A class $\mathcal{P}$ is* α-complete *if the problem of computing $\alpha(G)$, $G \in \mathcal{P}$, is NP-complete.*

A class $\mathcal{P}$ is α-polynomial *if there is a polynomial-time algorithm for computing $\alpha(G)$ for each $G \in \mathcal{P}$.*

In contrast to γ-polynomial classes, many hereditary classes are proved to be α-polynomial.

Problem 2. *Let $\mathcal{P}$ be a hereditary class with the set $Z(\mathcal{P})$ of minimal forbidden induced subgraphs. Is $\mathcal{P}$ an α-polynomial class or not?*

We may compare Problem 1 and Problem 2. For a graph G, we construct a split graph $H = Sp(G)$ as follows: $A = V(G)$ is a complete part of H, $B = E(G)$ is an empty part of H, and every vertex $e \in B$, $e = uv \in E(G)$, is incident to exactly two edges, namely, eu and ev [$u, v \in A = V(G)$]. In other words, $Sp(G)$ is obtained from G by subdivision of every edge in G by a new vertex and constructing complete subgraph on $V(G)$. We put $Sp_2 = \{Sp(G) : G \text{ is a graph}\}$.

For each connected split graph G there is a minimum dominating set D that is contained in the complete part A of G. If $H = Sp(G)$ then D is a minimum vertex covering of G. By Gallai's identity (see Lovász and Plummer [5]), $\alpha(G) = |V(G)| - |D|$.

Thus, computing of $\alpha(G)$, $G \in \mathcal{P}$, is equivalent to finding $\gamma(H)$, where $H \in Sp(\mathcal{P}) = \{Sp(G) : G \in \mathcal{P}\}$. Clearly, $Sp(G)$ is a very special subclass of split graphs. In general, the problem of finding $\alpha(H)$ is equivalent to finding $\gamma(G)$ for a split graph G. When H has a bounded order [equivalently, the clique number $\omega(G)$ of G is bounded above], both problems can be solved in polynomial time. We consider an extension of split graphs [$2K_2$-free graphs] and show that the condition $\omega(G) \leq$ const implies the existence of a polynomial-time algorithm for finding $\gamma(G)$.

The *clique number* $\omega(G)$ is the maximum order of a complete subgraph in G.

Definition 3. *A class $\mathcal{P}$ is ω-*bounded *if there is a constant c such that $\omega(G) \leq c$ for each $G \in \mathcal{P}$.*

Theorem 3. *Any ω-bounded class of $2K_2$-free graphs is γ-polynomial.*

Proof. Let $\omega(G) \leq c$ for each $G \in \mathcal{P}$. We consider an arbitrary graph $G \in \mathcal{P}$. Without loss of generality we may assume that G is connected.

If $\omega(G) \leq 2$ then $G \in \text{FIS}(2K_2, C_3)$ and, by Theorem 2, there is a polynomial time algorithm for finding $\gamma(G)$. So we assume that $\omega(G) \geq 3$. We choose a maximal complete subgraph H in G such that $|V(H)| = t \geq 3$.

We show that $D = V(H)$ is a domination set in G. Suppose that D is not dominate a vertex $u \in V(G)$. By the connectivity of G, there is a shortest path $(u, \dots, w, x, d)$, where $d \in D$, connecting u with D. Clearly, w is not adjacent to all vertices in D.

Since H is a maximal complete subgraph, there is a vertex $y \in V(H)$ that is not adjacent to x. For every vertex $y' \in D \setminus \{y\}$ the set $\{w, x, y, y'\}$ does not induce $2K_2$. So $x \sim y'$. Hence $D' = (D \setminus \{y\}) \cup \{x\}$ induces a clique in G and $|D'| = |D|$. The set D' dominates $D \cup \{w\}$ and D does not dominate w. By the choice of H, there is a vertex z which is dominated by D and does not dominated by D'. Clearly, $z \sim y$ and z does not adjacent to all vertices of D'.

Since $\{w, x, y, z\}$ does not induce $2K_2$, $w \sim z$. It follows from $|D| = t \geq 3$ that there are two vertices $u, v \in D \setminus \{y\}$. Then $\{u, v, w, z\}$ induces $2K_2$, a contradiction.

Thus, $D = V(H)$ is a domination set in G. We have $\gamma(G) \leq |D| \leq \omega(G) \leq c$. Since $\gamma(G) \leq c$ for each $G \in \mathcal{P}$, $\mathcal{P}$ is a γ-polynomial class. □

4. Further research

In conclusion, we propose some hereditary classes for further investigation. We suppose that most of them are γ-polynomial.

Let $\Gamma_1 = \{H \cup nK_1 : n \geq 0, H \in \{P_k, 2K_2, K_{1,3}, C_3\}, k \leq 4\}$. It can be checked that $Z(\Gamma_1) = \{G_1, G_2, \ldots, G_{11}\}$ (Figure 1).

Research Problem 1. *Let $\mathcal{P}$ = FIS(Z), where $Z \subseteq Z(\Gamma_1)$. Is $\mathcal{P}$ a γ-polynomial (γ-complete) class when $G_i, G_j \in Z$ in the following cases:*
$(i, j) = (1,5), (1,7), (1,9), (1,11), (3,5), (3,7), (3,11), (4,5), (4,7), (4,11), (5,8), (5,9), (6,9), (7,8), (7,9), (8,11), (9,11)$*?*

In other cases, $\mathcal{P}$ is a γ-complete class. We only have checked that $\mathcal{P}$ is a γ-polynomial class when $(i, j) = (1,7), (1,9)$ and $(3,7)$.

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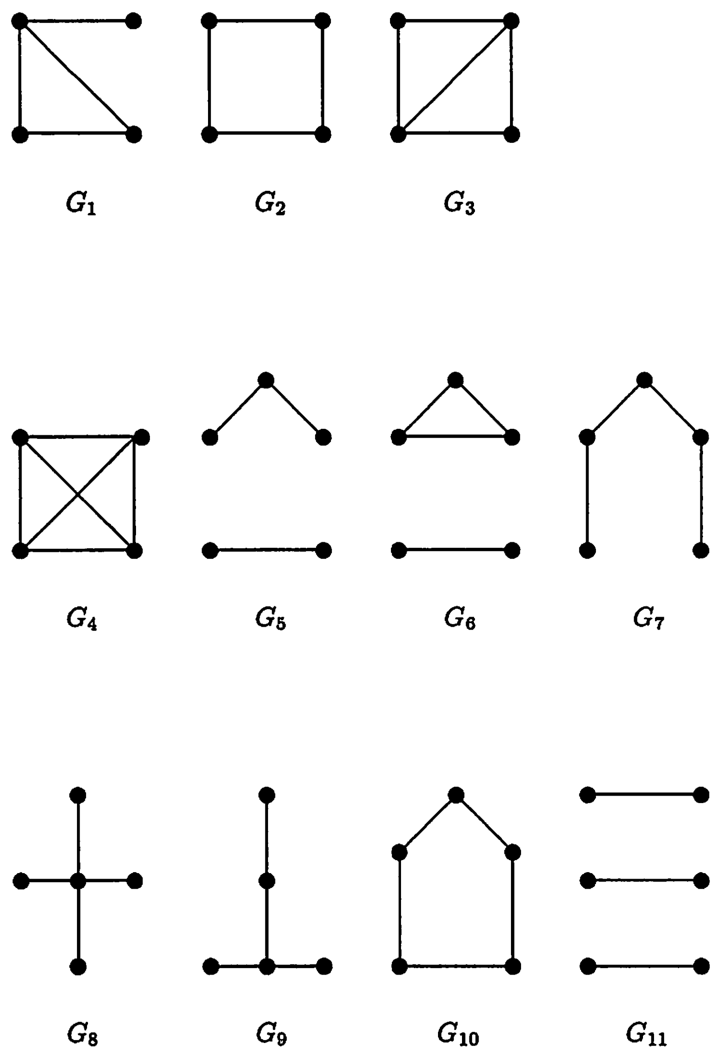

Figure 1. Graphs $G_1, G_2, \ldots, G_{11}$.