ON THE SUPER EDGE-MAGIC DEFICIENCY OF GRAPHS

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Dedicated to Professor Anna Lladó

ABSTRACT. A (p,q) graph G is called edge-magic if there exists a bijective function $f: V(G) \cup E(G) \to \{1,2,\ldots,p+q\}$ such that $f(u)+f(v)+f(uv)=k$ is a constant for any edge $uv \in E(G)$. Moreover, G is said to be super edge-magic if $f(V(G))=\{1,2,\ldots,p\}$. The question studied in this paper is for which graphs it is possible to add a finite number of isolated vertices so that the resulting graph is super edge-magic. If it is possible for a given graph G, then we say that the minimum such number of isolated vertices is the super edge-magic deficiency, $\mu_s(G)$ of G; otherwise we define it to be $+\infty$.

1. INTRODUCTION

The reader is directed to either Chartrand and Lesniak [1] or Hartsfield and Ringel [4] for all additional terminology not provided in this paper.

An *edge-magic labeling* of a (p,q) graph G is a bijective function

$$f: V(G) \cup E(G) \to \{1,2,\ldots,p+q\}$$

such that $f(u)+f(v)+f(uv)=k$ is a constant for any edge $uv \in E(G)$. In such a case, G is said to be *edge-magic*, and k is called the *valence* of f. Moreover, f is a *super edge-magic labeling* of G if $f(V(G))=\{1,2,\ldots,p\}$, and G is said to be *super edge-magic*.

The previous two definitions were first introduced by Kotzig and Rosa [6, 7] in 1970, and by Enomoto, Lladó, Nakamigawa and Ringel [2] in 1998, respectively. It is worthwhile to mention that Kotzig and Rosa called edge-magic labelings, magic valuations; the current term is due to Ringel [8].

Next, we provide the definitions of edge-magic and super edge-magic deficiencies of a graph.

The *edge-magic deficiency* of a graph G, $\mu(G)$, is defined as

$$\mu(G) = \min\{n \geq 0 : G \cup nK_1 \text{ is an edge-magic graph}\}.$$

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This definition was first introduced by Kotzig and Rosa [6, 7], who showed that $\mu(G)$ is well-defined. This motivates us to define the super edge-magic deficiency analogously.

Let G be a graph, and let

$$M(G) = \{n \geq 0 : G \cup nK_1 \text{ is a super edge-magic graph}\}.$$

Then the *super edge-magic deficiency* of G, $\mu_s(G)$, is defined to be

$$\mu_s(G) = \begin{cases} \min M(G), & \text{if } M(G) \neq \emptyset; \\ +\infty, & \text{if } M(G) = \emptyset. \end{cases}$$

It is a direct consequence of the above two definitions that the inequality $\mu(G) \leq \mu_s(G)$ holds for any graph G.

To conduct our study of the super edge-magic deficiency of graphs, the following results will prove to be useful.

The next lemma from [3] provides us with a necessary and sufficient condition for a graph to be super edge-magic, and it is this useful characterization that has become our preferred way of looking at these graphs.

Lemma 1.1. *A (p, q) graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1, 2, \ldots, p\}$ such that the set*

$$S = \{f(u) + f(v) : uv \in E(G)\}$$

consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with valence $k = p + q + s$, where $s = \min S$ and

$$S = \{k - (p+1), k - (p+2), \ldots, k - (p+q)\}.$$

Also, the authors proved the following theorem in [3].

Theorem 1.2. *Let G be a graph of size q having the property that for all non-empty sets V_1 and V_2 such that $V_1 \cup V_2 = V(G)$ and $V_1 \cap V_2 = \emptyset$,*

$$|\{uv \in E(G) : u \in V_1 \text{ and } v \in V_2\}|$$

is neither $\lfloor q/2 \rfloor$ nor $\lceil q/2 \rceil$. Then G is not super edge-magic.

As an immediate corollary, we obtain the following result.

Corollary 1.3. *Let G be a graph of size q having the property that for all non-empty sets V_1 and V_2 such that $V_1 \cup V_2 = V(G)$ and $V_1 \cap V_2 = \emptyset$,*

$$|\{uv \in E(G) : u \in V_1 \text{ and } v \in V_2\}|$$

is neither $\lfloor q/2 \rfloor$ nor $\lceil q/2 \rceil$. Then $\mu_s(G) = +\infty$.

To prove an easier to use (although less powerful) condition than the previous one, for a graph to have infinite super edge-magic deficiency, we will state and prove the following lemma.

Lemma 1.4. *If C is a cycle and $V(C)$ is partitioned into two sets V_1 and V_2, then $|\{uv \in E(C) : u \in V_1$ and $v \in V_2\}|$ is even.*

Proof. Take a cycle C and contract all the edges in $\langle V_1 \rangle$ and $\langle V_2 \rangle$. Then notice that the result is an eulerian bipartite multigraph M with

$$E(M) = \{uv \in E(C) : u \in V_1 \text{ and } v \in V_2\}.$$

Therefore, $|E(M)|$ is even. ∎

A graph G is said to be an *even graph* if all of its vertices have even degree. Thus, with this definition in hand, we obtain the next result as a corollary to the previous lemma and Corollary 1.3.

Theorem 1.5. *If G is an even graph of size q such that $q/2$ is odd, then $\mu_s(G) = +\infty$.*

Proof. Since G is an even graph, it follows that all components of G are eulerian. Hence, we can decompose G into a union of edge disjoint cycles. Now, any partition of $V(G)$ into two sets V_1 and V_2 induces partitions on each of the vertex sets of each cycle in our decomposition, and by the previous lemma, there are an even number of edges between each of these. Thus, there is an even number of edges joining the vertices of V_1 and V_2; however, $q/2$ is odd, so

$$|\{uv \in E(C) : u \in V_1 \text{ and } v \in V_2\}| \neq q/2.$$

Therefore, by Corollary 1.3, $\mu_s(G) = +\infty$. ∎

We remark that there exist graphs that satisfy the hypotheses of Corollary 1.3, but not those of the above theorem; as an example, if $G \cong K_{12}$, then $q(K_{12})/2 \neq q(K_{n,12-n})$ for $n = 1, 2, \ldots, 11$, since the resulting quadratic equation $n^2 - 12n + 33 = 0$ has no integer solutions.

The authors have not been very successful in using the machinery developed above to study graphs for which the clique number (the largest order $\omega(G)$ among the complete subgraphs of a graph G) is large in relation to the size of the graph. To do this, we have resorted to the theory of well-spread sets as introduced by Kotzig [5].

A set $X = \{x_1 < x_2 < \cdots < x_n\} \subset \mathbb{N}$ is a *well-spread set* (WS-set for short) if the sums $x_i + x_j$ for $i < j$ are all different. Furthermore, define the *smallest span of pairwise sums of cardinality* n, denoted by $\rho^*(n)$, to be

$$\rho^*(n) = \min\{x_n + x_{n-1} - x_2 - x_1 : \{x_1 < x_2 < \cdots < x_n\} \text{ is a WS-set}\}.$$

The following lemma found by Kotzig [5] states a few values of $\rho^*(n)$ for $n = 4, 5, \ldots, 8$ and provides a lower bound for any integer $n \geq 9$.

Lemma 1.6. *The smallest span of pairwise sums of cardinality n, $\rho^*(n)$ satisfies that $\rho^*(4) = 6$, $\rho^*(5) = 11$, $\rho^*(6) = 19$, $\rho^*(7) = 30$, $\rho^*(8) = 43$, and $\rho^*(n) \geq n^2 - 5n + 14$ for any integer $n \geq 9$.*

2. Super Edge-Magic Deficiencies of Some Graphs

Although we found the problem of computing $\mu(G)$ and $\mu_s(G)$ to be a difficult one, we have been successful in computing these parameters for some specific classes of graphs which we present next.

Theorem 2.1. *The edge-magic and super edge-magic deficiencies of the forest nK_2 are given by*

$$\mu_s(nK_2) = \mu(nK_2) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

Proof. First, note that Kotzig and Rosa [6] showed that the forest nK_2 is edge-magic if and only if n is odd. Hence, $\mu(nK_2) = 0$ if n is odd, and $\mu(nK_2) \geq 1$ if n is even. Actually, the edge-magic labeling that they provide in their proof for nK_2 when n is odd, is in fact, a super edge-magic labeling, which implies that $\mu(nK_2) = \mu_s(nK_2) = 0$ when n is odd. Thus, assume, without loss of generality, that n is even.

Now, define the graph $G \cong nK_2 \cup K_1$ with

$$V(G) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z\}$$

and

$$E(G) = \{x_i y_i : 1 \leq i \leq n\},$$

and consider the vertex labeling $g : V(G) \to \{1, 2, \ldots, 2n+1\}$ such that

$$g(w) = \begin{cases} i, & \text{if } w = x_i \text{ and } 1 \leq i \leq n; \\ 3n/2 + i + 1, & \text{if } w = y_i \text{ and } 1 \leq i \leq n/2; \\ n/2 + i, & \text{if } w = y_i \text{ and } n/2 + 1 \leq i \leq n; \\ 3n/2 + 1, & \text{if } w = z. \end{cases}$$

Then, by Lemma 1.1, g extends to a super edge-magic labeling of G with valence $3n/2 + 2$ and, consequently, $\mu_s(nK_2) \leq 1$.

Therefore, we conclude that $\mu(nK_2) = \mu_s(nK_2) = 1$ when n is even. ∎

The following theorem is due to Enomoto, Lladó, Nakamigawa and Ringel [2].

Theorem 2.2. *The cycle C_n is super edge-magic if and only if n is odd.*

The previous theorem allows us to compute the super edge-magic deficiency of cycles.

Theorem 2.3. *The super edge-magic deficiency of the cycle C_n is given by*

$$\mu_s(C_n) = \begin{cases} 0, & \text{if } n \equiv 1 \text{ or } 3 \pmod 4; \\ 1, & \text{if } n \equiv 0 \pmod 4; \\ +\infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. First, assume that n is odd. Then the cycle C_n is super edge-magic, implying that $\mu_s(C_n) = 0$. For $n \equiv 0 \pmod 4$, C_n is not super edge-magic, that is, $\mu_s(C_n) \geq 1$.

For the other inequality, define the graph $G \cong C_n \cup K_1$ with

$$V(G) = \{x_i : 1 \leq i \leq n/2\} \cup \{y_i : 1 \leq i \leq n/2\} \cup \{z\}$$

and

$$E(G) = \{x_i y_i : 1 \leq i \leq n/2\} \cup \{x_{i-1} y_i : 2 \leq i \leq n/2\} \cup \left\{x_{n/2} y_1\right\},$$

where $n \equiv 0 \pmod 4$. By Lemma 1.1, the following vertex labeling f extends to a super edge-magic labeling of G with valence $n/2 + 2$, where

$$f(v) = \begin{cases} i, & \text{if } v = x_i \text{ and } 1 \leq i \leq n/2; \\ n/2 + i, & \text{if } v = y_i \text{ and } 1 \leq i \leq n/4; \\ n/2 + i + 1, & \text{if } v = y_i \text{ and } n/4 + 1 \leq i \leq n/2; \\ 3n/4 + 1, & \text{if } v = z. \end{cases}$$

Thus, $\mu_s(C_n) \leq 1$, which leads to conclude that $\mu_s(C_n) = 1$.

Finally, the remaining case immediately follows from Theorem 1.5. ∎

We will use the following theorem to compute $\mu_s(K_n)$.

Theorem 2.4. *Let G be a graph that contains the complete subgraph K_n. If $|E(G)| < \rho^*(n)$, then $\mu_s(G) = +\infty$.*

Proof. We will use an indirect argument to prove the theorem. Suppose that there exists a graph G containing the complete subgraph $H \cong K_n$ with $|E(G)| < \rho^*(n)$ and such that $\mu_s(G) = m$, where $m \in \mathbb{N}$.

Now, assume that f is a super edge-magic labeling of $G \cup mK_1$, and let

$$S = \{f(u) + f(v) : uv \in E(G)\}.$$

Then S is a set of $|E(G)|$ consecutive integers, and hence

$$\{f(v) : v \in V(H)\} = \{x_1 < x_2 < \cdots < x_n\}$$

is a WS-set. Thus,

$$\begin{aligned} |E(G)| &= \max S - \min S + 1 \\ &\geq x_n + x_{n-1} - x_2 - x_1 + 1 \\ &\geq \rho^*(n) > |E(G)|, \end{aligned}$$

and therefore the desired contradiction has been reached. ∎

Obviously, the above theorem implies that if $|E(G)| < \rho^*(\omega(G))$ for a graph G, then $\mu_s(G) = +\infty$.

Now, as an immediate corollary of the previous theorem, we compute the super edge-magic deficiency of the complete graph K_n in the following way.

Theorem 2.5. *The super edge-magic deficiency of the complete graph satisfies that $\mu_s(K_n) = +\infty$ for every integer $n \geq 5$ and is finite for $n = 1, 2, 3$ or 4.*

Proof. The graphs K_1, K_2 and K_3 are trivially super edge-magic, and thus $\mu_s(K_1) = \mu_s(K_2) = \mu_s(K_3) = 0$. Also, K_4 is certainly not super edge-magic; however, $K_4 \cup K_1$ is super edge-magic as one simply needs to label the isolated vertex with 2, and the rest of the vertices with the remaining labels. Thus, $\mu_s(K_4) = 1$.

Finally, by Theorem 2.4, we conclude that $\mu_s(K_n) = +\infty$ for every integer $n \geq 5$. ∎

3. The Super Edge-Magic Deficiency of Complete Bipartite Graphs

Our first result in this section provides an upper bound for the super edge-magic deficiency of $K_{m,n}$ for every two positive integers m and n, implying that $\mu_s(K_{m,n}) < +\infty$.

Theorem 3.1. *The super edge-magic deficiency of the complete bipartite graph satisfies that $\mu_s(K_{m,n}) \leq (m-1)(n-1)$ for every two positive integers m and n.*

Proof. Let V_1 and V_2 be the partite sets of $K_{m,n}$, and let G be isomorphic to $K_{m,n} \cup (m-1)(n-1)K_1$. Then it suffices to present a super edge-magic labeling of G.

Thus, consider the vertex labeling $f : V(G) \to \{1, 2, \ldots, mn+1\}$ such that $f(V_1) = \{1, 2, \ldots, m\}$ and $f(V_2) = \{m+1, 2m+1, \ldots, nm+1\}$ that extends to a super edge-magic labeling of G with valence $m(2n+1)+3$ by Lemma 1.1. ∎

A computer search of small cases together with the next theorem leads the authors also to conjecture that $\mu_s(K_{m,n}) = (m-1)(n-1)$.

For the next proof, we will use the following notation: if $A \subseteq \mathbb{R}$ and $b \in \mathbb{R}$, then $A + b = \{a + b : a \in A\}$.

Theorem 3.2. *The super edge-magic deficiency of the graph $K_{m,2}$ is equal to $m-1$ for every positive integer m.*

Proof. First, notice that by the previous theorem, $\mu_s(K_{m,2}) \leq m-1$; so we assume that $\mu_s(K_{m,2}) = n$, and let $G \cong K_{m,2} \cup nK_1$ be the graph with

$$V(G) = \{u_1, u_2\} \cup \{v_1, v_2, \ldots, v_m\} \cup \{w_1, w_2, \ldots, w_n\}$$

and

$$E(G) = \{u_i v_j : i = 1, 2 \text{ and } j = 1, 2, \ldots, m\}.$$

Now, let $f : V(G) \cup E(G) \to \{1, 2, \ldots, 3m + n + 2\}$ be a super edge-magic labeling of G such that $f(u_1) < f(u_2)$ and $f(v_i) < f(v_j)$ if and only if $i < j$. Also, let $\alpha = f(u_1)$, $\beta = f(v_1)$, $\alpha + \delta = f(u_2)$, and $R = \{f(v_i) : i = 1, 2, \ldots, m\}$.

Now, the first part of this proof will consist of showing that

(1) $$R = \{\beta + 2\delta i + j : 0 \le i \le n \text{ and } 0 \le j \le \delta - 1\},$$

where n is such that $m = \delta(n + 1)$.

To do this, we prove by induction on i $(0 \le i \le n)$ that

$$\beta + 2i\delta, \ldots, \beta + (2i + 1)\delta - 1 \in R$$

and

$$\beta + (2i + 1)\delta, \ldots, \beta + (2i + 2)\delta - 1 \notin R.$$

Consider $S = \{f(u) + f(v) : uv \in E(G)\}$. Then the proof hinges on the repeated use of the following two facts:

(fact 1) the set S consists of $2m$ consecutive integers by Lemma 1.1; and
(fact 2) the sets $f(u_1) + R$ and $f(u_2) + R$ partition S.

Now, both $\alpha + \beta = f(u_1) + f(v_1)$ and $\alpha + \beta + \delta = f(u_2) + f(v_1)$ are in S. This implies that $\alpha + \beta, \ldots, \alpha + \beta + \delta - 1 \in S$ by fact 1, but $\alpha + \beta + \delta - 1 < \alpha + \beta + \delta = f(u_2) + \min(R)$, so $\alpha + \beta, \ldots, \alpha + \beta + \delta - 1 \in f(u_1) + R$ by fact 2, which in turn implies that

(2) $$\beta, \ldots, \beta + \delta - 1 \in R,$$

and $\alpha + \beta + \delta, \ldots, \alpha + \beta + 2\delta - 1 \in f(u_2) + R$.

Now, if $\beta + \delta, \ldots, \beta + 2\delta - 1 \in R$, then $\alpha + \beta + \delta, \ldots, \alpha + \beta + 2\delta - 1 \in f(u_1) + R$, which is a contradiction by fact 2. Thus,

(3) $$\beta + \delta, \ldots, \beta + 2\delta - 1 \notin R.$$

Hence, by (2) and (3), the claim is true for $i = 0$. Now, assume that it is true for $i = k$ and that $\max(S) > f(u_2) + \beta + (2k + 1)\delta - 1 = \alpha + \beta + (2k + 2)\delta - 1$, which is in R by inductive hypothesis. Thus,

(4) $$\alpha + \beta + (2k + 2)\delta \in S$$

by fact 1, but $\beta + (2k+1)\delta \notin R$ by inductive hypothesis, so $\alpha + \beta + (2k+2)\delta = f(u_2) + \beta + (2k + 1)\delta$ is not in $f(u_2) + R$, and hence it is in $f(u_1) + R$ by fact 2, which implies that $\beta + (2k + 2)\delta \in R$ and

(5) $$\alpha + \beta + (2k + 3)\delta = (\alpha + \delta) + \beta + (2k + 2)\delta \in f(u_2) + R \subseteq S.$$

Thus, $\alpha + \beta + (2k + 2)\delta, \ldots, \alpha + \beta + (2k + 3)\delta - 1 \in S$ by (4), (5) and fact 1. However, by inductive hypothesis, $\beta + (2k + 1)\delta, \ldots, \beta + (2k + 2)\delta - 1 \notin R$, and hence $(\alpha + \delta) + \beta + (2k + 1)\delta, \ldots, (\alpha + \delta) + \beta + (2k + 2)\delta - 1$ are not in $f(u_2) + R$, so they are in $f(u_1) + R$ by fact 2, that is, $\alpha + \beta + (2k + 2)\delta, \dots, \alpha + \beta + (2k+3)\delta - 1 \in f(u_1) + R$. Thus,

$$(6) \qquad \beta + (2k+2)\delta, \dots, \beta + (2k+3)\delta - 1 \in R.$$

Now, we prove by contradiction that

$$(7) \qquad \beta + (2k+3)\delta, \dots, \beta + (2k+4)\delta - 1 \notin R.$$

If (7) is false, then we have that the integers $(\alpha+\delta)+\beta+(2k+2)\delta$ through $(\alpha + \delta) + \beta + (2k+3)\delta - 1$ are all in $f(u_1) + R$; however, by (6), they are also in $f(u_2) + R$, which is a contradiction. Therefore, in light of (6) and (7), the claim follows from the principle of mathematical induction.

Finally, to conclude that $\mu_s(K_{m,2}) \geq m - 1$, we compute a lower bound for the minimum number of elements that would have to be added to the set $\{f(u_1), f(u_2)\} \cup R$ to obtain a set of consecutive integers.

Now, we glean from (1) that $|R| = m$, $\min(R) = f(v_1) = \beta$ and $\max(R) = f(v_m) = \beta + 2\delta n + \delta - 1$. Thus, we would have to add

$$(8) \qquad \max(R) - \min(R) - |R| + 1 = \delta n$$

elements to R to fill its gaps. Furthermore, since $f(u_2) - f(u_1) = \delta$ and $f(v_i) - f(v_{i-1}) = 1$ or $\delta + 1$, also by (1), we have only two cases to consider, namely, either $f(u_1) < f(u_2) < \min(R)$ or $\max(R) < f(u_1) < f(u_2)$. In either of these cases, we need $f(u_2) - f(u_1) = \delta - 1$ elements to fill the gap between $f(u_1)$ and $f(u_2)$, which combined with (8) leads to the desired conclusion that at least $(\delta - 1) + \delta n = m - 1$ elements are needed. ∎

4. The Deficiency of Unions of Graphs

The union of two graphs with finite super edge-magic deficiency does not always have finite super edge-magic deficiency. Indeed, K_3 is trivially super edge-magic, but $\mu_s(2K_3) = +\infty$ by Theorem 1.5. However, if one of the graphs is additionally bipartite, the situation is different as the results in this section indicate.

Theorem 4.1. *If G and H are super edge-magic graphs such that G is bipartite, then $\mu_s(G \cup H) < +\infty$.*

Proof. Consider such graphs G and H. Let the partite sets of G be V_1 and V_2, and g and h be super edge-magic labelings of G and H, respectively. Assume, without loss of generality, that $1 \in f(V_2)$, and define

$$\beta = \max\left\{\max_{v \in V_1} g(v), |V(H)| - M + m - 1\right\},$$

and

$$\alpha = M - m + 2\beta + 1,$$

where

$$m = \min\{g(u) + g(v) : uv \in E(G)\}$$

and

$$M = \max\{h(u) + h(v) : uv \in E(H)\}.$$

Then the function f defined by

$$f(v) = \begin{cases} g(v), & \text{if } v \in V_1 \\ g(v) + \alpha, & \text{if } v \in V_2 \\ h(v) + \beta, & \text{if } v \in V(H) \end{cases}$$

extends to a super edge-magic labeling of $G \cup H \cup rK_1$ for some integer r.

Now, we show that f is injective, by proving that $f(u) < f(v) < f(w)$ for any u, v, and w that are in V_1, $V(H)$, and V_2, respectively. This is true since $\max_{v \in V_1} g(v) < 1 + \beta$ and $|V(H)| + \beta < 1 + \alpha$ (observe that $\min_{v \in V(H)} h(v) = \min_{v \in V_2} g(v) = 1$ and $\max_{v \in V(H)} h(v) = |V(H)|$).

Finally, notice that

$$\{f(u) + f(v) : uv \in E(G \cup H)\}$$

consists of $|E(G \cup H)|$ consecutive integers, since

$$f(u) + f(v) = \begin{cases} g(u) + g(v) + \alpha, & \text{if } uv \in E(G), \\ h(u) + h(v) + 2\beta, & \text{if } uv \in E(H), \end{cases}$$

$M + 2\beta + 1 = m + \alpha$ and $r = \max f(V_2) - |V(G \cup H)|$. ∎

Notice that in the proof of the above theorem $\min f(V(G \cup H)) = \min f(V_1)$ and $\max f(V(G \cup H)) = \max f(V_2)$. Thus, the function $\hat{f}$ such that $\hat{f}(v) = f(v) - \min(f(V_1)) + 1$ for every $v \in V(G \cup H)$ extends to a super edge-magic labeling of $G \cup H \cup \hat{r}K_1$ with

$$\hat{r} = \max f(V_2) - \min f(V_1) + 1 - |V(G \cup H)|.$$

Therefore, $\mu_s(G \cup H) \leq \hat{r}$.

The above theorem also yields the following corollary.

Corollary 4.2. *If G and H are graphs such that $\mu_s(G) < +\infty$, $\mu_s(H) < +\infty$ and G is bipartite, then $\mu_s(G \cup H) < +\infty$.*

Proof. Let G and H be graphs that satisfy our hypothesis. Also, assume that G is non-trivial, for otherwise the result is trivial. The finite super edge-magic deficiencies of G and H imply that there exist $m, n \in \mathbb{N} \cup \{0\}$ such that $G \cup mK_1$ and $H \cup nK_1$ are super edge-magic. Moreover, $G \cup mK_1$ is bipartite. Therefore, $\mu_s(G \cup H \cup (m+n)K_1) < +\infty$ by the above theorem. ∎

As an immediate consequence to Theorem 3.2 and the above theorem, we have the following corollary.

Corollary 4.3. *The super edge-magic deficiency of $nK_{m,2}$ is finite for all positive integers m and n.*

5. The Deficiency of Forests

In [2], Enomoto, Lladó, Nakamigawa and Ringel conjectured that all trees are super edge-magic, so the most natural question is whether one can compute or at least bound $\mu_s(T)$ for a tree T. Indeed, in this section, we prove that every forest has finite super edge-magic deficiency.

Theorem 5.1. *If T is a tree, then $\mu_s(T) < +\infty$.*

Proof. Clearly, $\mu_s(K_1) = 0$, so let T be a non-trivial tree. Moreover, assume, without loss of generality, that the vertices of T are ordered pairs of integers (given a vertex $v = (x, y) \in V(T)$, we call x and y, respectively, the *level* and *position* of v, and we say that v is in level x) that satisfy the following four properties. First, the vertex $r = (0, 1)$, which we call the *root* of T, is in $V(T)$. Second, for all $v = (x, y) \in V(T)$, the level of v is the distance from r to v, that is, $x = d(r, v)$. Now, for $x \in \mathbb{N}$, let $l(x)$ be the number of vertices in level x, that is, $l(x) = |\{(x, y) \in V(T) : y \in \mathbb{N}\}|$ (notice that there exists a sufficiently large $n \in \mathbb{N}$ such that $l(m) = 0$ when $m \geq n$). Then the third property is that given a fixed level x of T, the vertices within it have consecutive positions which range from 1 to $l(x)$, that is, $\{y : (x, y) \in V(T)\} = \{1, 2, \ldots, l(x)\}$. Now, if $v \in V(T) - \{r\}$, then let $f(v)$, which we call the *father* of v, be the vertex adjacent to v in the (r, v)-path in T. Finally, the fourth property is that if the vertices u and v are in the same level, and the position of the father of u is less than or equal to the position of the father of v, then the position of u is less than or equal to the position of v.

Next, define the functions $g : V(T) - \{r\} \to \mathbb{Z}$ and $h : V(T) \to \mathbb{Z}$ as follows: if $v = (x, y) \in V(T) - \{r\}$, then

$$g(v) = \begin{cases} -\left(\sum_{k=1}^{n-1} l(2k)\right) - y + 1, & \text{if } x = 2n \text{ for some } n \in \mathbb{N}, \\ \left(\sum_{k=0}^{n-1} l(2k+1)\right) + y, & \text{if } x = 2n+1 \text{ for some } \in \mathbb{N}, \end{cases}$$

and $h(v) = g(v) - h(f(v))$ if $v \neq r$ and $h(r) = 0$.

Also, let $\lambda : V(T) \to \mathbb{N}$ be the function such that $\lambda(v) = h(v) + m$ for every $v \in V(T)$, and where $m = 1 - \min\{h(v) : v \in V(T)\}$. Then for all

$u, v \in V(T)$, we have that $\lambda(u) \geq 1$ and

$$\begin{aligned} & \{\lambda(u) + \lambda(v) : uv \in E(T)\} \\ = & \{\lambda(v) + \lambda(f(v)) : v \in V(T) - \{r\}\} \\ = & \{h(v) + h(f(v)) + 2m : v \in V(T) - \{r\}\} \\ = & \{g(v) + 2m : v \in V(T) - \{r\}\} \\ = & \left\{2m + 1 - \sum_{k=1}^{\infty} l(2k), \ldots, 0\right\} \cup \left\{1, \ldots, 2m + \sum_{k=0}^{\infty} l(2k+1)\right\} \end{aligned}$$

is a set of $|E(T)|$ consecutive integers.

To show that λ is injective, it suffices to show that h is injective. First, notice that $h(v)$ is positive if and only if the level of $v \in V(T)$ is odd. Next, let $v_1, v_2 \in V(T)$ be in level $x \geq 1$ and be such that the position of v_2 is greater than the position of v_1. Then we will prove by induction on x that $h(v_2) - h(v_1)$ is positive if x is odd and negative if x is even. First, if $x = 1$, the result is true since $h(1, y) = y$, where $1 \leq y \leq l(1)$. Now, assume that $x > 1$ is odd. Then, by inductive hypothesis and the fourth property that describes T, we have that $h(f(v_2)) - h(f(v_1)) < 0$ as the position of $f(v_2)$ is greater than the position of $f(v_1)$, and both are in the same even level. Moreover, $g(v_2) > g(v_1)$, thus

$$h(v_2) - h(v_1) = g(v_2) - g(v_1) - \{h(f(v_2)) - h(f(v_1))\} > 0.$$

Similarly, $h(v_2) - h(v_1) < 0$ if x is even. Therefore, for $x \geq 0$, we have that

$$|h(x,1)| < |h(x,2)| < \cdots < |h(x, l(x))|. \tag{9}$$

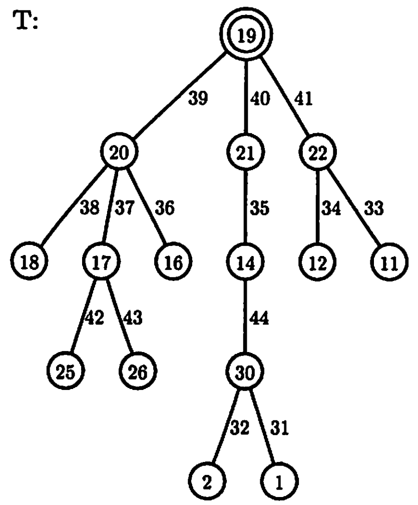
Next, we show that $h(x+2, 1) - h(x, l(x))$ is positive if x is odd and negative if x is even, also using induction. First, if $x = 0$ then $h(2,1) - h(0,1) = -g(f(2,1))$, which is negative since $f(2,1)$ is in level 1. Next, consider the case when x is odd. Then, by inductive hypothesis, $h(x+1, 1) - h(x-1, l(x-1)) < 0$ as $x - 1$ is even, which together with (9) leads us to

$$h(f(x+2,1)) \leq h(x+1,1) < h(x-1, l(x-1)) \leq h(f(x, l(x))).$$

Hence,

$$h(x+2,1) - h(x, l(x)) = 1 - h(f(x+2,1)) + h(f(x, l(x))) > 1$$

by the fact that $g(x, l(x)) + 1 = g(x+2, 1)$ and the definition of h. Similarly, $h(x+2, 1) - h(x, l(x)) < 1$ if $x \geq 2$ is even. Therefore, for $x \geq 0$, we have that $|h(x+2,1)| > |h(x, l(x))|$. This together with (9) implies that $h(2x_1 + 1, y) < h(2x_1 + 1, y_2)$ and $h(2x_1, y_1) > h(2x_2, y_2)$ if $x_1 = x_2$ and $y_1 < y_2$, or if $x_1 < x_2$, which combined with the fact that $h(0,1) = 0 < 1 = h(1,1)$ establishes that h is injective.



v	$g(v)$	$h(v)$	$\lambda(v)$
(0,1)		0	19
(1,1)	1	1	20
(1,2)	2	2	21
(1,3)	3	3	22
(2,1)	0	−1	18
(2,2)	−1	−2	17
(2,3)	−2	−3	16
(2,4)	−3	−5	14
(2,5)	−4	−7	12
(2,6)	−5	−8	11
(3,1)	4	6	25
(3,2)	5	7	26
(3,3)	6	11	30
(4,1)	−6	−17	2
(4,2)	−7	−18	1

FIGURE 1. Example for Theorem 5.1.

Finally, if v is the vertex with highest position in the highest odd level of T, then $\lambda^* = \lambda(v) = \max\{\lambda(v) : v \in V(T)\}$. Furthermore,

$$\min\{\lambda(v) : v \in V(T)\} = \min\{h(v) : v \in V(T)\} + m = 1.$$

Therefore, λ extends to a super edge-magic labeling of $T \cup (\lambda^* - |V(T)|)K_1$, and we conclude that $\mu_s(T) \leq \lambda^* - |V(T)| < +\infty$. ∎

We illustrate the construction on the above theorem with Figure 1, where $\lambda^* = 44$ and thus λ extends to a super edge-magic labeling of $T \cup 29K_1$.

Now, as a consequence to Theorem 5.1, Corollary 4.2 and the fact that all non-trivial trees are bipartite, we have the following corollary.

Corollary 5.2. *If F is a forest, then $\mu_s(F) < +\infty$.*

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