On Some Problems of A.K. Agarwal

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Abstract

The object of this paper is to give solutions to some of the problems suggested by A.K. Agarwal [n-color Analogues of Gaussian Polynomials, Ars Combinatoria 61 (2001), 97-117].

1 Introduction

Definition 1. An n-color partition is a partition in which a part of size n, $n \geq 1$ can come in n different colors which are denoted by $n_1, n_2, \ldots, n_n$. Let $P(\nu)$ denote the number of n-color partitions of n. Then it was proved [1] that

$$\sum_{\nu=0}^{\infty} P(\nu)q^{\nu} = \prod_{n=1}^{\infty}(1-q^n)^{-n}.$$

In [2] A.K. Agarwal has studied the following restricted n-color partition functions and obtained the following results. Let $\pi = (a_1)_{b_1} + \ldots + (a_m)_{b_m}$ be an n-color partition.

Definition 2. Let $p_1(r,k,m,\nu)$ denote the number of n-color partitions of ν into exactly m parts such that each subscript $b_1 \leq r$ and each part $a_i \leq k + b_i - 1$.

Definition 3. Let $p_2(r,k,m,\nu)$ denote the number of n-color partitions of ν into exactly m parts such that each subscript $b_i \leq r$ and each part $a_i \leq k$.

Theorem 1. [2]

$$\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} p_1(r,k,m,\nu) z^m q^\nu = \prod_{j=1}^{k} \frac{1}{(zq^j;q)_r}$$

Theorem 2. [2]

$$\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} p_2(r,k,m,\nu) z^m q^\nu = \prod_{\nu=1}^{k} (1-zq^\nu)^{-min(r,\nu)}$$

Agarwal [2] has given the following two different n-color analogues of the well-known Gaussian polynomials [3, Def. 3.1].

$$A_1(r,k,m,\nu) = \sum_{\nu=0}^{\infty} p_1(r,k,m,\nu) q^\nu$$
$$A_2(r,k,m,\nu) = \sum_{\nu=0}^{\infty} p_2(r,k,m,\nu) q^\nu$$

and obtained some properties of $A_1(r,k,m,\nu)$ and $A_2(r,k,m,\nu)$ analogues to those of Gaussian polynomials.

Agarwal [2] has also proved the following Theorems using generating functions.

Theorem 3. Let $P(D,\nu)$ denote the number of n-color partitions of ν into distinct parts. Let $B(\nu)$ denote the number of n-color partitions of ν such that even parts appear with even subscripts only. Then $P(D,\nu) = B(\nu)$ for all ν.

Theorem 4. Let $R(\nu)$ denote the number of strict plane partitions of ν. Let $Q(\nu)$ denote the number of n-color partitions of ν in which even parts appear with even subscripts and odd with odd subscripts. Then $R(\nu) = Q(\nu)$ for all ν.

The following questions were posed in [2].

Problem 1. Is it possible to find explicit expressions (in terms of q only) for $A_1(r,k,m,\nu)$ and $A_2(r,k,m,\nu)$?

Problem 2. We know that $\lim_{q\to 1} \left[{r \atop k}\right]$ is the binomial coefficient $\binom{r}{k}$. Do $A_1(r,k,m,1)$ and $A_1(r,k,m,1)$ have interpretations other than partition theoretic?

Problem 3. Is it possible to prove Theorems 3 and 4 combinatorially?

In this paper we give solutions to Problems 1, 2 and give a combinatorial proof of Theorem 3.

2 Solutions

Solutions of Problems 1 and 2. We know that

$$\sum_{m=0}^{\infty} A_1(r,k,m,q)z^m = \prod_{j=1}^{k} \frac{1}{(zq^j;q)_r}.$$

But

$$\frac{1}{(z;q)_r} = \sum_{j=0}^{\infty} \begin{bmatrix} r+j-1 \\ j \end{bmatrix} z^j. \qquad [3, \text{Eq. } (3.3.7)]$$

Hence

$$\sum_{m=0}^{\infty} A_1(r,k,m,q)z^m = \prod_{j=1}^{k} \sum_{i=0}^{\infty} \begin{bmatrix} r+i-1 \\ i \end{bmatrix} (zq^j)^i$$

$$= \sum_{i_1,\ldots,i_k=0}^{\infty} \begin{bmatrix} r+i_1-1 \\ i_1 \end{bmatrix} \cdots \begin{bmatrix} r+i_k-1 \\ i_k \end{bmatrix}$$

$$q^{i_1+2i_2+\ldots+ki_k}\ z^{i_1+\ldots+i_k}.$$

Equating the coefficients of z^m on both sides, we get

$$A_1(r,k,m,q) = \sum_{i_1+\ldots+i_k=m} \begin{bmatrix} r+i_1-1 \\ i_1 \end{bmatrix} \cdots \begin{bmatrix} r+i_k-1 \\ i_k \end{bmatrix}$$

$$q^{i_1+2i_2+\ldots+ki_k}$$

which is an explicit expression for $A_1(r,k,m,q)$ in terms of q only.

$$\lim_{q\to 1} A_1(r,k,m,q) = \sum_{i_1+\ldots+i_k=m} \binom{r+i_1-1}{i_1} \cdots \binom{r+i_k-1}{i_k}$$

since the Gaussian polynomial $\begin{bmatrix} n \\ m \end{bmatrix}$ tends to be the binomial coefficient $\binom{n}{m}$.

To obtain an explicit expression of $A_2(r,k,m,q)$ in terms of q only, we distinquish two cases.

Case 1. Let $r \le k$. Then

$$min(r,\nu) = \begin{cases} \nu & \text{for } \nu = 1,\ldots,r-1 \\ r & \text{for } \nu = r,\ldots,k. \end{cases}$$

We know that

$$\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} P_2(r,k,m,\nu)z^m q^\nu = \prod_{\nu=1}^{k} (1-zq^\nu)^{-min(r,\nu)}.$$

$$= (1-zq)^{-1}(1-zq^2)^{-2}\dots(1-zq^{r-1})^{-(r-1)}(1-zq^r)^{-r}\dots(1-zq^k)^{-r}$$
$$= (1+zq+z^2q^2+\dots)(1+zq^2+z^2q^4+\dots)^2\dots$$
$$(1+zq^{r-1}+z^2q^{2r-2}+\dots)^{r-1}(1+zq^r+z^2q^{2r}+\dots)^r\dots$$
$$(1+zq^k+z^2q^{2k}+\dots)^r$$
$$= \sum_{x_1=0}^{\infty} z^{\sum_{\alpha_j=1}^{j}\sum_{j=1}^{r-1} i_{j\alpha_j}} q^{\sum_{\alpha_j=1}^{j}\sum_{j=1}^{r-1} j\, i_{j\alpha_j}}$$
$$\sum_{x_2=0}^{\infty} z^{\sum_{\beta_l=1}^{l}\sum_{l=r}^{k} i_{l\beta_l}} q^{\sum_{\beta_l=1}^{l}\sum_{l=r}^{k} l\, i_{l\beta_l}}$$

where

$$x_1 = i_1, i_{21}, i_{22}, \dots, i_{r-11}, \dots, i_{r-1r-1}$$

and

$$x_2 = i_{r1}, \dots, i_{rr}, \dots, i_{k1}, \dots, i_{kr}.$$

Equating the coefficients of z^m on both sides and observing that $A_2(r,k,m,q) = \sum_{\nu=0}^{\infty} p_2(r,k,m,\nu)q^\nu$, we obtain for $r \le k$,

$$A_2(r,k,m,q) = \sum_{x_3=m} q^{j\, i_{j\alpha_j} + l\, i_{l\beta_l}}$$

where

$$x_3 = \sum_{\alpha_j=1}^{j}\sum_{j=1}^{r-1} i_{j\alpha_j} + \sum_{\beta_l=1}^{l}\sum_{l=r}^{k} i_{l\beta_l}.$$

$$\lim_{q\to 1} A_2(r,k,m,q) = \text{Number of solutions of } x_3 = m$$

Case 2. Let $r > k$. In this case

$$min(r,\nu) = \nu \text{ for all } 1 \le \nu \le k.$$

Hence

$$\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} p_2(r,k,m,\nu)z^m q^\nu = \prod_{\nu=1}^{k}(1-zq^\nu)^{-min(r,\nu)}$$
$$= (1-zq)^{-1}(1-zq^2)^{-2}\dots(1-zq^k)^{-k}$$
$$= \sum_{i_{a_j}=0}\sum_{a=1}^{k}\sum_{j=1}^{a} z^{i_{a_j}} q^{a i_{a_j}}$$

Equating the coefficients of z^m on both sides, we obtain for $r > k$,

$$A_2(r,m,k,q) = \sum_{y=m} q^{a i_{a_j}}$$

where

$$y = \sum_{a=1}^{k} \sum_{j=1}^{a} i_{a_j}.$$

Hence

$$\lim_{q \to 1} A_2(r,k,m,q) = \text{ Number of solutions of } y = m.$$

Combinatorial Proof of Theorem 3. Given an n-color partition of ν into distinct parts, replace each even part with odd subscript into two parts according to the following rule.

$$(2k)_{2l-1} = k_l + k_l$$

Repeat this process of splitting when k is even and l is odd till no even parts with odd subscripts are left.

eg. $8_1 \to 4_1 + 4_1 \to (2_1 + 2_1) + (2_1 + 2_1) \to 1_1 + \ldots + 1_1$

$8_5 \to 4_3 + 4_3 \to 2_2 + 2_2 + 2_2 + 2_2$.

Finally arrange the parts in decreasing order. This will be an n-color partition of ν in which even parts appear with only even subscripts.

Conversely, given an n-color partition of ν in which even parts appear with only even subscripts, add two repeated parts according to the following rule.

$$(n)_m + (n)_m = (2n)_{(2m-1)} \tag{1}$$

eg. $3_2 + 3_2 \to 6_3$, $\quad 4_3 + 4_3 \to 8_5$.

Repeat the above process of addition till there is no repetition of parts. Finally arrange the parts in decreasing order. The resulting partition is the required partition enumerated by $P(D,\nu)$.

We now illustrate our proof by an example.

$(24)_{15} + (19)_{10} + (18)_9 + 8_5 + 6_3 + 5_4 + 2_1$

$\to (12)_8 + (12)_8 + (19)_{10} + 9_5 + 9_5 + 4_3 + 4_3 + 3_2 + 3_2 + 5_4 + 1_1 + 1_1$

$\to (12)_8 + (12)_8 + (19)_{10} + 9_5 + 9_5 + 2_2 + 2_2 + 2_2 + 2_2 + 3_2 + 3_2 + 5_4 + 1_1 + 1_1$

$\to (19)_{10} + (12)_8 + (12)_8 + 9_5 + 9_5 + 5_4 + 3_2 + 3_2 + 2_2 + 2_2 + 2_2 + 2_2 + 1_1 + 1_1$.

Conversely, the last partition under the reverse map, after using (1) goes to

$$
\begin{aligned}
&(19)_{10} + (24)_{15} + (18)_9 + 5_4 + 6_3 + 4_3 + 4_3 + 2_1 \\
&\rightarrow (19)_{10} + (24)_{15} + (18)_9 + 5_4 + 6_3 + 8_5 + 2_1 \\
&\rightarrow (24)_{15} + (19)_{10} + (18)_9 + 8_5 + 6_3 + 5_4 + 2_1.
\end{aligned}
$$

References

[1] A.K. Agarwal and G.E. Andrews, Rogers-Ramanujan Identities for Partitions with N copies of N, *J. Combin. Theory Ser. A* **45(1)** (1987), 40–49.

[2] A.K. Agarwal, n-color Analogues of Gaussian Polynomials, *Ars Combinatoria* **61** (2001), 97–117.

[3] G.E. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and Its Applications, Vol. 2, Reading, MA, 1976 (Reprinted, Cambridge University Press, London, New York, 1984).