Hypergraphs and the Helly property.*

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Abstract

In this article we discuss about the Helly property and the strong Helly property in the hypergraphs. We give a characterization of neighborhood hypergraphs having the Helly and the strong Helly property. These properties are studied in both cartesian and strong products of hypergraphs.

1 Graph and Hypergraph Definitions

The Helly property is one of the most important concepts in hypergraph theory. Introduced in [6], this property has been extensively studied [2, 3, 7, 8, 10, 13, 15, 16, 17, 18, 19, 20, 21]. Many applications of the Helly property have been developed especially in image processing [9]. In this article we give some results about this property.
The general terminology concerning graphs and hypergraphs in this article is similar to the one used in [4, 5, 9]. All graphs in this paper are both finite and undirected. One will consider that these graphs are simple; graphs with no loops or multiple edges. All graphs will be considered as connected with no isolated vertex. We denote them $G = (V; E)$. Given a graph G, we denote the *neighborhood* of a vertex x by $\Gamma(x)$, i.e. the set formed by all the vertices adjacent to x is defined by:

$$\Gamma(x) = \{y \in V, \{x, y\} \in E\}.$$

The number of neighbors of x is the *degree* of x (denoted by dx).
A graph in which each pair of vertices are adjacent is a *complete graph*.
A *chain* in a graph is a sequence of distinct edges –one following another–, and the number of edges is the *length* of this chain.
A *cycle* is a chain such that the first vertex and the last vertex are the same.

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A *chord* of a cycle is an edge linking two non consecutive vertices of this cycle.
A cycle with a length equal to n will be denoted by C_n.
Let $G = (V, E)$ be a graph, a cycle C_{2n}, $(n > 2)$ with distinct vertices: $x_1, x_2, x_3, \ldots, x_i, \ldots, x_{2n}$ has a *well chord* if there exists an edge e linking x_i to x_{i+n}, $1 \leq i < n$. An example is given in figure 1.
A cycle C_n is *centered* if there exists a vertex of G adjacent to every vertex of C_n. (If this vertex is on the cycle, one will consider that it is adjacent to itself.)
A graph is *triangulated* if any cycle of length at least 4 has a chord.

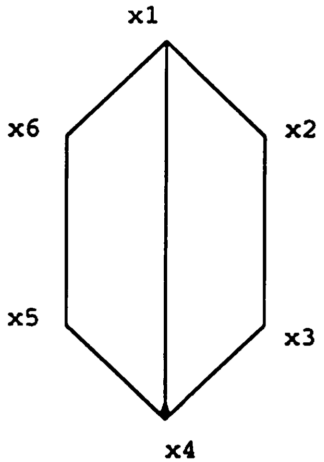

Figure 1: This figure shows a cycle C_6 with a well chord $\{x_1, x_4\}$.

$G' = (V'; E')$ is a *subgraph* of G when it is a graph satisfying $V' \subseteq V$ and $E' \subseteq E$. If $V' = V$ then G' is a *spanning subgraph*.
An *induced subgraph* (generated by A) $G(A) = (A; U)$, with $A \subseteq V$ and $U \subseteq E$ is a subgraph such that: for $x, y \in A$, when $\{x; y\} \in E$ implies $\{x; y\} \in U$.
A complete induced subgraph is a *clique*.
A graph $G = (V, E)$ is bipartite if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ and every edge joins a vertex V_1 to a vertex of V_2. We will denote a bipartite graph by $G(V_1, V_2)$.
A graph $G = (V, E)$ is *bipartite* if and only if it does not contain any cycle with an odd length.
A *hypergraph* H on a finite set S is a family $(E_i)_{i \in I}$, $I = \{1, 2, \ldots, n\}$ $n \in \mathbb{N}$ of non-empty subsets of S called hyperedges with:

$$\bigcup_{i \in I} E_i = \mathrm{S}.$$

Let us denote : $H = (\mathrm{S}; (E_i)_{i \in I})$.
Sometimes we will denote the set of vertices of H by $V(H)$ and the set of

hyperedges by $E(H)$.
The *rank* of H is the maximum cardinality of a hyperedge.
A hypergraph is *linear* if $|E_i \cap E_j| \leq 1$ for $i \neq j$.
A *loop* is a hyperedge with a cardinality equal to one.
A *simple hypergraph* is a hypergraph $H = (S; E = (E_i)_{i \in I})$ such that:

$$E_i \subset E_j \Longrightarrow i = j.$$

In this article, without losing generality we will suppose that any hypergraph is both without loop and without repeated hyperedges.
For $x \in S$, a *star* of H –with x as a center– is the set of hyperedges which contains x, and is called $H(x)$. The *degree of* x is the cardinality of the star $H(x)$.
A *partial hypergraph* on S is a subfamily $(E_j)_{j \in J}$ of $(E_i)_{i \in I}$.
A *subhypergraph* of the hypergraph H is the hypergraph $H(Y) = (Y, (E_i \cap Y \neq \emptyset)_{i \in I})$, (with $Y \subseteq S$).
The *dual* of a hypergraph $H = (E_1, E_2, \ldots, E_m)$ on S is a hypergraph $H^* = (X_1, X_2, \ldots, X_n)$ whose vertices $e_1, e_2, \ldots, e_m$ correspond to the hyperedges of H, and with hyperedges

$$X_i = \{e_j, x_i \in E_j\}$$

The *line graph*, $L(H)$ of hypergraph H is a graph whose set of vertices is the set of hyperedges of H and two vertices of $L(H)$ are adjacent if the corresponding hyperedges have a non empty intersection.
Let $H = (S; (E_i)_{i \in I})$ be a hypergraph. The *2-section (or section)* of H is the graph whose set of vertices is S and two vertices x, y are adjacent if there exists $i \in I$ such that $x, y \in E_i$. We will denote this graph by $[H]_2$.
A family of hyperedges is an *intersecting family* if every pair of hyperedges has a non empty intersection.
A hypergraph has the *Helly property* if each intersecting family has a non-empty intersection –belonging to a star–.
A hypergraph has the *the strong Helly property* if each subhypergraph has the Helly property.
A hypergraph H is *conformal* if all the maximal cliques of $[H]_2$ are hyperedges of H.
The *incidence graph* of a hypergraph $H = (S; E)$ is a bipartite graph with a vertex set $V = S \cup E$, where two vertices $x \in X$ and $e \in E$ are adjacent if and only if $x \in e$. We denote it $IG(H)$.
Let $G = (S; E)$ be a graph; we can associate a hypergraph called *neighborhood hypergraph* to this graph:

$$\mathbb{H}_G = (S, (E_x = \{x\} \cup \Gamma(x))).$$

One will say that the hyperedge E_x is generated by x.

2 Neighborhood hypergraph and Helly property.

We now characterize bipartite graphs such that the associated neighborhood hypergraph has the Helly property.

Theorem 1. *Let $G = (V; E)$ be a bipartite graph, and $\mathbb{H}_G$ its associated neighborhood hypergraph. $\mathbb{H}_G$ has the Helly property if and only if G does not contain C_4 and C_6.*

Proof. The condition is necessary. Suppose that G contains a C_4. H_G has the Helly property. Consequently C_4 is centered, so G contains a cycle C_3. Contradiction. If G contains C_6, either C_6 is centered and G contains C_3 or there exists a vertex adjacent to three non consecutive vertices of C_6, so G contains a cycle C_4 and thus it is a cycle C_3. Contradiction.
The condition is sufficient. We prove this assertion by induction on the hyperedge number from an intersecting family.
Let $(E_{x_i})_{i\in\{1,2,3\}}$ be an intersecting family of three hyperedges, by hypothesis x_1, x_2, x_3 cannot be on a cycle C_n with $n = 4, 5, or\ 6$. We denote $V = V_1 \cup V_2$. Two cases arise:

1 $x_1, x_2, x_3 \in V_1$. We have y_i adjacent to x_i, x_{i+1} (mod3). So $y_1 = y_2 = y_3 = y$, otherwise one would have C_4 or C_6.

2 $x_1, x_2 \in V_1$ and $x_3 \in V_2$ necessarily x_3 is adjacent to x_1 and x_2.

Consequently, there exists a vertex y adjacent to x_1, x_2, x_3 in the first case, and $y = x_3$ is adjacent to x_1 and x_2 in the second case.
Suppose that any intersecting family with $n - 1$ hyperedges is a star, and let $(E_{x_i})_{1\leq i\leq n}$ be an intersecting family. $(E_{x_i})_{2\leq i\leq n}$ is a star, so there exists y such that y is adjacent to $(x_i)_{2\leq i\leq n}$. Suppose that $y \in V_1$ and $(x_i)_{2\leq i\leq n} \subseteq V_2$.

- $x_1 \in V_1$. Then x_1 is adjacent to x_i $i \in \{2, 3, \ldots, n\}$. So $x_1 = y$ otherwise this would lead to a C_4

- $x_1 \in V_2$. Let u_i, $i \in \{2, 3, \ldots, n\}$ be the common neighbor of x_1, x_i. Suppose that for all $i \in \{2, 3, \ldots, n\}$, $u_i \neq y$.

  - There exists $i \neq j$ such that $u_i \neq u_j$, consequently this leads to a C_6.
  - For all $i, j \in \{2, 3, \ldots, n\}$, $u_i = u_j$, consequently this leads to a C_4.

  So there exists $i \in \{2, 3, \ldots, n\}$ such that $u_i = y$ and y is adjacent to x_1. We can conclude that $\mathbb{H}_G$ has the Helly property.

□

From this theorem we have the following:

Corollary 1. *Let $G = (V, E)$ be a bipartite graph and $\mathbb{H}_G$ its associated neighborhood hypergraph. $\mathbb{H}_G$ has the Helly property if and only if it has the strong Helly property.*

Proof. If $\mathbb{H}_G$ has the strong Helly property then obviously it has the Helly property.
Assume now that $\mathbb{H}_G$ has the Helly property. Let $H' = (V', E')$ be a subhypergraph of $\mathbb{H}_G$, the induced subgraph $G(V')$ does not contain neither C_4 nor C_6. Hence H' has the Helly property, so $\mathbb{H}_G$ has the strong Helly property. □

On the strong Helly property we have:

Theorem 2. *Let $H = (S, E)$ be a hypergraph. H has the strong Helly property if and only if every C_6 of $IG(H)$ is well chorded.*

Proof. If H has the strong Helly property it is easy to see that any C_6 of $IG(H)$ is well chorded.
Assume now that any C_6 of $IG(H)$ is well chorded. Let H' be a subhypergraph, we are going to prove this assertion by induction on the hyperedge number of H'.
Let $(E_i)_{i\in\{1,2,3\}}$ be an intersecting family. This family generates a cycle $(x_1, e_1, x_2, e_2, x_3, e_3, x_1)$ in the incidence graph of H'. This cycle is well chorded, consequently there exists a vertex x_i, $i \in \{1, 2, 3\}$ which belongs to $\bigcap_{i\in\{1,2,3\}} E_i$.
Assume that it is true for any intersecting family of H with $p - 1$ hyperedges. Let $(E_i)_{i\in\{1,2,3,4,\ldots,p\}}$ be an intersecting family with p hyperedges. The following families: $(E_i)_{i\in\{2,3,4,\ldots,p\}}$, $(E_i)_{i\in\{1,3,4,\ldots,p\}}$, $(E_i)_{i\in\{1,2,4,\ldots,p\}}$ are stars, by induction hypothesis. So one can stand for respectively by $H(u)$, $H(v)$, $H(w)$ these three stars. These three vertices are on a cycle $(u, E_{u,v}, v, E_{v,w}, w, E_{w,u}, u)$ ($E_{a,b}$ being a hyperedge containing the vertices a, b). These cycles being well chorded, u, v, or w belong to any hyperedge of $(E_i)_{i\in\{1,2,3,4,\ldots,p\}}$. Hence this family is a star. So H' has the Helly property. □

Corollary 2. *Let $G = (V; E)$ be a graph, and $\mathbb{H}_G$ its associated neighborhood hypergraph. $\mathbb{H}_G$ has the strong Helly property if and only if G does not contain C_4, C_5, C_6 and any sun S_3.*

Proof. Assume that $\mathbb{H}_G$ does not have the strong Helly property. From theorem 2 the incidence graph $IG(H)$ contains a cycle C_6 which is not well chorded. Let $x, \{x\} \cup \Gamma(x), y, \{y\} \cup \Gamma(y), z, \{z\} \cup \Gamma(z))$ be this cycle such that the vertices x, y, z belong to the graph G. Consequently these three vertices are on a cycle C_4,C_5 or C_6 or on a sun S_3 of G.
Suppose now that G does not contain C_4, C_5, C_6 and any sun S_3. Hence $IG(\mathbb{H}_G)$ does not contain any C_6. So every C_6 of $IG(\mathbb{H}_G)$ is well chorded. □

We say that the hypergraph $H = (S; E)$ has the *separation property* (briefly, SP) if for every pair of distinct vertices $x, y \in S$ there exists a hyperedge $E_i \in E$ such that either $x \in E_i$ and $y \notin E_i$ or $x \notin E_i$ and $y \in E_i$.

Corollary 3. *A hypergraph H with the SP property has the strong Helly property if and only if its dual H^* has the strong Helly property.*

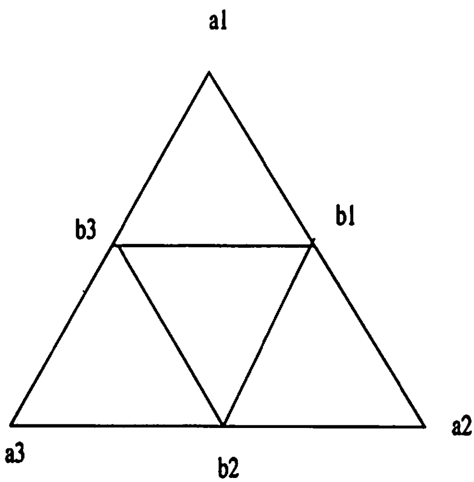

Figure 2: Sun S_3.

Proof. It is easy to see that the incidence graph of a hypergraph with the SP property is isomorphic to the incidence graph of its dual H^*. Moreover, from theorem above, H has the strong Helly property if and only if every C_6 of $IG(H)$ is well chorded. The corollary is proved. □

Corollary 4. *A hypergraph H with the SP property which has the strong Helly property is conformal.*

Proof. Easy from corollary 3 □

For the linear hypergraphs we have:

Proposition 1. *Let $H = (S; E)$ be a hypergraph. We have the two following properties:*

(i) For a linear hypergraph H, it follows that H has the Helly property if and only if $\mathbb{H}_{IG(H)}$ has the Helly property.

(ii) H is linear if and only if $IG(H)$ does not contain C_4.

Proof. [i] Suppose H linear.
$\mathbb{H}_{IG(H)}$ has the Helly property. Let $I = (E_i)_{i\in\{1,\dots,p\}}$ be an intersecting family of hyperedges of H. In $\mathbb{H}_{IG(H)}$ I is a set of vertices such that $(\{e_i\}\cup\Gamma(e_i))_{i\in\{1,\dots,p\}}$ is an intersecting family. $\mathbb{H}_{IG(H)}$ having the Helly property $(E_i)_{i\in\{1,\dots,p\}}$ has a non-empty intersection. Suppose that H has the Helly property and suppose that $\mathbb{H}_{IG(H)}$ does not satisfy the Helly property. From theorm 1 $IG(H)$ contains C_4 or C_6.
If $IG(H)$ contains C_4, there exists two vertices of C_4 $-e_1$, e_2 representing two

hyperedges of H- and two vertices x_1, x_2 of S belonging to C_4. So x_1, x_2 belong to E_1 and E_2. Consequently $|E_1 \cap E_2| > 1$, ($|E_1 \cap E_2|$ is the cardinality of $E_1 \cap E_2$) and H is not linear. Contradiction, $IG(H)$ does not contain C_4.
If $IG(H)$ contains C_6: x_1, e_1, x_2, e_3, x_3, e_2, x_1. There exists three vertices of C_6: e_1, e_2 and e_3 representing three hyperedges of H and three vertices: x_1, x_2 x_3 of S belonging to C_6. But H has the Helly property, so there exists $y \in S$ such that $y \in E_1 \cap E_2 \cap E_3$. Either $y = x_i$, $i = 1,2$ or 3. For example $y = x_1$, so x_1, e_1, x_2, e_3, x_1 is a C_4 contradiction, or $y \neq x_i$, $i = 1,2$ or 3, but for example $|E_1 \cap E_2| > 1$ and H is not linear, contradiction. Consequently $IG(H)$ does not contain C_6. We can conclude that $\mathbb{H}_{IG(H)}$ has the Helly property.
[ii] obvious from the proof of [i].

□

3 Products of Hypergraphs.

In this section we are going to define some products of hypergraphs. These generalize the products of graphs.
Let $H_1 = (S_1, E_1)$ and $H_2 = (S_2, E_2)$ be two hypergraphs, we define a hypergraph $H = (S, E)$ in the following way:
for each product the set of vertices is : $S = S_1 \times S_2$.

The product of hypergraph $H_1 \Box H_2 = (S, E) = H$ is the *cartesian product* if:

$$E = E(H_1 \Box H_2) = \underbrace{\{\{x\} \times e, x \in S_1 \text{ and } e \in E_2\}}_{A_1} \cup \underbrace{\{e \times \{x\}, e \in E_1 \text{ and } x \in S_2\}}_{A_2}.$$

The product of hypergraph $H_1 \times H_2 = (S, E) = H$ is the *strong product* if:

$$E = E(H_1 \times H_2) = \{e_i \times e_j, e_i \in E_1 \text{ and } e_j \in E_2\}.$$

Lemma 1. *$E(A_1) \cap E(A_2) = \emptyset$. Moreover for all $e \in E(A_1)$ and $e' \in E(A_2)$ $|e \cap e'| \leq 1$, ($E(A_i)$ being the hyperedges from A_i).*

Proof. Otherwise there exists $\{x\} \times e \in A_1$ and $e' \times \{y\} \in A_2$ such that $\{x\} \times e = e' \times \{y\}$, so $\{x\} = e'$ and $e = \{y\}$. Hence e' and e are both a loop. Contradiction. Suppose now that $|e \cap e'| \geq 2$. So there exists two distinct vertices (x_1, y_1) and (x_2, y_2) belonging to the intersection of e and e'. Consequently $x_1 = x_2$ and $y_1 = y_2$. Contradiction. □

Theorem 3. *Let $H = H_1 \Box H_2$ be the cartesian product of H_1 and H_2, H has the Helly property if and only if H_1 and H_2 have this property.*

Proof. Suppose that H_1 and H_2 have the Helly property. Let F be an intersecting family of H. Suppose that $F = (\{x\} \times e_i)_{i \in \{1,2,\ldots p\}}$, $(e_i)_{i \in \{1,2,\ldots p\}}$ is an intersecting family of H_2. So there exists $y \in S_2$ such that $y \in \bigcap_{i \in \{1,2,\ldots p\}} e_i$. Consequently $(x, y) \in \bigcap_{i \in \{1,2,\ldots p\}} (\{x\} \times e_i)$.

If $F = (e_i \times \{x\})_{i\in\{1,2,\ldots p\}}$ one will proceed as above.
Suppose now that $F = ((\{x\} \times e_i)_{i\in\{1,2,\ldots p\}}, (e_j \times \{y\})_{j\in\{1,2,\ldots k\}})$ is an intersecting family. It easy to see that $(x, y) \in \bigcap_{i\in\{1,2,\ldots p\}}(\{x\} \times e_i) \bigcap_{i\in\{1,2,\ldots p\}}(e_i \times \{y\}$ and hence H has the Helly property.
Suppose now that H has the Helly property. Let $(e_i)_{i\in\{1,2,\ldots p\}}$ be an intersecting family of H_2, $(\{x\} \times e_i)_{i\in\{1,2,\ldots p\}}$ is an intersecting family of H. So there exists $(x, y) \in S_1 \times S_2$ such that $(x, y) \in \bigcap_{i\in\{1,2,\ldots p\}}(\{x\} \times e_i)$. Consequently $y \in \bigcap_{i\in\{1,2,\ldots p\}}(e_i)$ and H_2 has the Helly property.
If $(e_i)_{i\in\{1,2,\ldots p\}}$ is an intersecting family of H_1 in the same way there exists $x \in S_1$ such that $x \in \bigcap_{i\in\{1,2,\ldots p\}}(e_i)$, and H_1 has the Helly property. □

Theorem 4. *Let $H_1 \Box H_2 = (S, E) = H$ be the cartesian product of $H_1 = (S_1, E_1)$ and $H_2 = (S_2, E_2)$ H is conformal if and only if H_1 and H_2 are conformal.*

Proof. Let us suppose that K is a maximal clique of $[H]_2$. Let (x_1, y_1) and (x_2, y_2) be two vertices of $V(K)$. From lemma1 these vertices belong to either a hyperedge of $E(A_1)$ or a hyperedge of $E(A_2)$. Without losing generality one can suppose that they belong to a hyperedge of $E(A_1)$. Hence $[(x_1, y_1)(x_2, y_2)] \in E(K)$ if and only if $x_1 = x_2$ and $[y_1, y_2] \in E([H_2]_2)$. So to every clique K of H one can associate a clique K' of H_2 (resp. H_1) and K' is contained in a hyperedge e of H_2 (resp. H_1) (by hypothesis). So K is contained in $\{x\} \times e$ (resp.$e \times \{x\}$).
In the same way we have the converse. □

Theorem 5. *Let $H = H_1 \times H_2$ be the strong product of H_1 and H_2, H has the Helly property if and only if H_1 and H_2 have this property.*

Proof. Let $F = (e_p \times e_q)_{(p,q)\in P\times Q}$ be a family of hyperedges of H. This family is an intersecting family if and only if the families $(e_p)_{p\in P}$ and $(e_q)_{q\in Q}$ are intersecting. Suppose that H has the Helly property. There exists $(x, y) \in S_1 \times S_2$ such that $(x, y) \in \bigcap_{(p,q)\in P\times Q}(e_p \times e_q)$ this is equivalent to $x \in \bigcap_{p\in P} e_p$ and $y \in \bigcap_{q\in Q} e_q$, and this is equivalent to say that H_1 and H_2 have the Helly property.

□

Recall that a hypergraph is *arboreal* if and only if it has the Helly property and its line graph $L(H)$ is triangulated. So we have the following:

Corollary 5. *Let $H = H_1 \times H_2$ be the strong product of H_1 and H_2, H is arboreal if and only if H_1 and H_2 are.*

Proof. It is sufficient to prove that $L(H)$ is triangulated if and only if both $L(H_1)$ and $L(H_2)$ are.
Let $C = \{(e_1 \times e'_1), (e_2 \times e'_2)\}$; $\{(e_2 \times e'_2), (e_3 \times e'_3)\}$; $\{(e_3 \times e'_3), (e_4 \times e'_4)\}$; $\ldots \{(e_{k-1} \times e'_{k-1}), (e_k \times e'_k), (e_1 \times e'_1)\}$ be a cycle of $L(H)$. This cycle gives rise to a cycle of $L(H_1)$: $C' = \{e_1, e_2\}$; $\{e_2, e_3\}$; $\{e_3, e_4\}$; $\ldots \{e_{k-1}, e_k\}$; $\{e_k, e_1\}$ and a cycle of $L(H_2)$: $C'' = \{e'_1, e'_2\}$; $\{e'_2, e'_3\}$; $\{e'_3, e'_4\}$; $\ldots \{e'_{k-1}, e'_k\}$; $\{e'_k, e'_1\}$. Conversely

a cycle of $L(H_1)$ and a cycle of $L(H_2)$ give rise to a cycle of $L(H)$. It is easy to see that C has a chord if and only if both C' and C'' have a chord. □

Theorem 6. *Let $H = H_1 \times H_2$ be the strong product of H_1 and H_2, H is conformal if and only if both H_1 and H_2 are.*

Proof. Let (x_i, y_i); (x_j, y_j) be two vertices of $H = H_1 \times H_2$. These two vertices belong to a hyperedge e if and only if there exists a hyperedge e_1 of H_1 and a hyperedge e_2 of H_2 such that $x_i, x_j \in e_1$ and $y_i, y_j \in e_2$. Consequently K is a clique of $[H]_2$ if and only if there exists a clique K_1 of $[H_1]_2$ and a clique K_2 of $[H_2]_2$ such that $K = K_1 \times K_2$. Hence $K = K_1 \times K_2 \subseteq e = e_1 \times e_2$ if and only if $K_1 \subseteq e_1$ and $K_2 \subseteq e_2$ □

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