On (g, f, n)-Critical Graphs

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Abstract

Let G be a graph, and let g and f be two integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for all $x \in V(G)$. A graph G is called a (g, f, n)-critical graph if $G - N$ has a (g, f)-factor for each $N \subseteq V(G)$ with $|N| = n$. In this paper, a necessary and sufficient condition for a graph to be (g, f, n)-critical is given. Furthermore, the properties of (g, f, n)-critical graph are studied.

Keywords: (g, f)-factor, factor-critical graph

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1 Introduction

All graphs considered in this paper are finite, undirected, and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G)$, an induced subgraph of G by S is denoted by $G[S]$ and $G - S = G[V(G) \setminus S]$. For any vertex x of G, the degree of x in G is denoted by $d_G(x)$, and the set of vertices adjacent to x in G is denoted by $N_G(x)$, and the order of G by $|G|$. Furthermore. $\delta(G) = \min\{d_G(x) : x \in V(G)\}$ and $N_G(S) = \sum_{x \in S} N_G(x)$ for $S \subseteq V(G)$.

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Let g and f be two integer-valued functions defined on $V(G)$. A (g, f)-*factor* of G is defined as a spanning subgraph F of G such that $g(x) \leq d_F(x) \leq f(x)$ for each $x \in V(G)$. If $g(x) = f(x)$ for all $x \in V(G)$, then a (g, f)-factor is called an f-*factor*. For an integer $k \geq 1$, an f-factor is a k-*factor* if $f(x) = k$ for all $x \in V(G)$. For two integers $a \leq b$, a (g, f)-factor is called an $[a, b]$-*factor* if $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$.

A graph G is said to be (g, f, n)-*critical* if $G - N$ has a (g, f)-factor for each $N \subseteq V(G)$ with $|N| = n$. If $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then a (g, f, n)-critical graph is an $[a, b, n]$-*critical graph*. If $g(x) = f(x)$ (resp. $g(x) = f(x) = k$) for all $x \in V(G)$, then a (g, f, n)-critical graph is called an (f, n)-*critical graph* (resp. a $[k, n]$-*critical graph*). In particular, a $[1, n]$-critical graph is simply called an n-*critical graph*.

Notation and definition not given in this paper can be found in [1].

M. D. Plummer [11] and L. Lovász [8] discussed the characterization and properties of 2-critical graph. Q. Yu [12] gave the characterization of n-critical graphs. O. Favaron [2] studied the properties of n-critical graph. G. Liu and Q. Yu [7] studied the characterization of $[k, n]$-critical graphs. The characterization of $[a, b, n]$-critical graph with $a < b$ was given by G. Liu and J. Wang [6]. In this paper, a necessary and sufficient condition for a graph to be (g, f, n)-critical is given. Furthermore, the properties of (g, f, n)-critical graphs are studied.

Let $\mathbb{Z}$ be the set of integers, and let S and T be disjoint subsets of $V(G)$. We write $e_G(S, T) = |\{xy \in E(G) : x \in S,\ y \in T\}|$. In particular, $e_G(x, T)$ means $e_G(\{x\}, T)$. For $g, f : V(G) \to \mathbb{Z}$, we denote $f(S) = \sum_{x \in S} f(x)$, $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$, and $g(T) = \sum_{x \in T} g(x)$. A component C of $G - (S \cup T)$ is called an *odd component* if $g(x) = f(x)$ for all $x \in V(C)$ and $f(V(C)) + e_G(V(C), T) \equiv 1 \pmod 2$. Moreover, a component C is called a *non-odd component* if C is not an odd component. Define

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) - h_G(S, T),$$

where $h_G(S, T)$ is the number of odd components C of $G - (S \cup T)$.

We use the following criterion in the next section.

Theorem A (L. Lovász [9]) *Let G be a graph. Let g and $f : V(G) \to \mathbb{Z}$ be two integers such that $g(x) \leq f(x)$ for each $x \in V(G)$. Then G has a (g, f)-factor if and only if*

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) - h_G(S, T) \geq 0$$

for all disjoint subsets S and T of $V(G)$.

2 Characterization of factor-critical graphs

Let $\mathcal{P}$ be a set of graphs with a given constant order. Then we say that G is *$(g, f, \mathcal{P})$-critical* if for every subgraph H of G isomorphic to a graph in $\mathcal{P}$, $G - V(H)$ has a (g, f)-factor.

We first give the following theorem.

Theorem 1 *Let G be a graph, and $\mathcal{P}$ a set of graphs of order n. Let g and $f : V(G) \to \mathbb{Z}$ be two functions such that $g(x) \leq f(x)$ for each $x \in V(G)$. Then G is $(g, f, \mathcal{P})$-critical if and only if*

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) - h_G(S, T) \geq f_S(n) \qquad (1)$$

for all disjoint subsets S and T of $V(G)$ such that $G[S]$ has a subgraph isomorphic to a graph in $\mathcal{P}$, where $f_S(n) = \max\{f(N) : N \subseteq S, |N| = n, \text{ and } G[N] \text{ has a subgraph in } \mathcal{P}\}$ and $f_\emptyset(n) = -\infty$.

Proof. Suppose that G is $(g, f, \mathcal{P})$-critical. Let $S \subseteq V(G)$ be any subset such that $G[S]$ has a subgraph N in $\mathcal{P}$ with $f(V(N)) = f_S(n)$. Then $G - N$ has a (g, f)-factor. Let $S' = S \setminus V(N)$. By Theorem A, $G - N$ has a (g, f)-factor if and only if

$$\delta_{G-N}(S', T) = f(S') + d_{(G-N)-S'}(T) - g(T) - h_{G-N}(S', T) \geq 0$$

for all disjoint subsets S' and T of $V(G) - N$. Since $d_{(G-N)-(S \setminus V(N))}(T) = d_{G-S}(T)$ and $h_G(S, T) = h_{G-N}(S \setminus V(N), T)$, we have

$$\begin{aligned}
0 &\leq \delta_{G-N}(S', T) \\
&= f(S \setminus V(N)) + d_{(G-N)-(S \setminus V(N))}(T) - g(T) - h_{G-N}(S \setminus V(N), T) \\
&= f(S) - f(V(N)) + d_{G-S}(T) - g(T) - h_G(S, T) \\
&= \delta_G(S, T) - f(V(N)) = \delta_G(S, T) - f_S(n).
\end{aligned}$$

Thus (1) holds.

Conversely, we assume that (1) holds. Let $N \in \mathcal{P}$ be a subgraph of G. For any subset $S' \subseteq V(G - N)$, write $S = S' \cup V(N)$. Then

$$\begin{aligned}
&\delta_{G-N}(S', T) \\
&= f(S \setminus V(N)) + d_{(G-N)-(S \setminus V(N))}(T) - g(T) - h_{G-N}(S \setminus V(N), T) \\
&= f(S) - f(V(N)) + d_{G-S}(T) - g(T) - h_G(S, T) \\
&= \delta_G(S, T) - f(V(N)) \geq \delta_G(S, T) - f_S(n) \geq 0.
\end{aligned}$$

Therefore $G - N$ has a (g, f)-factor by Theorem A. ∎

Recall that a graph G is said to be (g, f, n)-*critical* if for every subset $N \subseteq V(G)$ with $|N| = n$, $G - N$ has a (g, f)-factor. If $g(x) = f(x)$ for all $x \in V(G)$, then (g, f, n)-critical is called (f, n)-*critical*. The following results are easy consequences of Theorem 1.

Corollary 1 *Let G be a graph, $n \geq 0$ an integer, and g, $f : V(G) \to \mathbb{Z}$ two functions such that $g(x) \leq f(x)$ for each $x \in V(G)$. Then G is (g, f, n)-critical if and only if*

$$f(S) + d_{G-S}(T) - g(T) - h_G(S,T) \geq \max\{f(N) : N \subseteq S \text{ and } |N| = n\}$$

for all disjoint subsets S and T of $V(G)$ with $|S| \geq n$.

Corollary 2 *Let G be a graph, $n \geq 0$ an integer, and $f : V(G) \to \mathbb{Z}$. Then G is (f, n)-critical if and only if*

$$f(S) + d_{G-S}(T) - f(T) - h_G(S,T) \geq \max\{f(N) : N \subseteq S \text{ and } |N| = n\}$$

for all disjoint subsets S and T of $V(G)$ with $|S| \geq n$.

Corollary 3 *Let k and n be nonnegative integers. Then a graph G is $[k, n]$-critical if and only if*

$$k|S| + d_{G-S}(T) - k|T| - h_G(S,T) \geq kn$$

for all disjoint subsets S and T of $V(G)$ with $|S| \geq n$.

Corollary 4 *Let G be a graph, $n \geq 0$ an integer, and g, $f : V(G) \to \mathbb{Z}$ two functions such that $g(x) < f(x)$ for each $x \in V(G)$. Then G is (g, f, n)-critical if and only if*

$$f(S) + d_{G-S}(T) - g(T) \geq \max\{f(N) : N \subseteq S \text{ and } |N| = n\}$$

for all disjoint subsets S and T of $V(G)$ with $|S| \geq n$.

Corollary 4 is equivalent to the following corollary.

Corollary 5 *Let G be a graph, $n \geq 0$ an integer, and g, $f : V(G) \to \mathbb{Z}$ two functions such that $g(x) < f(x)$ for each $x \in V(G)$. Then G is (g, f, n)-critical if and only if*

$$f(S) + d_{G-S}(T_G(S)) - g(T_G(S)) \geq \max\{f(N) : N \subseteq S \text{ and } |N| = n\}$$

for all subset S of $V(G)$ with $|S| \geq n$, where $T_G(S) = \{x : x \in V(G) \setminus S$ and $d_{G-S}(x) \leq g(x)\}$.

Let $P_j(G-S) = |\{x \in V(G) : d_{G-S}(x) = j\}|$. Since $\sum_{x \in T_G(S)}(a - d_{G-S}(x)) = \sum_{j=0}^{a-1}(a-j)P_j(G-S)$, the following result is a special case of Corollary 5.

Theorem B (G. Liu and J. Wang [6]) *Let a and b be integers with $1 \le a < b$, and G a graph with $|G| \ge a+n+1$. Then G is $[a,b,n]$-critical if and only if*

$$b|S| + d_{G-S}(T_G(S)) - a|T_G(S)| \ge bn, \quad \text{or} \quad \sum_{j=0}^{a-1}(a-j)P_j(G-S) \le b|S| - bn$$

for all subset S of $V(G)$ with $|S| \ge n$.

3 Some properties of factor-critical graphs

Lemma 1 *Let G be a (g,f,n)-critical graph of order $|G| \ge n+1$ with $1 \le g(x) \le f(x)$ for all $x \in V(G)$. Then $d_G(x) \ge g(x)+n$ for all $x \in V(G)$.*

Proof. Suppose that G has a vertex v with $d_G(v) < g(v)+n$. Then take $N_0 \subset N_G(v)$ with $|N_0| = \min\{n, d_G(v)\}$. If $d_G(v) < n$, then take $N_1 \subset V(G) - (N_G(v) \cup \{v\})$ with $|N_1| = n - d_G(v)$. Otherwise, let $N_1 = \emptyset$. Write $N = N_0 \cup N_1$. Then $|N| = n$ and $d_{G-N}(v) < g(v)$, which implies that $G-N$ has no (g,f)-factor. This is a contradiction. ∎

Theorem 2 *Let $a \ge 1$ and $n \ge 1$ be two integers. Let $f : V(G) \to \mathbb{Z}$ be a function such that $f(x) \ge a$ for all $x \in V(G)$. Then an (a,f,n)-critical graph G is $(a+n)$-connected.*

In particular, an n-critical graph is $(n+1)$-connected.

Proof. We use the induction on $|G|$. Since G is (a,f,n)-critical, $|G| \ge a+n+1$. Suppose first that $|G| = a+n+1$. If G is not $(a+n)$-connected, then there exists a subset $S \subset V(G)$ with $|S| = a+n-1$ such that $G-S$ is disconnected. Let $S' \subseteq S$ be any subset with $|S'| = n$. Then $G-S'$ has no (a,f)-factor, which contradicts that G is (a,f,n)-critical.

We next assume that $|G| > a+n+1$. Let x be any vertex of G. Since G is (a,f,n)-critical, $G-\{x\}$ is $(a,f,n-1)$-critical. Hence $G-\{x\}$ is $(a+n-1)$-connected by the induction hypothesis, which implies that G is $(a+n)$-connected. ∎

Theorem 3 *Let G be a (g,f,n)-critical graph with $n \geq 1$ and $g(x) \geq 1$ for all $x \in V(G)$. Then G is $(g,f,n-1)$-critical unless $|G|-n$ is even and G has at least $|G|-n+1$ vertices with $g(x)=f(x)\equiv 1 \pmod 2$.*

Note that the condition "unless $|G|-n$ is even and G has at least $|G|-n+1$ vertices with $g(x)=f(x)\equiv 1 \pmod 2$" cannot be dropped. If G is the exceptional graph, then there exists $N \subseteq V(G)$ with $|N|=n-1$ such that $g(x)=f(x)\equiv 1 \pmod 2$ for all $x \in V(G)\setminus N$. Since $|G-N|$ is odd, $G-N$ has no f-factor and hence G cannot be $(g,f,n-1)$-critical.

Proof of Theorem 3. Since G is (g,f,n)-critical, by Corollary 1,

$$\begin{aligned}\delta_G(S,T) &= f(S)+d_{G-S}(T)-g(T)-h_G(S,T)\\ &\geq \max\{f(N) : N\subseteq S \text{ and } |N|=n\}\end{aligned}$$

for all disjoint subsets S and T of $V(G)$ with $|S|\geq n$.

Consequently, we need only to show that for all disjoint subsets S and T of $V(G)$ with $|S|=n-1$

$$\delta_G(S,T)=f(S)+d_{G-S}(T)-g(T)-h_G(S,T)\geq f(S).$$

If $|U|=|V(G)\setminus(S\cup T)|=0$, then $h_G(S,T)=0$. Since G is (g,f,n)-critical, we have $d_G(x)\geq g(x)+n$ for all $x\in V(G)$ by Lemma 1. It follows from $|S|=n-1$ that $d_{G-S}(x)\geq g(x)+n-|S|=g(x)+1$ for all $x\in T$. Thus

$$\delta_G(S,T)=f(S)+d_{G-S}(T)-g(T)\geq f(S)+|T|\geq f(S).$$

Hence we may assume that $|U|>0$. Let u be any vertex of U, and let $C_1,\ldots,C_m$ be the components of $G-(S\cup T)$. Since G is (g,f,n)-critical and $|S\cup\{u\}|=n$, we have

$$\begin{aligned}\delta_G(S,T) &= f(S)+d_{G-S}(T)-g(T)-h_G(S,T)\\ &= \delta_G(S\cup\{u\},T)+h_G(S\cup\{u\},T)-h_G(S,T)-f(u)+e_G(u,T)\\ &\geq f(S\cup\{u\})+h_G(S\cup\{u\},T)-h_G(S,T)-f(u)+e_G(u,T),\end{aligned}$$

which implies

$$\delta_G(S,T)\geq f(S)+h_G(S\cup\{u\},T)-h_G(S,T)+e_G(u,T). \qquad (2)$$

If there exists a non-odd component C_i, then $h_G(S\cup\{u\},T)\geq h_G(S,T)$ for each $u\in V(C_i)$ and hence $\delta_G(S,T)\geq f(S)$ holds by (2). Thus we may assume that all $C_1,\ldots,C_m$ are odd components. If $e_G(u',T)\geq 1$ for some $u'\in V(C_i)$, then by (2) and $h_G(S\cup\{u'\},T)-h_G(S,T)\geq -1$, we have $\delta_G(S,T)\geq f(S)+h_G(S\cup\{u'\},T)-h_G(S,T)+e_G(u',T)\geq f(S)$.

Hence we may consider the case $e_G(V(C_i),T) = 0$ for all $i = 1, \dots, m$. Then for any $u \in V(C_i)$, it follows from Lemma 1 that $0 = e_G(u,T) = d_G(u) - e_G(u, S \cup U) \geq g(u) + n - |S| - (|C_i| - 1) = g(u) + 2 - |C_i| \geq 3 - |C_i|$, which implies $|C_i| \geq 3$ for each $i = 1, \dots, m$. We divide into two cases.

Case 1. $T = \emptyset$.

Since G is (g, f, n)-critical and $|S \cup \{u\}| = n$, we have

$$\delta_G(S \cup \{u\}, \emptyset) = f(S \cup \{u\}) - h_G(S \cup \{u\}, \emptyset) \geq f(S \cup \{u\}),$$

implying $h_G(S \cup \{u\}, \emptyset) = 0$. Now, we prove that $h_G(S, \emptyset) = 0$.

Since all $C_1, \dots, C_m$ are odd components of $G - S$, and $h_G(S \cup \{u\}, \emptyset) = 0$, we obtain $h_G(S, \emptyset) = m \leq 1$. Suppose that $m = 1$. Then $G - S = \{C_1\}$, $g(u) = f(u)$ for all $u \in V(C_1)$, and $f(V(C_1)) + e_G(V(C_1), \emptyset) = f(V(C_1)) \equiv 1 \pmod 2$. If there exists a vertex $u'' \in V(C_1)$ such that $f(u'')$ is even, then $f(V(C_1) \setminus \{u''\}) \equiv 1 \pmod 2$. Hence there exists at least one component C in $G - (S \cup \{u''\})$ with $f(V(C))$ odd. This contradicts $h_G(S \cup \{u\}, \emptyset) = 0$ for any $u \in U$. Thus $g(u) = f(u) \equiv 1 \pmod 2$ for each $u \in V(C_1)$. Consequently, G has at least $|C_1| = |G - S| = |G| - (n-1)$ vertices u with $g(u) = f(u) \equiv 1 \pmod 2$. If $|C_1|$ is even, then $f(V(C_1)) \equiv |C_1| \equiv 0 \pmod 2$. This contradicts the fact C_1 is an odd component of $G - S$. Thus $|C_1| = |G - S| = |G| - (n-1)$ is odd, implying $|G| - n$ is even.

Therefore G has at least $|C_1| = |G| - (n-1)$ vertices with $g(u) = f(u) \equiv 1 \pmod 2$, and $|G| - n$ is even. This contradicts the assumption of Theorem 3.

Finally, we get $h_G(S, \emptyset) = 0$. Hence $\delta_G(S, \emptyset) \geq f(S) - h_G(S, \emptyset) = f(S)$ by (2).

Case 2. $T \neq \emptyset$.

Take $y \in T$. Since G is (g, f, n)-critical, $d_G(y) \geq g(y) + n$, and $e_G(V(C_i), T) = 0$ for each $i = 1, \dots, m$, we have

$$\begin{aligned}
\delta_G(S,T) &= f(S) + d_{G-S}(T) - g(T) - h_G(S,T) \\
&= \delta_G(S \cup \{y\}, T \setminus \{y\}) + d_G(y) - g(y) - e_G(y,S) - f(y) \\
&\quad + e_G(y,T) + h_G(S \cup \{y\}, T \setminus \{y\}) - h_G(S,T) \\
&\geq f(S \cup \{y\}) + d_G(y) - g(y) - e_G(y,S) - f(y) + e_G(y,T) \\
&\quad + h_G(S \cup \{y\}, T \setminus \{y\}) - h_G(S,T) \\
&= f(S) + d_G(y) - g(y) - e_G(y,S) + e_G(y,T) \\
&\geq f(S) + n - e_G(y,S) + e_G(y,T) \geq f(S) + e_G(y,T) \geq f(S).
\end{aligned}$$

The proof is complete. ∎

From Theorem 3, we immediately obtain the following results.

Corollary 6 *Let G be a (g, f, n)-critical graph with $n \geq 1$, $g(x) \geq 1$, and $f(x)$ even for all $x \in V(G)$. Then for any integer m with $0 \leq m < n$, G is also (g, f, m)-critical. In particular, G has a (g, f)-factor.*

The following result is a special case of Corollary 6 for $g(x) = f(x) = 2r$ and $n = 1$.

Theorem C (P. Katerinis [3]) *Let G be a graph of order at least two, and r a positive integer. If $G - \{x\}$ has a $2r$-factor for each $x \in V(G)$, then G has a $2r$-factor.*

Corollary 7 *Let G be a (g, f, n)-critical graph with $n \geq 1$ and $1 \leq g(x) < f(x)$ for all $x \in V(G)$. Then for any integer m with $0 \leq m < n$, G is also (g, f, m)-critical.*

This reads to the following theorem.

Theorem D (G. Liu and J. Wang [6]) *Let G be a $[a, b, n]$-critical graph with $1 \leq a < b$ and $n \geq 1$. Then for any integer m with $0 \leq m < n$, G is also $[a, b, m]$-critical.*

Theorem 4 *Let G be a (g, f, n)-critical graph with $n \geq 2$ and $g(x) \geq 1$ for all $x \in V(G)$. Then G is also $(g, f, n-2)$-critical.*

Proof. By Corollary 1, we need only to show that for all disjoint subsets S and T of $V(G)$ with $|S| = n - 1$ or $|S| = n - 2$

$$\begin{aligned}\delta_G(S,T) &= f(S) + d_{G-S}(T) - g(T) - h_G(S,T)\\ &\geq \max\{f(N) : N \subseteq S \text{ and } |N| = n-2\}.\end{aligned}$$

As the same with the proof of Theorem 3, we have $|U| = |V(G) \backslash (S \cup T)| \geq 1$.

Case 1. $|S| = n - 1$.

Let u be any vertex of U. Since $|S \cup \{u\}| = n$ and G is (g, f, n)-critical, we have

$$\begin{aligned}\delta_G(S,T) &= f(S) + d_{G-S}(T) - g(T) - h_G(S,T)\\ &= \delta_G(S \cup \{u\}, T) - f(u) + e_G(u,T) + h_G(S \cup \{u\}, T) - h_G(S,T)\\ &\geq f(S \cup \{u\}) - f(u) + e_G(u,T) + h_G(S \cup \{u\}, T) - h_G(S,T),\end{aligned}$$

that is, $\delta_G(S,T) \geq f(S) + e_G(u,T) + h_G(S \cup \{u\}, T) - h_G(S,T)$. By $f(x) \geq 1$ for all $x \in S$ and $h_G(S \cup \{u\}, T) - h_G(S,T) \geq -1$, we have

$$\begin{aligned}\delta_G(S,T) &\geq f(S) + h_G(S \cup \{u\}, T) - h_G(S,T) + e_G(u,T)\\ &\geq \max\{f(N) : N \subseteq S \text{ and } |N| = n-2\} + 1 + h_G(S \cup \{u\}, T)\\ &\quad - h_G(S,T) + e_G(u,T)\\ &\geq \max\{f(N) : N \subseteq S \text{ and } |N| = n-2\},\end{aligned}$$

as desired.

Case 2. $|S| = n - 2$.

Suppose that $|U| = 1$. Let $U = \{u\}$. Since $d_{G-S}(x) \geq g(x) + 1 \geq 2$ hold for each $x \in V(G)$ by Lemma 1, we have $e_G(u, T) \geq 2$, implying $|T| \geq 1$. Thus

$$\delta_G(S, T) \geq f(S) + d_{G-S}(T) - g(T) - 1 \geq f(S) + |T| - 1 \geq f(S).$$

Therefore we may assume that $|U| \geq 2$.

Let $C_1, \ldots, C_m$ be the components of $G - (S \cup T)$, and let u_1, u_2 be any two vertices of U. Since $|S \cup \{u_1, u_2\}| = n$ and G is (g, f, n)-critical, we have

$$\begin{aligned}
\delta_G(S, T) &= f(S) + d_{G-S}(T) - g(T) - h_G(S, T) \\
&= \delta_G(S \cup \{u_1, u_2\}, T) - f(\{u_1, u_2\}) + e_G(\{u_1, u_2\}, T) \\
&\quad + h_G(S \cup \{u_1, u_2\}, T) - h_G(S, T) \\
&\geq f(S \cup \{u_1, u_2\}) - f(\{u_1, u_2\}) + e_G(\{u_1, u_2\}, T) \\
&\quad + h_G(S \cup \{u_1, u_2\}, T) - h_G(S, T).
\end{aligned}$$

that is,

$$\delta_G(S, T) \geq f(S) + e_G(\{u_1, u_2\}, T) + h_G(S \cup \{u_1, u_2\}, T) - h_G(S, T). \quad (3)$$

By Lemma 1, we have $e_G(u, T) = d_G(u) - e_G(u, S \cup U) \geq g(u) + n - |S| - (|C_i| - 1) = g(u) + 3 - |C_i|$ for each $u \in V(C_i)$. If u_1 or u_2, say $u_1 \in V(C_i)$, satisfies $|C_i| \leq g(u_1) + 1$, then $e_G(u_1, T) \geq 2$. This inequality together with (3) and $h_G(S \cup \{u_1, u_2\}, T) - h_G(S, T) \geq -2$ implies $\delta_G(S, T) \geq f(S)$. Hence we have $|C_i| \geq g(u) + 2 \geq 3$ for each $i = 1, \ldots, m$.

We divide into two subcases.

Subcase 2.1. There exists a non-odd component C_i of $G - (S \cup T)$.

With out loss of generality, we may assume that $u_1, u_2 \in V(C_i)$. Since C_i is a non-odd component of $G - (S \cup T)$, we have $h_G(S \cup \{u_1, u_2\}, T) - h_G(S, T) \geq 0$. Hence $\delta_G(S, T) \geq f(S)$ by (3).

Subcase 2.2. Each C_i is an odd component of $G - (S \cup T)$ with $|C_i| \geq 3$, where $i = 1, \cdots, m$.

If there exists $u_1 \in V(C_i)$ such that $e_G(\{u_1\}, T) \geq 1$, then for any $u_2 \in V(C_i) \setminus \{u_1\}$, we obtain $h_G(S \cup \{u_1, u_2\}, T) - h_G(S, T) + e_G(\{u_1, u_2\}, T) \geq -1 + 1 \geq 0$. Hence $\delta_G(S, T) \geq f(S)$ holds by (3). Consequently, we may assume that $e_G(C_i, T) = 0$ for all $i = 1, \ldots, m$. By $|C_i| \geq 3$, there exist two vertices $u_1, u_2 \in V(C_i)$ with $f(u_1) \equiv f(u_2) \pmod 2$ for each i, $1 \leq i \leq m$. Then $f(V(C_i)) + e_G(V(C_i), T) = f(V(C_i)) \equiv f(V(C_i) \setminus \{u_1, u_2\}) \equiv 1 \pmod 2$, which means $G[C_i - \{u_1, u_2\}]$ has at least one odd component of

$G - (S \cup \{u_1, u_2\} \cup T)$. Thus $h_G(S \cup \{u_1, u_2\}, T) - h_G(S, T) \geq 0$ and so $\delta_G(S, T) \geq f(S)$ by (3).

Finally, our proof is complete. ∎

By Theorem 4, we immediately obtain the following result.

Corollary 8 *Let G be a (g, f, n)-critical graph with $n \geq 2$ and $g(x) \geq 1$ for all $x \in V(G)$. Then for any integer m with $0 \leq m < n$ and $m \equiv n$ (mod 2), G is also (g, f, m)-critical. In particular, G has a (g, f)-factor for $n \equiv 0$ (mod 2).*

The following theorem gives a relationship of properties of edge-inclusion (or edge-deletion) and (g, f, n)-critical graphs.

Theorem 5 *Let G be a (g, f, n)-critical graph with $1 \leq n < |G|$ and $1 \leq g(x) < f(x)$ for all $x \in V(G)$. Suppose that H_1 and H_2 are any two edge-disjoint subgraphs of G with $|E(H_1) \cup E(H_2)| \leq n$ and $d_{H_1}(x) \leq f(x)$ for all $x \in V(H_1)$. Then G has a (g, f)-factor F such that $E(H_1) \subseteq E(F)$ and $E(H_2) \cap E(F) = \emptyset$.*

By Theorem 5, we obtain the following results.

Corollary 9 *Let G be a (g, f, n)-critical graph with $n \geq 1$ and $1 \leq g(x) < f(x)$ for all $x \in V(G)$. For any n edges of G, let H be a subgraph of G induced by the n edges and $d_H(x) \leq f(x)$ for all $x \in V(H)$. Then G has a (g, f)-factor which includes the n edges.*

Corollary 10 *Let G be a (g, f, n)-critical graph with $n \geq 1$ and $1 \leq g(x) \leq f(x)$ for all $x \in V(G)$. Then the subgraph obtained from G by deleting any n edges has a (g, f)-factor.*

Corollary 10 implies the following result, which is due to G. Liu and J. Wang.

Theorem E (G. Liu and J. Wang [6]) *Let G be an $[a, b, n]$-critical graph with $n \geq 1$ and $1 \leq a < b$. Then the subgraph obtained from G by deleting any n edges has an $[a, b]$-factor.*

In order to show Theorem 5, we use the following.

Theorem F (P. B. C. Lam, G. Liu, G. Li, and W. C. Shiu [5]) *Let G be a graph and let H_1 and H_2 be two edge-disjoint subgraphs of G. Let g and $f : V(G) \to \mathbb{Z}$ be two functions such that $g(x) < f(x)$ for each $x \in V(G)$.*

Then G has a (g,f)-factor such that $E(H_1) \subseteq E(F)$ and $E(H_2) \cap E(F) = \emptyset$ if and only if

$$f(S) + d_{G-S}(T) - g(T) \geq R_S(H_1) + R_T(H_2)$$

for any two disjoints subsets S and T of $V(G)$, where $R_S(H_1) = \sum_{x \in S \cap V(H_1)} d_{H_1}(x) - e_{H_1}(S,T)$ and $R_T(H_2) = \sum_{x \in T \cap V(H_2)} d_{H_2}(x) - e_{H_2}(S,T)$.

Proof of Theorem 5. In order to prove Theorem 5, we need to show that for any two disjoint subsets S and T of $V(G)$

$$f(S) + d_{G-S}(T) - g(T) \geq R_S(H_1) + R_T(H_2)$$

by Theorem F. Note that $R_S(H_1) + R_T(H_2) \leq \sum_{x \in S \cap V(H_1)} d_{H_1}(x) + \sum_{x \in T \cap V(H_2)} d_{H_2}(x) \leq 2|E(H_1)| + 2|E(H_2)| \leq 2n$ by the assumption of this theorem. We divide into two cases.

Case 1. $|S| \geq n$.

Since G is (g,f,n)-critical, by Corollary 1, for any disjoint subsets S and T of $V(G)$ with $|S| \geq n$, we obtain $f(S) + d_{G-S}(T) - g(T) \geq \max\{f(N) : N \subseteq S \text{ and } |N| = n\} \geq 2n \geq R_S(H_1) + R_T(H_2)$ as desired.

Case 2. $|S| < n$.

Since G is (g,f,n)-critical, we have $d_G(x) \geq g(x) + n$ for all $x \in V(G)$ by Lemma 1. Thus $d_{G-S}(x) \geq g(x) + n - |S|$ holds. Then it follows this inequality that

$$f(S) + d_{G-S}(T) - g(T) \geq f(S) + (n - |S|)|T|. \tag{4}$$

Suppose first that $|T| \geq 2$. Then by (4), $f(S) + d_{G-S}(T) - g(T) \geq f(S) + (n - |S|)|T| \geq 2|S| + 2(n - |S|) = 2n \geq R_S(H_1) + R_T(H_2)$ as desired.

We next consider the case $|T| = 1$ and put $T = \{t\}$. By the assumption of this theorem, $n \geq |E(H_1)| + |E(H_2)| \geq |E(H_1)| + d_{H_2}(t) \geq |E(H_1)| + R_T(H_2)$. It follows from this inequality, (4), and $|T| = 1$ that

$$\begin{aligned} f(S) + d_{G-S}(T) - g(T) &\geq f(S) + n - |S| \\ &\geq f(S) + |E(H_1)| + R_T(H_2) - |S|. \end{aligned} \tag{5}$$

If $|S| \leq |E(H_1)|$, then it follows from (5) and the assumption of this theorem that $f(S) + d_{G-S}(T) - g(T) \geq f(S) + R_T(H_2) \geq \sum_{x \in S \cap V(H_1)} d_{H_1}(x) + R_T(H_2) \geq R_S(H_1) + R_T(H_2)$. If $|S| > |E(H_1)|$, then by (5) and $f(S) \geq 2|S|$, we have $f(S) + d_{G-S}(T) - g(T) \geq f(S) - |S| + |E(H_1)| + R_T(H_2) \geq |S| + |E(H_1)| + R_T(H_2) > 2|E(H_1)| + R_T(H_2) \geq R_S(H_1) + R_T(H_2)$.

If $|T| = 0$, then $R_T(H_2) = 0$. By the assumption of this theorem, $f(S) \geq d_{H_1}(S) \geq R_S(H_1)$ hold. Substituting this to (4) yields $f(S) + d_{G-S}(T) - g(T) \geq R_S(H_1) + R_T(H_2)$.

Finally, the proof is complete. ∎

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