A HELLY THEOREM FOR INTERSECTIONS OF SETS STARSHAPED VIA STAIRCASE n-PATHS

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ABSTRACT. For $n \geq 1$, let $p(n)$ denote the smallest natural number r for which the following is true: For $\mathcal{K}$ any finite family of simply connected orthogonal polygons in the plane and points x and y in $\cap\{K : K \text{ in } \mathcal{K}\}$, if every r (not necessarily distinct) members of $\mathcal{K}$ contain a common staircase n-path from x to y, then $\cap\{K : K \text{ in } \mathcal{K}\}$ contains such a staircase path. It is proved that $p(1) = 1, p(2) = 2, p(3) = 4, p(4) = 6$, and $p(n) \leq 4 + 2p(n-2)$ for $n \geq 5$.

The numbers $p(n)$ are used to establish the following result. For $\mathcal{K}$ any finite family of simply connected orthogonal polygons in the plane, if every $3p(n+1)$ (not necessarily distinct) members of $\mathcal{K}$ have an intersection which is starshaped via staircase n-paths, then $\cap\{K : K \text{ in } \mathcal{K}\}$ is starshaped via staircase $(n+1)$-paths. If $n = 1$, a stronger result holds.

1. INTRODUCTION.

We begin with some definitions and comments from [1] and [5]. Let S be a nonempty set in the plane. Set S is called an *orthogonal polygon* if and only if S is a connected union of finitely many convex polygons (possibly degenerate) whose edges are parallel to the coordinate axes. Let λ be a simple polygonal path in the plane whose edges $[v_{i-1}, v_i], 1 \leq i \leq n$, are parallel to the coordinate axes. Path λ is called a *staircase path* if and only if the associated vectors alternate in direction. That is, for an appropriate labeling, for i odd the vectors $\overrightarrow{v_{i-1}v_i}$ have the same horizontal direction, and for i even the vectors $\overrightarrow{v_{i-1}v_i}$ have the same vertical direction. Edge $[v_{i-1}, v_i]$ will be called *north*, *south*, *east*, or *west* according to the direction of vector $\overrightarrow{v_{i-1}v_i}$. Similarly, we use the terms north, south, east, west, northeast, northwest, southeast, southwest to describe the relative position of points. For $n \geq 1$, if the staircase path λ is a union of at most n edges, then λ is called a *staircase $n-$path*. For points x and y in set S, we say x *sees* y (x is *visible* from y) via staircase n-paths if and only if there is a

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staircase n-path in S which contains both x and y. Similarly, for subsets A, B of S, we say A sees B via staircase n-paths if and only if a sees b via staircase n-paths for all a in A, b in B. Set S is called *staircase n – convex* provided for every x, y in S, x sees y via staircase n-paths. Similarly, set S is *starshaped via staircase n – paths* if and only if for some point p in S, p sees each point of S via staircase n-paths, and the set of all such points p is the *staircase n – kernel* of S. Of course, parallel definitions hold for staircase paths. Set S is *horizonally convex* if and only if for each x, y in S with $[x, y]$ horizontal, it follows that $[x, y] \subseteq S$. *Vertically convex* is defined analogously. Finally, S is an *orthogonally convex* polygon if and only if S is an orthogonal polygon which is both horizontally convex and vertically convex. Using [12, Lemma 1], an orthogonal polygon S is orthogonally convex if and only if it is staircase convex.

Many results in convexity that involve the usual notion of visibility via straight line segments have interesting analogues that use the idea of visibility via staircase paths. (See [12], [3], [4], [5], [6], [7], [13], [14].) A recent example concerns a Helly-type theorem by N. A. Bobylev [1]. The planar version of the theorem states that, for $\mathcal{K}$ a family of compact sets in the plane, if every three (not necessarily distinct) members of $\mathcal{K}$ have an intersection which is nonempty and starshaped via segments, then the intersection of all the sets is nonempty and starshaped via segments as well. In a staircase analogue proved in [2], for $\mathcal{K}$ a finite family of simply connected orthogonal polygons in the plane, if every three (not necessarily distinct) members of $\mathcal{K}$ have a nonempty intersection which is starshaped via staircase paths, then the intersection of all the sets is a nonempty simply connected orthogonal polygon which is starshaped via staircase paths.

In this paper, we pursue a staircase n-path analogue of the earlier result and establish the existence of Helly number $h(n)$ such that the following is true: For $\mathcal{K}$ a finite family of simply connected orthogonal polygons in the plane, if every $h(n)$ members of $\mathcal{K}$ have an intersection which is starshaped via staircase n-paths, then $\cap\{K : K \text{ in } \mathcal{K}\}$ is starshaped via staircase $(n+1)$-paths. Along the way, we obtain a similar result on the existence of $x - y$ n-paths, where x and y belong to $\cap\{K : K \text{ in } \mathcal{K}\}$.

As in [2], the proof will employ the following Helly-type result by Molnár [11] which appears in [8]: If $\mathcal{C}$ is a family of simply connected compact sets in the plane such that every two (not necessarily distinct) members of $\mathcal{C}$ have a connected intersection and every three (not necessarily distinct) members of $\mathcal{C}$ have a nonempty intersection, then $\cap\{C : C \text{ in } \mathcal{C}\}$ is nonempty and simply connected. We will also use some results by Topală [14] on visibility via staircase n-paths. Concerning notation, throughout the paper, *bdry S* will denote the boundary of set S. For distinct points x and y, $L(x, y)$ will be the line determined by x and y. If λ is a simple path containing x and

$y, \lambda(x,y)$ will represent the subpath of λ from x to y. The reader may refer to Valentine [15], to Lay [10], to Danzer, Grünbaum, Klee [8], and to Eckhoff [9] for discussions concerning Helly-type theorems, visibility via segments, and starshaped sets.

2. THE RESULTS.

The following definitions will be useful.

Definitions. For every $n \geq 1$, let $p(n)$ denote the smallest natural number r for which the following is true: For $\mathcal{K}$ any finite family of simply connected orthogonal polygons in the plane and for points x and y in $\cap\{K : K \text{ in } \mathcal{K}\}$, if every r (not necessarily distinct) members of $\mathcal{K}$ contain a common staircase n-path from x to y, then $\cap\{K : K \text{ in } \mathcal{K}\}$ contains such a path.

Similarly, let $s(n)$ denote the smallest natural number r for which the following holds: For $\mathcal{K}$ any finite family of simply connected orthogonal polygons in the plane and for A and B rectangular regions (possibly degenerate) in $\cap\{K : K \text{ in } \mathcal{K}\}$, if every r (not necessarily distinct) members of $\mathcal{K}$ contain a common staircase n-path from A to B, then $\cap\{K : K \text{ in } \mathcal{K}\}$ contains such a path.

The following sequence of lemmas will establish bounds for $p(n)$ and $s(n)$.

Lemma 1. *Let K be a simply connected orthogonal polygon in the plane, with A and B disjoint (and possibly degenerate) rectangular regions contained in K. For some fixed $n \geq 1$, let $\mathcal{P}$ denote the collection of staircase n-paths λ in K from A to B, where $\lambda \cap A$ and $\lambda \cap B$ are singleton sets. Let $A' = \{a : a \text{ in } \lambda \cap A \text{ for some } \lambda \text{ in } \mathcal{P}\}$, $B' = \{b : b \text{ in } \lambda \cap B \text{ for some } \lambda \text{ in } \mathcal{P}\}$. If $\mathcal{P}$ is nonempty, then sets A' and B' are closed connected subsets of bdry A and bdry B, respectively. Moreover, if A is fully two-dimensional, then for α a union of three appropriately chosen edges of bdry A (selected according to the relative positions of A and B), $A' \subseteq \alpha$. A parallel statement holds for B.*

Proof. For the moment, assume that sets A and B are fully two-dimensional. Since $A \cap B = \emptyset$, without loss of generality assume that each point of A is strictly west of the vertical line determined by the western edge of B. Observe that, for λ in P with associated endpoints a in A and b in B, a lies on the north, east, or south edge of A. We let α denote the union of these three edges. Order α along *bdry* A in a clockwise direction from the northeast vertex of A to the southwest vertex of A. Similarly, b lies on the north, west, or south edge of B, and we order the union β of these edges along *bdry* B in a counterclockwise direction from the northeast corner to the southwest corner of B.

To see that A' is connected, let a_1, a_2 belong to A', with $\lambda_i = \lambda_i(a_i, b_i)$ the associated member of $\mathcal{P}, i = 1, 2$. The region bounded by $\lambda_1 \cup \lambda_2 \cup \alpha(a_1, a_2) \cup \beta(b_1, b_2)$ lies in the simply connected set K. Choose any a_0 in $\alpha(a_1, a_2), a_0 \neq a_1, a_2$, to prove that a_0 is an endpoint of some path in $\mathcal{P}$. If a_0 lies on the east edge of A, choose the first point q of $\lambda_1 \cup \lambda_2 \cup \beta(b_1, b_2)$ which lies to the east of a_0. If q lies on one of λ_1 or λ_2, say λ_1, since $a_0 \neq a_1, q$ is not on the last segment of λ_1. Then $[a_0, q] \cup \lambda(q, b_1)$ is a staircase path in K having at most n segments. Clearly the path is in $\mathcal{P}$, so $a_0 \in A'$. If q lies on $\beta(b_1, b_2)$, then $[a_0, q]$ is an appropriate staircase path, and again $a_0 \in A'$. If a_0 is on the north (or south) edge of A, choose the first point q of $\lambda_1 \cup \lambda_2$ north (or south) of a_0 to obtain an appropriate path. We conclude that for a_1, a_2 in $A', \alpha(a_1, a_2) \subseteq A'$, so A' is a connected subset of α. A parallel proof produces an analogous result for B' and β.

In case one of A' or B' is a segment, a simplified version of the proof above produces the result.

Finally, a standard convergence argument shows that A' and B' are closed. Let $\{a_i\}$ be a sequence in A' converging to a_0, to show a_0 is in A'. For each a_i we let $a_i = a_{i0}, \ldots, a_{in} = b_i$ denote the vertices of a staircase n-path in $\mathcal{P}$. Passing to appropriate subsequences if necessary, without loss of generality assume $\{a_{ij} : 1 \leq i\}$ converges to a_j for each $1 \leq j \leq n$. It is easy to see that $a_0, \ldots, a_n \equiv b_0$ are vertices of a staircase n-path λ_0 in K, with $a_0 \in bdry\ A, b_0 \in bdry\ B$. Moreover, $\lambda_0 \cap A = \{a_0\}, \lambda_0 \cap B = \{b_0\}$, so λ_0 is in $\mathcal{P}$. Hence $a_0 \in A'$ and A' is closed. By a similar argument, B' is closed as well, finishing the proof of the lemma.

Lemma 2. *For $s(n)$ and $p(n)$ defined previously, $s(2) = p(2) = 2$.*

Proof. Let $\mathcal{K}$ be a finite family of simply connected orthogonal polygons in the plane, and let A and B be rectangular regions (possibly degenerate) contained in $\cap\{K : K \text{ in } \mathcal{K}\}$. Assume that every two members of $\mathcal{K}$ contain a common staircase 2-path from A to B. We assert that $\cap\{K : K \text{ in } \mathcal{K}\}$ contains such a path. Assume $A \cap B = \emptyset$, for otherwise the result is trivial. For convenience, we use the language of full two-dimensional sets A and B, although one or both of these sets may be degenerate. Without loss of generality, assume that each point of A is strictly west of the vertical line determined by the western edge of B. In case every two of the K sets share a common 1-path from A to B, then to each K in $\mathcal{K}$, let $\mathcal{P}_K$ denote the set of segments λ in K from A to B, where $\lambda \cap A$ and $\lambda \cap B$ are singleton sets. Let $A'_K = \{a : a \text{ in } \lambda \cap A \text{ for some } \lambda \text{ in } \mathcal{P}_K\}$. Clearly (or by Lemma 1) each A'_K is a closed segment. Every two of these sets have a nonempty intersection, so by Helly's theorem on the real line $\cap\{A'_K : K \text{ in } \mathcal{K}\}$ is nonempty. For a_0 in this intersection, there is an associated b_0 east of a_0

and in B such that $[a_0, b_0] \subseteq \cap\{K : K \text{ in } \mathcal{K}\}$. Then $[a_0, b_0]$ satisfies the assertion.

Otherwise, let a denote the northeast vertex of A, b the southwest vertex of B. There are two cases to consider, determined by the relative positions of a and b.

Case 1. Assume that b is on or north of the horizontal line at a. If every set K in $\mathcal{K}$ contains both an east - north 2-path (or segment) from A to B and a north - east 2-path (or segment) from A to B, then every K contains the rectangular region (possibly degenerate) with vertices a and b. Then either $a - b$ staircase 2-path satisfies the lemma.

Otherwise, we may assume that some set K in $\mathcal{K}$ contains no north - east 2-path from A to B. Let λ denote the east - north 2-path (or segment) from a to b. Every K must contain an east - north 2-path (or segment) from A to B. The first edge of such a path is on or south of λ (that is, on or south of the horizontal line which supports λ on the south), while the second edge of such a path is on or east of λ (on or east of the vertical line which supports λ on the east). It is easy to see that for each K_i we may select a corresponding λ_i as close as possible to λ. Choose K_0 such that its associated λ_0 (among all the λ_i paths) is as far as possible from λ. We assert that λ_0 lies in every K_i set. If $\lambda = \lambda_0$, this is immediate. Otherwise, $\lambda \not\subseteq K_0$. This implies that K_0 contains no north - east 2-path from A to B. Certainly for any K_i, the associated λ_i is at least as close to λ as λ_0 is to λ. Hence λ_i is in the region bounded by λ, λ_0, and the appropriate edges of A and B. (See Figure 1.) Moreover, K_i and K_0 share an east - north 2-path λ_i' from A to B, and since λ_0 is the closest such path to λ in K_0, λ_0 is in the region bounded by λ_i' and λ (and appropriate edges of A and B). Then λ_0 lies in the simply connected region of K_i determined by λ_i and λ_i' (and edges of A and B), so $\lambda_0 \subseteq K_i$. Path λ_0 satisfies the assertion.

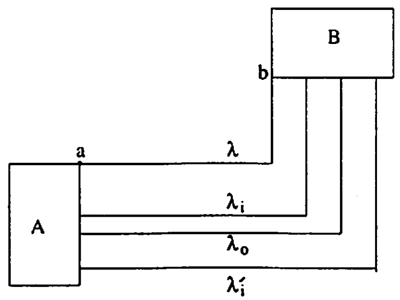

Figure 1

Case 2. Assume that b is strictly south of the horizontal line at a. Staircase 2-paths from A to B may be north - east, east, or one of east - north, south - east, depending on the relative positions of the southern edges of A and B. For the moment, assume that the southern edge of A is south of (or on) the horizontal line determined by the southern edge of B. Then $A - B$ staircase 2-paths may be north - east, east, or east - north. Since we are assuming that some two K_i sets contain no common 1-path, for at least one K_i set, say K_1, we may assume that K_1 contains no north - east 2-path from A to B. Let b' denote the first point of B east of a. Let $\lambda = [a, b']$ and repeat the argument in Case 1 to obtain an $A - B$ staircase 2-path λ_0 in $\cap\{K : K \text{ in } \mathcal{K}\}$.

If the southern edge of A is north of the horizontal line determined by the southern edge of B, a parallel argument yields an appropriate staircase 2-path in $\cap\{K : K \text{ in } \mathcal{K}\}$. We conclude that $s(2) \leq 2$.

That $p(2) \leq 2$ follows from the special case in which A and B are singleton sets. It is easy to find examples to show that the bound 2 cannot be reduced. (See Example 1.) Hence $p(2) = s(2) = 2$, finishing the proof of the lemma.

Lemma 3. *For $p(n)$ defined previously, $p(3) = 4$ and $p(4) = 6$.*

Proof. Let $\mathcal{K}$ be a finite family of simply connected orthogonal polygons in the plane, with points x and y in $\cap\{K : K \text{ in } \mathcal{K}\}$. Without loss of generality, assume that y is northeast of x.

For $n = 3$, assume that every 4 members of $\mathcal{K}$ share a common staircase 3-path from x to y, to show that $\cap\{K : K \text{ in } \mathcal{K}\}$ contains such a path. For each K_i, we select a corresponding staircase 3-path λ_i (if it exists) from x to y in K_i such that the first segment is east and is as long as possible among all such paths in K_i. Similarly, select a staircase 3-path μ_i (if it exists) from x to y whose first segment is north and is as long as possible among all such paths in K_i. Observe that for each K_i at least one of λ_i, μ_i must exist. If some K_i has no associated λ_i, let K_1 denote such a set. Otherwise, choose K_1 whose associated λ_1 has the shortest segment at x (among all the K_i sets in $\mathcal{K}$). Similarly, if some K_i has no associated μ_i, let K_2 denote such a set. Otherwise, choose K_2 whose associated μ_2 has the shortest segment at x (among all K_i in $\mathcal{K}$). By hypothesis, $K_1 \cap K_2$ contains a 3-path from x to y, so at least one of K_1 or K_2 has an associated λ_1 or μ_2 path. Define the rectangular region A (possibly degenerate) as follows: If both λ_1 and μ_2 exist, let A denote the nondegenerate rectangular region at x determined by the first edge (at x) of λ_1 and the first edge (at x) of μ_2. Otherwise, exactly one of λ_1, μ_2 exists, and we let A be the degenerate rectangular region determined by the associated edge at x. It is easy to see that each set K contains A.

Let $B = \{y\}$. Since every 4 of the K sets contain a common staircase 3-path from x to y, for every 2 of the K sets, say K_i and $K_j, K_1 \cap K_2 \cap K_i \cap K_j$ contains a staircase 3-path from x to y. Such a path necessarily has its first segment entirely contained in A, producing a 2-path in $K_i \cap K_j$ from A to B. By Lemma 2, $\cap\{K : K \text{ in } \mathcal{K}\}$ contains a 2-path $\lambda_0 = [a, z] \cup [z, y]$ from A to B. Clearly λ_0 may be extended to a 3-path from x to y in $A \cup \lambda_0 \subseteq \cap\{K : K \text{ in } \mathcal{K}\}$. This 3-path satisfies the lemma, so $p(3) \leq 4$. Example 1 of the paper will show that the number 4 is best.

Finally, we adapt the argument above to the case for $n = 4$. Assume that every 6 members of $\mathcal{K}$ share a common staircase 4-path from x to y to show that $\cap\{K : K \text{ in } \mathcal{K}\}$ contains such a path. For each K_i, select a corresponding staircase 4-path λ_i (if it exists) from x to y in K_i such that its first segment is east and is as long as possible. Select a staircase 4-path μ_i (if it exists) from x to y in K_i such that its first segment is north and is as long as possible. Choose K_1, K_2 in the manner described previously to define rectangular region A at x. By a parallel argument, using 4-paths from y to x, select K_3 and K_4 to define rectangular region B at y. Every set K contains $A \cup B$, and every staircase 4-path from x to y in $K_1 \cap K_2 \cap K_3 \cap K_4$ has its first segment entirely contained in A, its last segment entirely contained in B. By hypothesis, every 6 members of $\mathcal{K}$ share a common staircase 4-path from x to y. Thus for every 2 of the sets, say K_i and $K_j, K_1 \cap K_2 \cap K_3 \cap K_4 \cap K_i \cap K_j$ contains a staircase 4-path from x to y. Since the first segment lies in A and the last segment lies in B, this produces a staircase 2-path from A to B in $K_i \cap K_j$. Since this is true for every K_i, K_j in $\mathcal{K}$, by Lemma 2, $\cap\{K : K \text{ in } \mathcal{K}\}$ contains a staircase 2-path λ_0 from A to B. Path λ_0 may be extended to a staircase 4-path from x to y in $A \cup \lambda_0 \cup B \subseteq \cap\{K : K \text{ in } \mathcal{K}\}$, and this 4-path satisfies the lemma. Thus $p(4) \leq 6$. Example 1 will show that the number 6 is best, finishing the proof.

Finally, we are ready to establish the following general result.

Theorem 1. *For $p(n)$ and $s(n)$ defined previously,*

$$p(1) = 1,$$
$$s(1) = s(2) = p(2) = 2,$$
$$p(3) = 4,$$
$$p(4) = 6. \quad \textit{Inductively,}$$
$$s(n) \leq 2p(n) \quad \textit{for } n \geq 3, \quad \textit{and}$$
$$p(n) \leq 4 + s(n-2) \quad \textit{for } n \geq 4.$$
$$\textit{Hence } p(n) \leq 4 + 2p(n-2) \; \textit{for } n \geq 5.$$

Proof. The proof is by induction on n. When $n = 1$, the result for points is trivial. The result for sets follows immediately from the proof of Lemma 2 and easy examples. When $n = 2$, the result follows from Lemma 2, and when $3 \leq n \leq 4$, the result for points follows from Lemma 3. Inductively, we assume that the result is true for points and n-paths when $3 \leq n \leq m$ and for sets and n-paths when $2 \leq n \leq m - 1$.

We will prove that the result holds for sets and n-paths when $n = m \geq 3$. Let $\mathcal{K}$ be a finite family of simply connected orthogonal polygons in the plane, and let A and B be rectangular regions (possibly degenerate) contained in $\cap\{K : K \text{ in } \mathcal{K}\}$. Assume that every $2\ p(m)$ members of $\mathcal{K}$ contain a common staircase m-path from A to B, to show that $\cap\{K : K \text{ in } \mathcal{K}\}$ contains such a path. If $A \cap B \neq \emptyset$, the result is trivial, so assume A and B are disjoint. Furthermore, for convenience we assume that each set A, B is fully two-dimensional, for otherwise a simplified version of the argument yields the result.

For each collection $\mathcal{C}$ of $p(m)$ sets from $\mathcal{K}$, let $\mathcal{P}_\mathcal{C}$ denote the collection of staircase m-paths λ in $\cap\{K : K \text{ in } \mathcal{C}\}$ from A to B, where $\lambda \cap A, \lambda \cap B$ are singleton sets.

Let $A'_\mathcal{C} = \{a : a \text{ in } \lambda \cap A \text{ for some } \lambda \text{ in } \mathcal{P}_\mathcal{C}\}, B'_\mathcal{C} = \{b : b \text{ in } \lambda \cap B \text{ for some } \lambda \text{ in } \mathcal{P}_\mathcal{C}\}$. Clearly the component of $\cap\{K : K \text{ in } \mathcal{C}\}$ which contains A and B is a simply connected orthogonal polygon. Hence by Lemma 1, each of the sets $A'_\mathcal{C}$ and $B'_\mathcal{C}$ is connected. Moreover, $A'_\mathcal{C}$ lies in a union of three edges of *bdry* A and may be labeled as an interval. A parallel statement holds for $B'_\mathcal{C}$.

Since every $2p(m)$ members of $\mathcal{K}$ share a common staircase m-path, every two of the $A'_\mathcal{C}$ sets have a nonempty intersection. By Helly's theorem on the real line, the intersection of all the $A'_\mathcal{C}$ sets is nonempty as well. Choose a_0 in this intersection. Using a parallel argument, choose b_0 in the intersection of all the $B'_\mathcal{C}$ sets. Then every $p(m)$ members of $\mathcal{K}$ share a common staircase m-path from a_0 to b_0. By our induction hypothesis for points and m-paths, $\cap\{K : K \text{ in } \mathcal{K}\}$ contains a staircase m-path from a_0 to b_0, and this provides a staircase m-path from A to B in $\cap\{K : K \text{ in } \mathcal{K}\}$. Hence the result is true for sets and m-paths. That is, $s(m) \leq 2p(m)$.

Finally, we will show that the result holds for points and n-paths when $n = m + 1 \geq 4$. Again let $\mathcal{K}$ be a finite family of simply connected orthogonal polygons in the plane, and let x, y belong to $\cap\{K : K \text{ in } \mathcal{K}\}$. Without loss of generality assume that y is northeast of x. Assume that every $4 + s(m - 1)$ members of $\mathcal{K}$ share a common staircase $(m + 1)$-path from x to y, to show that $\cap\{K : K \text{ in } \mathcal{K}\}$ contains such a path. Using an argument like the one in Lemma 3, for each K_i select a corresponding staircase $(m + 1)$-path λ (if it exists) from x to y in K_i such that the first segment is east and is as long as possible among all such paths in K_i. Similarly, select a staircase $(m+1)$-path μ_i (if it exists) from x to y whose first segment is north and is as long as possible. Choose K_1, K_2 in the manner discussed in Lemma 3 to define rectangular region A. By a parallel argument, select K_3, K_4 to define rectangular region B.

Sets A, B lie in $\cap\{K : K \text{ in } \mathcal{K}\}$, and every $(m+1)$-path from x to y in $K_1 \cap K_2 \cap K_3 \cap K_4$ has its first segment entirely in A, its last segment entirely in B. Since every $4+s(m-1)$ sets in $\mathcal{K}$ contain a common staircase $(m+1)$-path from x to y, every $s(m-1)$ members of $\mathcal{K}$ share with $K_1 \cap K_2 \cap K_3 \cap K_4$ such a path, producing a common staircase $(m-1)$-path from A to B. By our induction hypothesis for sets and $(m-1)$-paths, all sets in $\mathcal{K}$ share a staircase $(m-1)$-path λ from A to B. Clearly λ may be extended to a staircase $(m+1)$-path from x to y in $A \cup \lambda \cup B \subseteq \cap\{K : K \text{ in } \mathcal{K}\}$. Hence the result is true for points and $(m+1)$-paths.

By induction, the results are true for all $n \geq 1$.

The following example establishes lower bounds for $p(n)$ and $s(n), n \geq 2$.

Example 1. For $i \geq 1$, let C_i denote the square region in the plane having (diagonal) vertices (i,i) and $(i+1, i+1)$. (See Figure 2.) For $n \geq 2$, let $C = \cup\{C_i : 1 \leq i \leq n\}$, and let $D_i, D_{i+(n-1)}$ denote the two square regions which share one edge with C_i and one edge with $C_{i+1}, 1 \leq i \leq n$. Finally, for $1 \leq i \leq 2(n-1)$, define $K_i = \cup\{C \cup D_j : 1 \leq j \leq 2(n-1), j \neq i\}$.

It is easy to see that every $2(n-1)-1 = 2n-3$ of the K_i sets share a staircase n-path from $x = (1,1)$ to $y = (n+1, n+1)$. However, $\cap\{K_i : 1 \leq i \leq 2(n-1)\}$ contains no such path. Hence $2(n-1) \leq p(n)$ and, of course, $2(n-1) \leq s(n)$ as well.

Observe that $2 \leq p(2) \leq s(2), 4 \leq p(3)$, and $6 \leq p(4)$, producing the exact values for $p(2)$ and $s(2)$ in Lemma 2 and the exact values for $p(3)$ and $p(4)$ in Lemma 3.

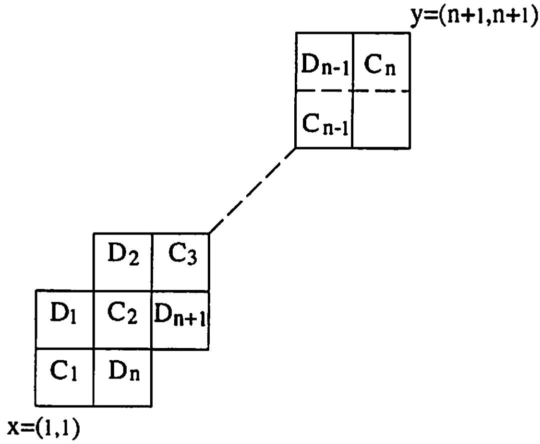

Figure 2

The following lemma will help us to establish a result for intersections of sets starshaped via staircase n-paths.

Lemma 4. *Let $\mathcal{K}$ be a finite family of simply connected orthogonal polygons in the plane, and let j, n be fixed natural numbers. Assume that every $3j$ (not necessarily distinct) members of $\mathcal{K}$ have a nonempty intersection which is starshaped via staircase n-paths. Then there exists some point z_0 in $\cap\{K : K \text{ in } \mathcal{K}\}$ such that the following is true: For every point s in $\cap\{K : K \text{ in } \mathcal{K}\}$ and for every j sets $K_1, \ldots, K_j$ in $\mathcal{K}, z_0$ sees s via staircase $(n+1)$-paths in $K_1 \cap \ldots \cap K_j$.*

Proof. By [2, Theorem 1], the set $S \equiv \cap\{K : K \text{ in } \mathcal{K}\}$ is a (nonempty) simply connected orthogonal polygon which is starshaped via staircase paths.

Adapting an approach in Bobylev [1] and in [2], for each j members $K_1, \ldots, K_j$ of $\mathcal{K}$, define $M'(1, \ldots, j) = \{x : x \text{ sees } S \text{ via staircase } n\text{-paths in } K_1 \cap \ldots \cap K_j\}$. Because the sets $M'(1, \ldots, j)$ need not be well behaved, we augment them as follows: For x, y in $M'(1, \ldots, j)$, if an associated rectangular region with vertices x and y lies in $K_1 \cap \ldots \cap K_j$, join the region to $M'(1, \ldots, j)$. That is, let $M(1, \ldots, j) = M'(1, \ldots j) \cup \{z : z \text{ lies in a minimal rectangular region in } K_1 \cap \ldots \cap K_j \text{ having two associated vertices in } M'(1, \ldots, j)\} \subseteq K_1 \ldots \cap K_j$. Let $\mathcal{M}$ denote the collection of all the sets $M(1, \ldots, j)$. We will show that each set M is $\mathcal{M}$ is simply connected and compact, that every two of these sets have a connected intersection, and that every three have a nonempty intersection. Then by Molnár's theorem [11] it will follow that $\cap\{M : M \text{ in } \mathcal{M}\}$ is nonempty and simply connected. For z_0 in this intersection, we will prove that z_0 satisfies the lemma.

To see that each set $M(1,\ldots j)$ in $\mathcal{M}$ is simply connected, let δ be a simple closed curve in $M(1,\ldots,j)$, and let point p belong to the (open) region bounded by δ. We will show that $p \in M(1,\ldots,j)$. Choose points q, r on δ such that (q,r) is a horizontal segment at p lying in the region bounded by δ. Then q, r lie in (possibly degenerate) rectangular regions Q, R in $K_1 \cap \ldots \cap K_j$ with appropriate vertices q_1, q_2 and r_1, r_2 for Q and R, respectively, in $M'(1,\ldots,j)$. If $p \in Q \cup R$, the argument is finished. Otherwise, one of Q or R, say Q, is west of p, while the other rectangle R is east of p. (See Figure 3.) Fix point s in S. By a result of Topală [14, Proposition 2], there is a staircase 2-path λ_q in $K_1 \cap \ldots \cap K_j$ from q_1 to q_2 such that s sees each point of λ_q via staircase n-paths in $K_1 \cap \ldots \cap K_j$. Similarly, there is a staircase 2-path λ_r in $K_1 \cap \ldots \cap K_j$ such that s sees each point of λ_r via staircase n-paths in $K_1 \cap \ldots \cap K_j$. Choose $a_q \in \lambda_q \cap L(q,r)$ and $a_r \in \lambda_r \cap L(q,r)$. Clearly $p \in [a_q, a_r]$. Also, δ lies in the simply connected set $K_1 \cap, \ldots, \cap K_j$, so $[a_q, a_r]$ does, too. By [14, Proposition 1], s sees via staircase n-paths in $K_1 \cap \ldots \cap K_j$ each point of $[a_q, a_r]$. Hence s sees p via staircase n-paths in $K_1 \cap \ldots \cap K_j$. Since this is true for every s in $S, p \in M'(1,\ldots,j) \subseteq M(1,\ldots,j)$, the desired result.

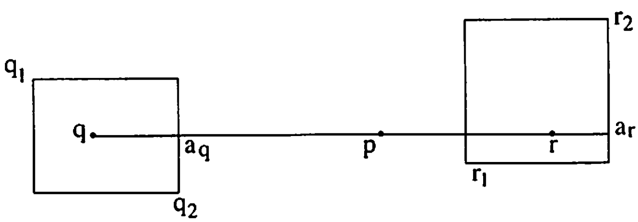

Figure 3

It is not hard to see that $M(1,\ldots,j)$ is compact. Using arguments like those in [6, Lemma 1] and [5, Theorem 1], for each point s in S, the set $\{x : x \text{ sees } s \text{ via staircase } n\text{-paths in } K_1 \cap \ldots \cap K_j\}$ is a finite union of closed rectangular regions, hence compact. Set $M'(1,\ldots,j)$, as an intersection of these compact sets, is compact as well. We join $M'(1,\ldots,j)$ to a finite union of rectangular regions to produce $M(1,\ldots,j)$, so $M(1,\ldots,j)$ is compact, too.

It is easy to see that every three of the sets in $\mathcal{M}$ have a nonempty intersection. For convenience of notation, let M_1, M_2, M_3 denote any three of these sets. Since every $3j$ members of $\mathcal{K}$ have an intersection which is starshaped via staircase n-paths, the members of $\mathcal{K}$ associated with M_1, M_2, M_3 have this property. For x in the staircase n-kernel of their intersection, x sees S (in fact, x sees all points of the intersection) via staircase n-paths in $\cap\{K : K \text{ associated with } M_1, M_2, \text{ or } M_3\}$. Hence x belongs to $M_1' \cap M_2' \cap M_3' \subseteq M_1 \cap M_2 \cap M_3$, so $M_1 \cap M_2 \cap M_3$ is nonempty.

It remains to show that every two of the M sets have a connected intersection. We will show that for every two of the M sets, say M_1 and M_2, $M_1 \cap M_2$ is starshaped. Select z in the staircase n-kernel of $\cap\{K : K$ associated with M_1 or $M_2\}$. By the argument above, $z \in M_1' \cap M_2'$. We will show that z is in the staircase kernel (in fact, in the staircase n-kernel) of $M_1 \cap M_2$. Let point p belong to $M_1 \cap M_2$ to show that z sees p via staircase n-paths in $M_1 \cap M_2$. Since z is in the staircase n-kernel of $\cap\{K : K$ associated with M_1 or $M_2\}$ and $p \in M_1 \cap M_2 \subseteq \cap\{K : K$ associated with M_1 or $M_2\}$, z sees p via a staircase n-path λ in $\cap\{K : K$ associated with M_1 or $M_2\}$. Without loss of generality, assume that λ is chosen with a minimal number m of segments, $m \leq n$. We must show that $\lambda \subseteq M_1 \cap M_2$.

The following proposition will be useful

Proposition 2. Let x and y be points in $M' = M'(1,\ldots,j)$ joined by a staircase path μ in $K_1 \cap \ldots \cap K_j$. Then $\mu \subseteq M$.

Proof. First consider the case in which x and y are joined by a staircase 2-path δ in $K_1 \cap \ldots \cap K_j$. If δ is a segment, then $\delta = \mu$. By [14, Proposition 1], for each s in S, s sees each point of δ by a staircase n-path in $K_1 \cap \ldots \cap K_j$, so $\delta \subseteq M'$. Thus $\mu = \delta \subseteq M' \subseteq M$. If δ is not a segment and is the only $x - y$ staircase 2-path in $K_1 \cap \ldots \cap K_j$, then by [14, Theorem 1], each s in S sees δ via staircase n-paths in $K_1 \cap \ldots \cap K_j$, and again $\delta \subseteq M'$. It is easy to see that each point of μ lies in a minimal rectangular region in $K_1 \cap \ldots \cap K_j$ determined by points in $\delta \subseteq M'$, so the region bounded by $\mu \cup \delta$ lies in M, and $\mu \subseteq M$. If δ' is a second $x - y$ 2-path in $K_1 \cap \ldots \cap K_j$, then the associated rectangular region bounded by $\delta \cup \delta'$ lies in $K_1 \cap \ldots \cap K_j$. Since x and y are in M', this rectangular region lies in M, and again $\mu \subseteq M$.

To finish the proof, we proceed by induction on the length of μ. In case μ is a staircase 2-path (or a staircase 1-path) the argument is finished by comments above. Assume that μ is a staircase k-path for $k \geq 3$. In case x and y are joined by a staircase 2-path in $K_1 \cap \ldots \cap K_j$, again the argument is finished by the argument above. Hence assume that points x and y are not joined by a staircase 2-path in $K_1 \cap \ldots \cap K_j$. If $k = 3$, then k is minimal and, by [14, Proposition 3], each point of S sees each point of μ by staircase n-paths in $K_1 \cap \ldots \cap K_j$. Then $\mu \subseteq M' \subseteq M$.

Inductively, assume that the result is true for all $k, 3 \leq k < i$, to prove for i. If i is minimal, again we are through by [14, Proposition 3]. Otherwise, choose an $x - y$ staircase path μ' in $K_1 \cap \ldots \cap K_j$ having a minimal number of segments. By [14, Proposition 3], all points of S see all points of μ' via staircase n-paths in $K_1 \cap \ldots \cap K_j$, so $\mu' \subseteq M'$. Let $x = x_0, x_1, \ldots, x_i = y$ be the vertices of μ, $i \geq 4$. In case μ' meets $\mu(x_1, x_{i-1})$ at some w, then each of $\mu(x, w)$ and $\mu(w, y)$ has fewer than i segments and has its endpoints in M'. By our induction hypothesis, $\mu = \mu(x, w) \cup \mu(w, y) \subseteq M$, the desired result. Otherwise, μ' meets μ only in $[x, x_1) \cup (x_{i-1}, x_i]$. Without loss of generality, assume that y is northeast of x and μ' is southeast of μ. (See Figure 4.) Then the edge $[x_2, x_3]$ is vertical. The associated line meets μ' at some point w'. Path $[w', x_3] \cup \mu(x_3, y)$ has two fewer segments than μ, and it is a $w - y$ staircase path in $K_1 \cap \ldots \cap K_j$ whose endpoints are in M'. Using our induction hypothesis, all of its points are in M, and $\mu(x_2, y) \subseteq M$. Similarly, the edge $[x_1, x_2]$ is vertical, and the associated line meets μ' at some w''. Path $\mu(x_0, x_2) \cup [x_2, w'']$ has two segments, lies in $K_1 \cap \ldots \cap K_j$, and has its endpoints in M'. Again by our induction hypothesis, all points of the path are in M. Thus we conclude that $\mu = \mu(x_0, x_2) \cup \mu(x_2, y) \subseteq M$. This finishes the induction and completes the proof of the proposition.

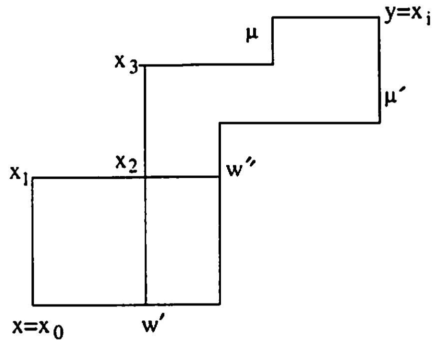

Figure 4.

We will use Proposition 1 to show that $\lambda \subseteq M_1 \cap M_2$. Recall that z is in the staircase n-kernel of $\cap\{K : K$ associated with M_1 or $M_2\}, z \in M_1' \cap M_2'$, and p belongs to $M_1 \cap M_2 \subseteq \cap\{K : K$ associated with M_1 or $M_2\}$. Also, z sees p via the staircase m-path λ in $\cap\{K : K$ associated with M_1 or $M_2\}$, and m is minimal, $m \leq n$. First we will show that $\lambda \subseteq M_1$. For convenience, let $M_1 = M(1, \ldots, j)$. Assume that p lies in the (possibly degenerate) region P in $M_1 \subseteq K_1 \cap \ldots \cap K_j$ with appropriate vertices p_1, p_2 of P in M_1'. Also for convenience, we assume that P is fully two-dimensional, for otherwise, a simplified version of the same argument finishes the proof. Notice that exactly one of p_1 or p_2 lies along each edge of P. (See Figure 5.) If $z \in P$, then $\lambda \subseteq P \subseteq M_1$, the desired result. Otherwise, $z \notin P$. If necessary, extend the east segment of λ to create staircase path $\lambda_0(z, w)$ from point z to point w in *bdry* P, with $\lambda \subseteq \lambda_0(z, w)$. Let v be the first point of $\lambda(z, p) = \lambda_0(z, p)$ in P. It is easy to see that for one of p_1 or p_2, say p_1, and for one of v or w, say $v, \lambda' \equiv \lambda_0(z, v) \cup [v, p_1]$ is a staircase path in $K_1 \cap \ldots \cap K_j$ from z to p_1. Since z and p_1 belong to M_1', by the proposition, the staircase path λ' lies in M_1. Since all points of λ are in $\lambda' \cup P \subseteq M_1, \lambda \subseteq M_1$ also.

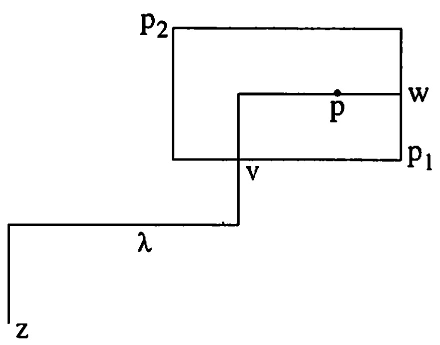

Figure 5.

A parallel argument holds for M_2, so $\lambda \subseteq M_1 \cap M_2$. Hence z sees each point p of $M_1 \cap M_2$ via staircase n-paths in $M_1 \cap M_2$, and z is in the staircase n-kernel of $M_1 \cap M_2$. We conclude that every two members M_1, M_2 of $\mathcal{M}$ have a connected intersection, the desired result.

Thus the family of sets $\mathcal{M}$ satisfies the hypothesis of Molnár's theorem, and by that theorem it follows that $\cap\{M : M$ in $\mathcal{M}\}$ is nonempty. For z_0 in this intersection, we assert that z_0 satisfies the lemma.

We will show that for every point s in $\cap\{K : K \text{ in } \mathcal{K}\}$ and for every j sets $K_1, \ldots, K_j$ in $\mathcal{K}$, z_0 sees s via staircase $(n+1)$-paths in $K_1 \cap \ldots \cap K_j$: If $z_0 \in M'(1, \ldots, j)$, the result is immediate. If $z_0 \notin M'(1, \ldots, j)$, then still $z_0 \in M(1, \ldots, j)$, so z_0 lies in a minimal (and possibly degenerate) rectangular region R in $K_1 \cap \ldots \cap K_j$ with appropriate vertices r_1, r_2 in $M'(1, \ldots, j)$. Let μ_1, μ_2 be staircase n-paths in $K_1 \cap \ldots \cap K_j$ from s to r_1, from s to r_2, respectively. Observe that the region R' determined by $R \cup \mu_1 \cup \mu_2$ lies in the simply connected set $K_1 \cap \ldots \cap K_j$. Moreover, by [6, Lemma 2], R' is orthogonally convex. It is easy to see that at least one of μ_1, μ_2, say μ_1, contains a point w on a horizontal or vertical segment at z_0. We assume that w is the first such point of μ_1 relative to the order on μ_1 from s to r_1. (See Figure 6.) Then $\mu_1(s, w) \cup [w, z_0]$ is a staircase path. Furthermore, since R' is orthogonally convex, $[w, z_0] \subseteq R' \subseteq K_1 \cap \ldots \cap K_j$. If w belongs to R, then $\mu_1(s, w) \cup [w, z_0]$ has at most $n + 1$ segments. Otherwise, $\mu_1(s, w) \cup [w, z_0]$ has at most n segments. Either way, $\mu_1(s, w) \cup [w, z_0]$ is a staircase $(n + 1)$-path in $K_1 \cap \ldots \cap K_j$, satisfying the assertion and completing the proof of the lemma.

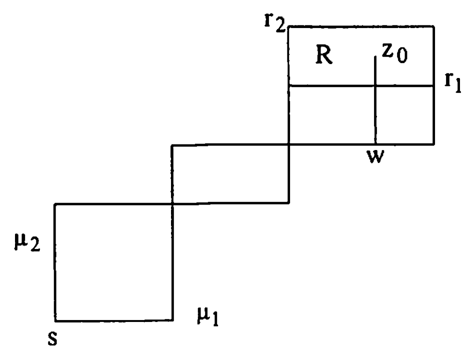

Figure 6.

We are ready to establish the following theorem for intersections of orthogonally starshaped sets.

Theorem 3. *Let $\mathcal{K}$ be a finite family of simply connected orthogonal polygons in the plane, and let $n \geq 1$. If every $3p(n + 1)$ members of $\mathcal{K}$ have a nonempty intersection which is starshaped via staircase n-paths then $\cap\{K : K \text{ in } \mathcal{K}\}$ is nonempty and is starshaped via staircase $(n + 1)$-paths.*

Proof. By Lemma 4, select point z_0 in $\cap\{K : K \text{ in } \mathcal{K}\}$ such that the following is true: For every point s in $\cap\{K : K \text{ in } \mathcal{K}\}$ and for every $p = p(n+1)$ sets $K_1, \ldots, K_p$ in $\mathcal{K}$, z_0 sees s via staircase $(n+1)$-paths in $K_1 \cap \ldots \cap K_p$. Thus every $p(n+1)$ members of $\mathcal{K}$ contain a common staircase $(n+1)$-path from z_0 to s. By definition of $p(n+1)$, it follows that $\cap\{K : K \text{ in } \mathcal{K}\}$ contains such a path. Since this holds for every s in $\cap\{K : K \text{ in } \mathcal{K}\}, z_0$ is in the staircase $(n+1)$-kernel of $\cap\{K : K \text{ in } \mathcal{K}\}$ and $\cap\{K : K \text{ in } \mathcal{K}\}$ is starshaped via staircase $(n+1)$-paths, finishing the proof.

We conclude with the observation that the bound $3p(n+1)$ in the theorem may not be best. In particular, it certainly is not best for $n = 1$, as the following result demonstrates.

Theorem 4. *Let $\mathcal{K}$ be a family of simply connected orthogonal polygons in the plane. If every 3 (not necessarily distinct) members of $\mathcal{K}$ have a nonempty intersection and every 2 (not necessarily distinct) members of $\mathcal{K}$ have an intersection which is starshaped via staircase 1-paths, then $S = \cap\{K : K \text{ in } \mathcal{K}\}$ is nonempty and starshaped via staircase 1-paths. The result is best possible.*

Proof. By Molnár's theorem [11], $\cap\{K : K \text{ in } \mathcal{K}\} \neq \emptyset$. If every set K is a segment, the result follows immediately from Helly's theorem on the real line. Hence assume that some set K_1 is not a segment. It follows that K_1 has a one point kernel $\{p\}$ and consists of two nondegenerate perpendicular segments at p. Let L, M denote the corresponding perpendicular lines at p. Then $\cap\{K : K \text{ in } \mathcal{K}\} \subseteq L \cup M$. Clearly each K_i is orthogonally convex, so $\cap\{K : K \text{ in } \mathcal{K}\}$ is, too. If $S \subseteq L$ or $S \subseteq M$, then S is a convex segment, finishing the proof. Otherwise, for each $K_i, K_i \cap K_1$ contains a point x_i of $L \backslash M$ and a point y_i of $M \backslash L$. Since $K_i \cap K_1$ is starshaped via staircase 1-paths, it has a one point kernel $\{p_i\}$. Moreover, $p_i \in K_1 \subseteq M \cup L$, so $p_i = p$. We conclude that $p \in S$ and $p \in ker S$, finishing the argument.

It is easy to see that the result in Theorem 3 is best possible.

Example 2. Let v_0, v_1, v_2, v_3 denote the four vertices of a square, ordered in a clockwise direction. As in [2, Example 1], sets $[v_0, v_1], [v_1, v_2], [v_2, v_3] \cup [v_3, v_0]$ show that the number 3 in Theorem 1 is best. Sets $[v_0, v_1] \cup [v_1, v_2]$ and $[v_2, v_3] \cup [v_3, v_0]$ show that the number 2 is best as well.

REFERENCES

[1] N. A. Bobylev, *The Helly theorem for starshaped sets*, Journal of Mathematical Sciences **105** (2001) 1819-1825.

[2] Marilyn Breen, *A Helly theorem for intersections of orthogonally starshaped sets*, Archiv der Mathematik (to appear).

[3] ———, *A Krasnosel'skii theorem for staircase paths in orthogonal polygons*, Journal of Geometry **51** (1994), 22-30.

[4] ———, *An improved Krasnosel'skii theorem for orthogonal polygons which are starshaped via staircase paths*, Journal of Geometry **51** (1994), 31-35.

[5] ———, *Staircase k-kernels for orthogonal polygons*, Archiv der Mathematik **63** (1994), 182-190.

[6] ———, *Staircase kernels in orthogonal polygons*, Archiv der Mathematik **59** (1992), 588-594.

[7] Victor Chepoi, *On staircase starshapedness in rectilinear spaces*, Geometriae Dedicata **63** (1996), 321-329.

[8] Ludwig Danzer, Branko Grünbaum, Victor Klee, *Helly's theorem and its relatives*, Convexity, Proc. Sympos. Pure Math. **7** (1962), Amer. Math. Soc., Providence, RI, 101-180.

[9] Jürgen Eckhoff, *Helly, Radon, and Carathéodory type theorems*, Handbook of Convex Geometry vol A, ed. P.M. Gruber and J.M. Wills, North Holland, New York (1993), 389-448.

[10] Steven R. Lay, *Convex Sets and Their Applications*, John Wiley, New York, 1982.

[11] J. Molnár, *Über den zweidimensionalen topologischen Satz von Helly* (in Hungarian, Russian, and German summary), Mat. Lapok **8** (1957), 108-114.

[12] Rajeev Motwani, Arvind Raghunathan, and Huzur Saran, *Covering orthogonal polygons with star polygons: the perfect graph approach*, Journal of Computer and System Sciences **40** (1990), 19-48.

[13] Oleg Topală, *Intersections of maximal orthogonally starshaped polygons*, Journal of Geometry **70** (2001), 164-167.

[14] ———, *On intersections of maximal orthogonally k-starshaped polygons*, Geometriae Dedicata **78** (1999), 271-278.

[15] F.A. Valentine, *Convex Sets*, McGraw-Hill, New York (1964).