Matrices with maximum exponent in the class of central symmetric primitive matrices *

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Abstract

The extremal matrix problem of symmetric primitive matrices has been completely solved in [Sci. Sinica Ser. A 9(1986) 931-939] and [Linear Algebra Appl. 133(1990) 121-131]. In this paper, we determine the maximum exponent in the class of central symmetric primitive matrices, and give a complete characterization of those central symmetric primitive matrices whose exponents actually attain the maximum exponent.

Keywords: Primitive matrix, Symmetric primitive matrix, Exponent, Extremal matrix, Associated graph

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1 Introduction

An $n \times n$ $(0,1)$-matrix A over the binary Boolean algebra $\{0,1\}$ is said to be primitive if $A^k > 0$ for some positive integer k. The least such k is called the exponent of A, denoted by $\gamma(A)$. The associated graph of symmetric matrix $A = (a_{ij})$, denoted by $G(A)$, is the graph with a vertex set $V(G(A)) = \{1, 2, \cdots, n\}$ such that there is an edge from i to j in $G(A)$

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if and only if $a_{ij} = 1$. A graph G is called to be primitive if there exists an integer $k > 0$ such that for all ordered pairs of vertices $i, j \in V(G)$ (not necessarily distinct), there is a walk from i to j with length k. The least such k is called the exponent of G, denoted by $\gamma(G)$. It is well known (see e.g. [1]) that a symmetric matrix A is primitive if and only if its associated graph $G(A)$ is primitive, namely, $G(A)$ is connected and contains at least one odd cycle. In this case we have $\gamma(A) = \gamma(G(A))$.

The extremal matrix problem (EMP) is a main problem in the study of exponents. As remarked in [2], the problem of complete characterization of the extremal matrices of certain matrix classes is usually very difficult. In 1980, Brualdi and Ross [3] settled the EMP for the class of $n \times n$ primitive nearly reducible matrices. In 1986, Shao [4] settled the EMP for the class of $n \times n$ symmetric primitive Boolean matrices. In 1990, Liu et al. [5] settled the EMP for the class of $n \times n$ symmetric primitive Boolean matrices with zero trace. Huang [6] settled the EMP for the class of $n \times n$ primitive circulant matrices. In 1991, Liu and Shao [7] settled the EMP for the class of $n \times n$ primitive matrices with exactly d nonzero diagonal entries. In 1999, B. Zhou and B. Liu [8] settled the EMP for the class of $n \times n$ doubly stochastic primitive matrices. In 2003, B. Zhou [9] settled the EMP for the class of $n \times n$ symmetric primitive Boolean matrices whose graphs having given odd girth.

In this paper, we consider a particular class of symmetric primitive matrices—central symmetric primitive matrices. An $n \times n$ symmetric primitive matrix $A = (a_{ij})$ is said to be a central symmetric primitive matrix if $a_{ij} = a_{n+1-i,n+1-j}$ $(i, j = 1, 2, \cdots, n)$. Clearly, If A is an $n \times n$ central symmetric primitive matrix, then its associated graph $G(A)$ is primitive and for each ordered pair of vertices i and j (not necessarily distinct), there is an edge $[i, j]$ if and only if there is an edge $[n+1-i, n+1-j]$. The vertex $n+1-i$ is called the central symmetric vertex of i, denoted by i^d. If $n \equiv 1 (\mathrm{mod}\, 2)$ and the vertex $i = \frac{n+1}{2}$, then $i = i^d$; otherwise, we always have $i \neq i^d$ for $i \in V(G(A))$. If $W = i_1 i_2 \cdots i_m$ is a walk from a vertex i_1 to a vertex i_m in $G(A)$, then $i_1^d i_2^d \cdots i_m^d$ is a walk from i_1^d to i_m^d in $G(A)$, and denoted by W^d. In particular, if P is the shortest path joining a vertex i and a vertex j in $G(A)$, then P^d is the shortest path joining i^d and j^d in $G(A)$; if C is a cycle in G, then C^d is also a cycle in G.

Denote by $CSP(n)$ the set of all $n \times n$ central symmetric primitive matrices. Using the graph theoretical method we determine the maximum exponent in $CSP(n)$, and characterize the extremal matrices completely. Since the cases $n = 1$ and $n = 2$ are trivial, we always assume that $n \geq 3$ in this paper.

2 The maximum exponent

In this section we determine the maximum exponent in $CSP(n)$. Let G be a primitive graph. For $i, j \in V(G)$, the exponent from i to j, denoted by $\gamma(i, j)$, is defined to be the least integer k such that there exists a walk of length p from i to j for every $p \geq k$. The following two lemmas are contained in [4] and [5] , respectively.

Lemma 2.1 [4] *If G is a primitive graph, then $\gamma(G) = \max_{i,j \in V(G)} \gamma(i, j)$.*

Lemma 2.2 [5] *Let G be a primitive graph, and let $i, j \in V(G)$. If there are two walks from i to j with lengths k_1 and k_2, respectively, where $k_1 + k_2 \equiv 1 (\mathrm{mod}\, 2)$, then $\gamma(i, j) \leq \max\{k_1, k_2\} - 1$.*

We will make use of the following notation. Let G be a graph. If W is a walk in G, then $|W|$ denotes the length of W. If P is a path (i.e. a walk without repeated vertices) in G and $i, j \in V(P)$, then iPj denotes the subpath of P joining i and j. In particular, $|iPi| = 0$. If C is an odd cycle and $i, j \in V(C)$, then C contains two walks joining i and j, and these walks are of different length since $|C|$ is odd. We denote these walks by $iC'j$ and $iC''j$ where $|iC'j| < |iC''j|$. Note that if $i = j$, then $|iC'j| = 0$ and $iC''j = C$. The concatenation of a walk W_1 from a vertex i to a vertex t, and a walk W_2 from t to a vertex j is denoted by $W_1 + W_2$. We denote the distance between two vertices i and j of G by $d(i, j)$. If G' and G'' be two subgraphs of G, then $P(G', G'')$ denotes the shortest path between G' and G'', and its length $d(G', G'') = |P(G', G'')| = \min\{d(i, j) : i \in V(G'), j \in V(G'')\}$. Clearly, $d(G', G'') = |P(G', G'')| \geq 1$ if and only if $V(G') \cap V(G'') = \phi$.

Lemma 2.3 *Let i be any vertex of $G = G(A)$ with $A \in CSP(n)$. Let C_i be any odd cycle such that $d(i, C_i) = \min\{d(i, C) : C$ is an odd cycle in $G\}$. Assume that $d(i, C_i) = m \geq 1$. Let $P = i_0 i_1 \cdots i_m$ be any shortest path between i and C_i, where $i_0 = i$ and $i_m \in V(C_i)$. Then the following hold:*

(i) *If $V(P^d) \cap V(P) \neq \phi$, then $V(P^d) \cap V(P) = \{i_t\} = \{i_t^d\}$ $(0 \leq t \leq m)$.*

(ii) *If $V(P^d) \cap V(C_i) \neq \phi$, then $V(P^d) \cap V(C_i) = \{i_m^d\}$.*

(iii) $|V(P^d) \cap (V(P) \cup V(C_i))| \leq 1$.

Proof. We first assume that $V(P^d) \cap V(P) \neq \phi$ and prove (i). Let i_t be any vertex in $V(P^d) \cap V(P)$. Then there exists $i_k^d \in V(P^d)$ such that

$i_k^d = i_t$ (and thus $i_t^d = i_k$). Suppose that $k \neq t$. If $k > t$, then

$$d(i, C_i^d) \leq |i_0 P i_t| + |i_k^d P^d i_m^d| = m - (k - t) < m = d(i, C_i).$$

If $k < t$, then

$$d(i, C_i^d) \leq |i_0 P i_k| + |i_t^d P^d i_m^d| = m - (t - k) < m = d(i, C_i).$$

Thus in any case we have $d(i, C_i^d) < d(i, C_i)$, contradicting the definition of C_i. Hence $k = t$, and hence (i) holds.

We now assume that $V(P^d) \cap V(C_i) \neq \phi$ and prove (ii). Let i_t^d be any vertex in $V(P^d) \cap V(C_i)$. Then $i_t \in V(P) \cap V(C_i^d)$. If $0 \leq t \leq m - 1$, then

$$d(i, C_i^d) \leq |i_0 P i_t| = t \leq m - 1 < m = d(i, C_i).$$

This contradicts the definition of C_i. Hence $t = m$, and (ii) holds.

Now, if $V(P^d) \cap V(P) = \phi$ or $V(P^d) \cap V(C_i) = \phi$, then from (i) and (ii) above we have

$$|V(P^d) \cap (V(P) \cup V(C_i))| \leq |V(P^d) \cap (V(P)| + |V(P^d) \cap V(C_i)| \leq 1,$$

and so (iii) holds. If $V(P^d) \cap V(P) \neq \phi$ and $V(P^d) \cap V(C_i) \neq \phi$, then from (i) and (ii) above we have $V(P^d) \cap V(P) = \{i_t\} = \{i_t^d\}$ $(0 \leq t \leq m)$ and $V(P^d) \cap V(C_i) = \{i_m^d\}$. Notice that if there exists a vertex i satisfying $i = i^d$, then the vertex i is unique, that is $i = \frac{n+1}{2}$. We now show that $t = m$. If $t \neq m$, then $i_m \neq i_m^d$ (since $i_t^d = i_t$), so the walks

$$i_t P i_m + i_m C_i' i_m^d + i_m^d P^d i_t^d \quad \text{and} \quad i_t P i_m + i_m C_i'' i_m^d + i_m^d P^d i_t^d$$

are two cycles, and one of the cycles is an odd cycle, denoted by C_t^*. Hence $d(i, C_t^*) = t < m = d(i, C_i)$, contradicting the definition of C_i. Therefore $t = m$. This implies that

$$V(P^d) \cap (V(P) \cup V(C_i)) = (V(P^d) \cap V(P)) \cup (V(P^d) \cap V(C_i)) = \{i_m\} = \{i_m^d\},$$

and hence (iii) also holds. □

Lemma 2.4 *Let i and j be two vertices of $G = G(A)$ with $A \in CSP(n)$. Let P_{ij} be any shortest path joining i and j, and let $C_{P_{ij}}$ be any odd cycle such that $d(P_{ij}, C_{P_{ij}}) = \min\{d(P_{ij}, C) : C$ is an odd cycle in $G\}$. Assume that $d(P_{ij}, C_{P_{ij}}) = m \geq 1$. Let $P_* = i_0 i_1 \cdots i_m$ be any shortest path between P_{ij} and $C_{P_{ij}}$, where $i_0 \in V(P_{ij})$ and $i_m \in V(C_{P_{ij}})$. Then the following hold:*

(i) *If* $V(P_*^d) \cap V(P_*) \neq \phi$, *then* $V(P_*^d) \cap V(P_*) = \{i_t\} = \{i_t^d\}$ $(0 \leq t \leq m)$.

(ii) *If* $V(P_*^d) \cap V(C_{P_{ij}}) \neq \phi$, *then* $V(P_*^d) \cap V(C_{P_{ij}}) = \{i_m^d\}$.

(iii) $|V(P_*^d) \cap (V(P_*) \cup V(C_{P_{ij}}))| \leq 1$.

(iv) *If* $V(P_*^d) \cap V(P_{ij}) \neq \phi$, *then* $V(P_*^d) \cap V(P_{ij}) = \{i_0^d\}$.

(v) $|V(P_*^d) \cap (V(P_{ij}) \cup V(P_*) \cup V(C_{P_{ij}}))| \leq 2$.

Proof. By the definition of $C_{P_{ij}}$ we have that $d(i_0, C_{P_{ij}}) = \min\{d(i_0, C) : C \text{ is an odd cycle in } G\}$, and P_* is the shortest path between i_0 and $C_{P_{ij}}$. Hence by Lemma 2.3 we conclude that (i), (ii) and (iii) hold.

We now assume that $V(P_*^d) \cap V(P_{ij}) \neq \phi$ and prove (iv). Let i_t^d be any vertex in $V(P_*^d) \cap V(P_{ij})$. If $1 \leq t \leq m$, then

$$d(P_{ij}, C_{P_{ij}}^d) \leq |i_t^d P_*^d i_m^d| = m - t < m = d(P_{ij}, C_{P_{ij}}),$$

contradicting the definition of $C_{P_{ij}}$. Hence $t = 0$, and (iv) holds.

We know from (iii) and (iv) above that

$$\begin{aligned} & |V(P_*^d) \cap (V(P_{ij}) \cup V(P_*) \cup V(C_{P_{ij}}))| \\ \leq \ & |V(P_*^d) \cap V(P_{ij})| + |V(P_*^d) \cap (V(P_*) \cup V(C_{P_{ij}}))| \leq 2. \end{aligned}$$

Hence (v) also holds. □

Lemma 2.5 *Let i be any vertex of $G = G(A)$ with $A \in CSP(n)$. Then $\gamma(i,i) \leq n-1$. In particular, $\gamma(i,i) \leq n-2$ when $n \equiv 0 (\text{mod } 2)$.*

Proof. It is trivial if i is the vertex with a loop. Let i be any vertex without loop. Clearly, there is a walk from i to i with length 2. Let C_i be any odd cycle such that $d(i, C_i) = \min\{d(i, C) : C \text{ is an odd cycle in } G\}$, and let P be the shortest path between i and C_i. Then there is a walk $W = P + C_i + P$ from i to i, its length $|W| = 2|P| + |C_i|$ is an odd number not less than 3. So by Lemma 2.2 we have

$$\gamma(i,i) \leq \max\{2|P| + |C_i|, 2\} - 1 = 2|P| + |C_i| - 1.$$

If $|P| = 0$. Then $\gamma(i,i) \leq |C_i| - 1 \leq n - 1$. In particular, if $n \equiv 0 (\text{mod } 2)$, then $|C_i| \leq n-1$, and hence $\gamma(i,i) \leq n-2$.

If $|P| \geq 1$. Then by Lemma 2.3 we have that

$$\begin{aligned} n &\geq |V(P^d) \cup V(P) \cup V(C_i)| \\ &= |V(P^d)| + |V(P) \cup V(C_i)| - |V(P^d) \cap (V(P) \cup V(C_i))| \\ &\geq (|P| + 1) + (|P| + |C_i|) - 1 \\ &= 2|P| + |C_i|. \end{aligned}$$

Therefore $\gamma(i,i) \leq 2|P| + |C_i| - 1 \leq n - 1$. In particular, if $n \equiv 0 (\mathrm{mod}\, 2)$, then the odd number $2|P| + |C_i| \leq n - 1$, and hence $\gamma(i,i) \leq n - 2$. □

Lemma 2.6 *Let i and j be two vertices of $G = G(A)$ with $A \in CSP(n)$. Then $\gamma(i,j) \leq n - 1$.*

Proof. Let P_{ij} be any shortest path joining i and j, and let $C_{P_{ij}}$ be any odd cycle such that $d(P_{ij}, C_{P_{ij}}) = \min\{d(P_{ij}, C) : C \textit{ is an odd cycle in } G\}$. Let P_* be any shortest path between P_{ij} and $C_{P_{ij}}$. We consider two cases.

Case 1: $|P_*| = 0$. Let $x, y \in V(P_{ij}) \cap V(C_{P_{ij}})$ (perhaps $x = y$), where x (y) is the first (last) vertex on $C_{P_{ij}}$ along P_{ij}. Then the lengths of walks $iP_{ij}x + xC'_{P_{ij}}y + yP_{ij}j$ and $iP_{ij}x + xC''_{P_{ij}}y + yP_{ij}j$ have different parity and not greater than n. So by Lemma 2.2 we have that $\gamma(i,j) \leq n - 1$. In particular, if $|V(P_{ij}) \cap V(C_{P_{ij}})| \geq 2$, then $\gamma(i,j) \leq n - 2$.

Case 2: $|P_*| = m \geq 1$. Let $P_* = i_0 i_1 \cdots i_m$, where $i_0 \in V(P_{ij})$ and $i_m \in V(C_{P_{ij}})$. We have by Lemma 2.4 that $|V(P_*^d) \cap (V(P_{ij}) \cup V(P_*) \cup V(C_{P_{ij}}))| \leq 2$.

Subcase 2.1: $|V(P_*^d) \cap (V(P_{ij}) \cup V(P_*) \cup V(C_{P_{ij}}))| \leq 1$. Then

$$\begin{aligned} n &\geq |V(P_*^d) \cup (V(P_{ij}) \cup V(P_*) \cup V(C_{P_{ij}}))| \\ &= |V(P_*^d)| + |V(P_{ij}) \cup V(P_*) \cup V(C_{P_{ij}})| \\ &\qquad - |V(P_*^d) \cap (V(P_{ij}) \cup V(P_*) \cup V(C_{P_{ij}}))| \\ &\geq (|P_*^d| + 1) + (|P_{ij}| + |P_*| + |C_{P_{ij}}|) - 1 \\ &= |P_{ij}| + |C_{P_{ij}}| + 2|P_*|. \end{aligned}$$

Since the lengths of the walks

$$iP_{ij}i_0 + i_0P_*i_m + C_{P_{ij}} + i_mP_*i_0 + i_0P_{ij}j \quad \text{and} \quad P_{ij}$$

have different parity, and

$$|iP_{ij}i_0 + i_0P_*i_m + C_{P_{ij}} + i_mP_*i_0 + i_0P_{ij}j| = |P_{ij}| + |C_{P_{ij}}| + 2|P_*| > |P_{ij}|,$$

it follows from Lemma 2.2 that $\gamma(i,j) \le |P_{ij}|+|C_{P_{ij}}|+2|P_*|-1$. Therefore $\gamma(i,j) \le |P_{ij}|+|C_{P_{ij}}|+2|P_*|-1 \le n-1$. In particular, if $|V(P_*^d)\cap(V(P_{ij})\cup V(P_*)\cup V(C_{P_{ij}}))| = 0$, then $\gamma(i,j) \le n-2$ since $|P_{ij}|+|C_{P_{ij}}|+2|P_*|+1 \le n$.

Subcase 2.2: $|V(P_*^d)\cap(V(P_{ij})\cup V(P_*)\cup V(C_{P_{ij}}))| = 2$. Since $V(P_*^d)\cap(V(P_{ij})\cup V(P_*)\cup V(C_{P_{ij}})) = (V(P_*^d)\cap V(P_{ij}))\cup(V(P_*^d)\cap(V(P_*)\cup V(C_{P_{ij}})))$, it follows from Lemma 2.4 that $V(P_*^d)\cap V(P_{ij}) = \{i_0^d\}$ and $V(P_*^d)\cap(V(P_*)\cup V(C_{P_{ij}})) = \{i_t^d\}$, where $1 \le t \le m$.

We now show that $i_0 \ne i_0^d, \quad i_t = i_t^d$ and $|i_0 P_{ij} i_0^d| \ge 2$.

If $i_0 = i_0^d$, then $i_0^d \in V(P_*^d)\cap V(P_*) \subseteq V(P_*^d)\cap(V(P_*)\cup V(C_{P_{ij}}))$, we obtain the contradiction $|V(P_*^d)\cap(V(P_{ij})\cup V(P_*)\cup V(C_{P_{ij}}))| = 1$. Hence $i_0 \ne i_0^d$.

If $i_t \ne i_t^d$, then $V(P_*^d)\cap V(P_*) = \phi$, and $V(P_*^d)\cap V(C_{P_{ij}}) = V(P_*^d)\cap(V(P_*)\cup V(C_{P_{ij}})) = \{i_t^d\}$. It follows from (ii) of Lemma 2.4 that $t = m$. So the walks $i_0P_*i_m + i_mC'_{P_{ij}}i_m^d + i_m^dP_*^di_0^d + i_0^dP_{ij}i_0$ and $i_0P_*i_m + i_mC''_{P_{ij}}i_m^d + i_m^dP_*^di_0^d + i_0^dP_{ij}i_0$ are two cycles, and one of these cycles is an odd cycle. This contradictions the definition of $C_{P_{ij}}$. Thus $i_t = i_t^d$.

If $|i_0P_{ij}i_0^d| < 2$, then $|i_0P_{ij}i_0^d| = 1$ (since $i_0 \ne i_0^d$). It follows from $i_t = i_t^d$ that the cycle $i_0P_*i_t + i_t^dP_*^di_0^d + i_0^dP_{ij}i_0$ is an odd cycle, contradicting the definition of $C_{P_{ij}}$. Hence $|i_0P_{ij}i_0^d| \ge 2$.

We now assume without loss of generality that $P_{ij} = iP_{ij}i_0 + i_0P_{ij}i_0^d + i_0^dP_{ij}j$. We consider the walks $W_1 = iP_{ij}i_0 + P_* + C_{P_{ij}} + i_mP_*i_t + i_t^dP_*^di_0^d + i_0^dP_{ij}j$ and $W_2 = iP_{ij}i_0 + i_0P_*i_t + i_t^dP_*^di_0^d + i_0^dP_{ij}j$.

Clearly, $|W_1| = |iP_{ij}i_0| + |i_0^dP_{ij}j| + |C_{P_{ij}}| + 2|P_*|$, $|W_2| = |iP_{ij}i_0| + |i_0^dP_{ij}j| + 2|i_0P_*i_t|$. So $|W_1|$ and $|W_2|$ have different parity, and $|W_1| > |W_2|$. Hence by Lemma 2.2 we have

$$\begin{aligned}\gamma(i,j) &\le |iP_{ij}i_0| + |i_0^dP_{ij}j| + |C_{P_{ij}}| + 2|P_*| - 1\\ &= |P_{ij}| - |i_0P_{ij}i_0^d| + |C_{P_{ij}}| + 2|P_*| - 1\\ &\le |P_{ij}| + |C_{P_{ij}}| + 2|P_*| - 3.\end{aligned}$$

By a similar argument as in subcase 2.1, we have $|P_{ij}|+|C_{P_{ij}}|+2|P_*| \le n+1$. Thus $\gamma(i,j) \le |P_{ij}| + |C_{P_{ij}}| + 2|P_*| - 3 \le n-2$. □

Theorem 2.1 *Let* $\gamma_n = \max\{\gamma(A) : A \in CSP(n)\}$. *Then* $\gamma_n = n-1$.

Proof. Let A be any matrix in $CSP(n)$, and let $i, j \in V(G(A))$. Then by Lemma 2.1, Lemma 2.5 and Lemma 2.6, we have

$$\gamma(A) = \gamma(G(A)) = \max\{\gamma(i,j) : i,j \in V(G(A))\} \leq n-1.$$

Now let $G = (V, E)$ be a graph, where $V = \{1, 2, \cdots, n\}$, $E = \{[i, i+1] : 1 \leq i \leq n-1\} \cup \{[1,1], [n,n]\}$, and let $A(G)$ be the adjacency matrix of G. Clearly, $A(G) \in CSP(n)$, and $\gamma(A(G)) = \gamma(G) = \max_{i,j \in V} \gamma(i,j) = \gamma(1,n) = d(1,n) = n-1$. Therefore $\gamma_n = n-1$. □

3 The extremal matrices

In this section we give a complete characterization of those central symmetric primitive matrices having the maximum exponent $n-1$, that is the below Theorem 3.1. Before proving it we establish some lemmas.

Lemma 3.1 *Let G be a primitive graph, and let i be any vertex of G. If i is the vertex without loop, then there is an odd cycle C_i^0 in G such that $\gamma(i,i) = 2d(i, C_i^0) + |C_i^0| - 1$.*

Proof. Since G is primitive, G contains at least one odd cycle. Then there exists an odd cycle C_i^0 such that

$$2d(i, C_i^0) + |C_i^0| = \min\{2d(i, C) + |C| : C \textit{ is an odd cycle in } G\}.$$

Clearly, there are two walks from i to i with lengths $2d(i, C_i^0) + |C_i^0|$ and 2, respectively. Since i is the vertex without loop, $2d(i, C_i^0) + |C_i^0| \geq 3$. By Lemma 2.2 we have $\gamma(i,i) \leq \max\{2d(i, C_i^0) + |C_i^0|, 2\} - 1 = 2d(i, C_i^0) + |C_i^0| - 1$. Conversely, it is clear that there is no any walk from i to i with length $2d(i, C_i^0) + |C_i^0| - 2$, so $\gamma(i,i) \geq 2d(i, C_i^0) + |C_i^0| - 1$. Hence $\gamma(i,i) = 2d(i, C_i^0) + |C_i^0| - 1$. □

Lemma 3.2 *Let i and j be two vertices of $G = G(A)$ with $A \in CSP(n)$, and let P_{ij} be any shortest path joining i and j. If $V(P_{ij}^d) = V(P_{ij})$. Then $j = i^d$, and $|iP_{ij}t| = |t^dP_{ij}j|$ for each vertex $t \in V(P_{ij})$.*

Proof. If $j \neq i^d$, then $i \neq j^d$, and we obtain the contradiction $d(i^d, j^d) = |i^dP_{ij}j^d| < d(i,j)$. Hence $j = i^d$, and hence for each vertex $t \in V(P_{ij})$, we have $|iP_{ij}t| = d(i,t) = d(t^d, i^d) = |t^dP_{ij}^d i^d| = |t^dP_{ij}j|$. □

Lemma 3.3 *Let $G = G(A)$ with $A \in CSP(n)$. Assume that $\gamma(G) = n-1$ and $\gamma(i,i) = n-1$ for some vertex i of G. Let C_i be any odd cycle such that $d(i, C_i) = \min\{d(i,C) : C$ is an odd cycle in $G\}$. Then the following hold:*

(i) *If $V(C_i) \cap V(C_i^d) \neq \phi$, then $C_i = C_i^d$.*

(ii) *If $V(C_i) \cap V(C_i^d) = \phi$, then $|C_i| = |C_i^d| = 1$.*

Proof. Clearly, if $i \in V(C_i)$, then $n-1 = \gamma(i,i) \leq \max\{|C_i|, 2\} - 1 = |C_i| - 1$ (since $n \geq 3$). We conclude that $|C_i| = n$ and so $C_i = C_i^d$.

Now let $i \notin V(C_i)$, and let $P = i_0 i_1 \cdots i_m$ be the shortest path between i and C_i, where $i_0 = i$ and $i_m \in V(C_i)$. Since $|P| \geq 1$ and $\gamma(i,i) = n-1$, according to the proof in Lemma 2.5 we have that $|V(P^d) \cap (V(P) \cup V(C_i))| = 1$ and $|V(P^d) \cup V(P) \cup V(C_i)| = 2|P| + |C_i| = n$. Hence $V(C_i^d) \subseteq V(P^d) \cup V(P) \cup V(C_i) = V(G)$. Since $V(C_i^d) \cap V(P^d) = \{i_m^d\}$ and $V(C_i^d) \cap V(P) \subseteq \{i_m\} \subseteq V(C_i)$, it follows that $V(C_i^d) \backslash \{i_m^d\} \subseteq V(C_i)$.

We now assume that $V(C_i) \cap V(C_i^d) \neq \phi$ and prove that $C_i = C_i^d$. Clearly, we only need to show that $i_m^d \in V(C_i)$. It is trivial if $|C_i| = 1$. Let $|C_i| \geq 3$. Suppose that $i_m^d \notin V(C_i)$. Then $i_m \notin V(C_i^d)$ and, by (ii) of Lemma 2.3 we have $V(P^d) \cap V(C_i) = \phi$. So $|V(P^d) \cap V(P)| = |V(P^d) \cap (V(P) \cup V(C_i))| = 1$, and we do have that a path $P(i_m, i_m^d)$ from i_m to i_m^d (in $C_i \cup C_i^d$) with odd length $|C_i|$. By (i) and (ii) of Lemma 2.3 we conclude that $V(P^d) \cap V(P) = \{i_t\} = \{i_t^d\}$, where $0 \leq t \leq m-1$. So the walk $i_t P i_m + P(i_m, i_m^d) + i_m^d P^d i_t^d$ is an odd cycle, denoted by C_t^*, and so $d(i, C_t^*) = t < m = d(i, C_i)$, contradicting the definition of C_i. Thus $i_m^d \in V(C_i)$, and hence $C_i^d = C_i$.

We now assume that $V(C_i) \cap V(C_i^d) = \phi$ and prove $|C_i| = |C_i^d| = 1$. Since $V(C_i^d) \backslash \{i_m^d\} \subseteq V(C_i)$, we have $V(C_i^d) \backslash \{i_m^d\} = \phi$. Hence $V(C_i^d) = \{i_m^d\}$, and hence $|C_i| = |C_i^d| = 1$. The lemma now follows. □

Lemma 3.4 *Let $G = G(A)$ with $A \in CSP(n)$. Assume that $\gamma(G) = n-1$ and $\gamma(i,j) = n-1$ for two vertices $i, j \in V(G)$. Let P_{ij} be any shortest path joining i and j, and let $C_{P_{ij}}$ be any odd cycle such that $d(P_{ij}, C_{P_{ij}}) = \min\{d(P_{ij}, C) : C$ is an odd cycle in $G\}$. Then the following hold:*

(i) *If $V(C_{P_{ij}}) \cap V(C_{P_{ij}}^d) \neq \phi$, then $C_{P_{ij}} = C_{P_{ij}}^d$.*

(ii) *If $V(C_{P_{ij}}) \cap V(C_{P_{ij}}^d) = \phi$, then $|C_{P_{ij}}| = |C_{P_{ij}}^d| = 1$.*

Proof. Let P_* be any shortest path between P_{ij} and $C_{P_{ij}}$. We consider two cases.

Case 1: $|P_*| = 0$. Since $\gamma(i,j) = n-1$, according to the proof in Lemma 2.6 we have that $|V(C_{P_{ij}}) \cap V(P_{ij})| = 1$ and $\gamma(i,j) \le |P_{ij}| + |C_{P_{ij}}| - 1$. It follows that $|P_{ij}| + |C_{P_{ij}}| = n$ and $|V(P_{ij}) \cup V(C_{P_{ij}})| = n$. So $V(C^d_{P_{ij}}) \subseteq V(P_{ij}) \cup V(C_{P_{ij}})$.

We first assume that $V(C_{P_{ij}}) \cap V(C^d_{P_{ij}}) \ne \phi$ and prove $C_{P_{ij}} = C^d_{P_{ij}}$. Let $V(C_{P_{ij}}) \cap V(P_{ij}) = \{x\}$. If $V(C^d_{P_{ij}}) \backslash V(C_{P_{ij}}) \ne \phi$, then $V(C^d_{P_{ij}}) \backslash V(C_{P_{ij}}) = \{y\} \subseteq V(P_{ij})$, and $y \ne x$. So we have two paths P' and P'' (in $C_{P_{ij}} \cup C^d_{P_{ij}}$) from x to y with lengths $|P'| = 2$ and $|P''| = |C_{P_{ij}}|$, respectively. By Lemma 2.2 we have $\gamma(i,j) \le (|C_{P_{ij}}| + |P_{ij}| - |xP_{ij}y|) - 1 \le n-2$, a contradiction. Hence $V(C^d_{P_{ij}}) \backslash V(C_{P_{ij}}) = \phi$, and hence $C_{P_{ij}} = C^d_{P_{ij}}$.

We now assume that $V(C_{P_{ij}}) \cap V(C^d_{P_{ij}}) = \phi$. Then $V(C^d_{P_{ij}}) \subseteq V(P_{ij})$. Hence $|V(C^d_{P_{ij}})| = 1$, and hence $|C_{P_{ij}}| = |C^d_{P_{ij}}| = 1$.

Case 2: $|P_*| = m > 0$. Let $P_* = i_0 i_1 \cdots i_m$, where $i_0 \in V(P_{ij})$ and $i_m \in V(C_{P_{ij}})$. According to the proof in Lemma 2.6 we have that $|V(P^d_*) \cap (V(P_{ij}) \cup V(P_*) \cup V(C_{P_{ij}}))| = 1$ and $|V(P^d_*) \cup V(P_{ij}) \cup V(P_*) \cup V(C_{P_{ij}})| = |P_{ij}| + |C_{P_{ij}}| + 2|P_*| = n$. Hence $V(C^d_{P_{ij}}) \subseteq V(P^d_*) \cup V(P_{ij}) \cup V(P_*) \cup V(C_{P_{ij}}) = V(G)$. Since $|P_*| > 0$, $d(P_{ij}, C^d_{P_{ij}}) \ge d(P_{ij}, C_{P_{ij}}) = |P_*| > 0$. Since $V(C^d_{P_{ij}}) \cap V(P^d_*) = \{i^d_m\}$ and $V(C^d_{P_{ij}}) \cap V(P_*) \subseteq \{i_m\} \subseteq V(C_{P_{ij}})$, it follows that $V(C^d_{P_{ij}}) \backslash \{i^d_m\} \subseteq V(C_{P_{ij}})$.

We assume that $V(C_{P_{ij}}) \cap V(C^d_{P_{ij}}) \ne \phi$ and prove $C_{P_{ij}} = C^d_{P_{ij}}$. Clearly, we only need to show that $i^d_m \in V(C_{P_{ij}})$. Suppose that $i^d_m \notin V(C_{P_{ij}})$. Then by (ii) of Lemma 2.4 we have $V(C_{P_{ij}}) \cap V(P^d_*) = \phi$. So $|V(P^d_*) \cap (V(P_{ij}) \cup V(P_*))| = |V(P^d_*) \cap (V(P_{ij}) \cup V(P_*) \cup V(C_{P_{ij}}))| = 1$, and we do have two paths P' and P'' (in $C_{P_{ij}} \cup C^d_{P_{ij}}$) from i_m to i^d_m with lengths $|P'| = 2$ and $|P''| = |C_{P_{ij}}|$, respectively.

If $V(P^d_*) \cap V(P_*) = \phi$, then $|V(P^d_*) \cap V(P_{ij})| = |V(P^d_*) \cap (V(P_{ij}) \cup V(P_*))| = 1$. It follows from (iv) of Lemma 2.4 that $V(P^d_*) \cap V(P_{ij}) = \{i^d_0\}$. Hence one of the walks

$$i_0 P_* i_m + P' + i^d_m P^d_* i^d_0 + i^d_0 P_{ij} i_0 \quad \text{and} \quad i_0 P_* i_m + P'' + i^d_m P^d_* i^d_0 + i^d_0 P_{ij} i_0$$

is odd cycle, contradicting the definition of $C_{P_{ij}}$ and $d(P_{ij}, C_{P_{ij}}) = m > 0$.

If $V(P^d_*) \cap V(P_*) \ne \phi$, then from (i) and (ii) of Lemma 2.4 we have $V(P^d_*) \cap V(P_*) = \{i^d_t\} = \{i_t\}$, where $0 \le t \le m-1$. So the walk $C^*_t =$

$i_tP_*i_m+P''+i^d_mP^d_*i^d_t$ is an odd cycle, and $d(P_{ij}, C^*_t) = t < m$, contradicting the definition of $C_{P_{ij}}$.

Thus $i^d_m \in V(C_{P_{ij}})$, and hence $C_{P_{ij}} = C^d_{P_{ij}}$.

We now assume that $V(C_{P_{ij}}) \cap V(C^d_{P_{ij}}) = \phi$. We then have $V(C^d_{P_{ij}}) \backslash \{i^d_m\} = \phi$ (since $V(C^d_{P_{ij}}) \backslash \{i^d_m\} \subseteq V(C_{P_{ij}})$). Hence $V(C^d_{P_{ij}}) = \{i^d_m\}$, and hence $|C_{P_{ij}}| = |C^d_{P_{ij}}| = 1$. The lemma now follows. □

Lemma 3.5 *Let $G = G(A)$ with $A \in CSP(n)$. Assume that $\gamma(G) = n-1$ and $\gamma(i,j) = n-1$ for two vertices $i, j \in V(G)$. Let P_{ij} be any shortest path joining i and j, and let $C_{P_{ij}}$ be any odd cycle such that $d(P_{ij}, C_{P_{ij}}) = \min\{d(P_{ij}, C) : C$ is an odd cycle in $G\}$. Assume that $|P_{ij}| < n-1$ and $V(C_{P_{ij}}) \cap V(C^d_{P_{ij}}) \neq \phi$. Then $\gamma(i,i) = \gamma(j,j) = n-1$.*

Proof. Let P_* be the shortest path between P_{ij} and $C_{P_{ij}}$. According to the proof in Lemma 3.4 we have $V(P^d_*) \cup V(P_{ij}) \cup V(P_*) \cup V(C_{P_{ij}}) = V(G)$ and $|P_{ij}| + |C_{P_{ij}}| + 2|P_*| = n$ (In particular, if $|P_*| = 0$, then $V(P_{ij}) \cup V(C_{P_{ij}}) = V(G)$ and $|P_{ij}| + |C_{P_{ij}}| = n$). Since $V(C_{P_{ij}}) \cap V(C^d_{P_{ij}}) \neq \phi$, we have by (i) of Lemma 3.4 that $C_{P_{ij}} = C^d_{P_{ij}}$. Since $\gamma(i,j) = n-1$ and $|P_{ij}| < n-1$, we have that there is no any vertex in $V(G) \backslash V(C_{P_{ij}})$ with loop, and $V(C) \subseteq V(C_{P_{ij}})$ for any odd cycle C in G.

Let $i_0 = V(P_*) \cap V(P_{ij})$ (In particular, if $|P_*| = 0$ then $i_0 = V(P_{ij}) \cap V(C_{P_{ij}})$). Clearly, $d(i, C_{P_{ij}}) \leq |iP_{ij}i_0| + |P_*|$, $d(j, C_{P_{ij}}) \leq |P_*| + |i_0P_{ij}j|$. So we can use Lemma 2.2 to obtain $\gamma(i,j) \leq d(i, C_{P_{ij}}) + |C_{P_{ij}}| + d(j, C_{P_{ij}}) - 1$.

If $d(i, C_{P_{ij}}) < |iP_{ij}i_0| + |P_*|$. Then $\gamma(i,j) < |P_{ij}| + 2|P_*| + |C_{P_{ij}}| - 1 = n-1$, a contradiction. Hence $d(i, C_{P_{ij}}) = |iP_{ij}i_0| + |P_*|$.

Now Let C^0_i be the odd cycle such that $\gamma(i,i) = 2d(i, C^0_i) + |C^0_i| - 1$ (see Lemma 3.1). Then by $V(C^0_i) \subseteq V(C_{P_{ij}})$ we have that $d(i, C^0_i) \leq d(i, C_{P_{ij}}) + \frac{|C_{P_{ij}}| - |C^0_i|}{2}$ and $d(j, C^0_i) \leq d(j, C_{P_{ij}}) + \frac{|C_{P_{ij}}| - |C^0_i|}{2}$.

If $d(i, C^0_i) < d(i, C_{P_{ij}}) + \frac{|C_{P_{ij}}| - |C^0_i|}{2}$. Then $\gamma(i,j) \leq d(i, C^0_i) + d(j, C^0_i) + |C^0_i| - 1 < d(i, C_{P_{ij}}) + \frac{|C_{P_{ij}}| - |C^0_i|}{2} + d(j, C_{P_{ij}}) + \frac{|C_{P_{ij}}| - |C^0_i|}{2} + |C^0_i| - 1 = d(i, C_{P_{ij}}) + d(j, C_{P_{ij}}) + |C_{P_{ij}}| - 1 \leq |P_{ij}| + 2|P_*| + |C_{P_{ij}}| - 1 = n-1$, a contradiction. Hence $d(i, C^0_i) = d(i, C_{P_{ij}}) + \frac{|C_{P_{ij}}| - |C^0_i|}{2} = |iP_{ij}i_0| + |P_*| + \frac{|C_{P_{ij}}| - |C^0_i|}{2}$, and hence $\gamma(i,i) = 2d(i, C^0_i) + |C^0_i| - 1 = 2(|iP_{ij}i_0| + |P_*| +$

$\frac{|C_{P_{ij}}|-|C_i^0|}{2}) + |C_i^0| - 1 = 2|iP_{ij}i_0| + 2|P_*| + |C_{P_{ij}}| - 1$.

Similarly, we have $\gamma(j,j) = 2|i_0P_{ij}j| + 2|P_*| + |C_{P_{ij}}| - 1$. So $\gamma(i,i) + \gamma(j,j) = 2(|P_{ij}| + 2|P_*| + |C_{P_{ij}}|) - 2 = 2(n-1)$. Since $\gamma(i,i) \leq n-1$ and $\gamma(j,j) \leq n-1$, it follows that $\gamma(i,i) = \gamma(j,j) = n-1$. □

Lemma 3.6 *Let $G = G(A)$ with $A \in CSP(n)$. Assume that $\gamma(G) = n-1$ and $\gamma(i,j) = n-1$ for two vertices $i, j \in V(G)$. Let P_{ij} be any shortest path joining i and j, and let $C_{P_{ij}}$ be any odd cycle such that $d(P_{ij}, C_{P_{ij}}) = \min\{d(P_{ij}, C) : C$ is an odd cycle in $G\}$. Let P_* be any shortest path between P_{ij} and $C_{P_{ij}}$. Assume that $|P_*| = m \geq 1$ and $V(P_*) \cap V(P_*^d) \neq \phi$. Then $\gamma(i,i) = \gamma(j,j) = n-1$.*

Proof. Let $P_* = i_0i_1 \cdots i_m$. We have by (i) of Lemma 2.4 that $V(P_*^d) \cap V(P_*) = \{i_t\} = \{i_t^d\}$ $(0 \leq t \leq m)$. We consider two cases.

Case 1: $V(C_{P_{ij}}) \cap V(C_{P_{ij}}^d) \neq \phi$. Then by Lemma 3.5 we have $\gamma(i,i) = \gamma(j,j) = n-1$.

Case 2: $V(C_{P_{ij}}) \cap V(C_{P_{ij}}^d) = \phi$. Then by (ii) of Lemma 3.4 we have $|C_{P_{ij}}| = |C_{P_{ij}}^d| = 1$. According to the proof in Lemma 3.4 we have $V(P_*^d) \cup V(P_{ij}) \cup V(P_*) \cup V(C_{P_{ij}}) = V(G)$ and $|P_{ij}| + |C_{P_{ij}}| + 2|P_*| = n$. So $V(P_*^d) \cup V(P_{ij}) \cup V(P_*) = V(G)$ and $|P_{ij}| + 2|P_*| + 1 = n$ (since $|C_{P_{ij}}| = 1$). This implies that G contains exactly two odd cycles $C_{P_{ij}}$ and $C_{P_{ij}}^d$.

Now we show that $d(i_0, C_{P_{ij}}^d) = |P_*|$. Clearly, $d(i_0, C_{P_{ij}}^d) \leq |i_0P_*i_t| + |i_t^dP_*^di_m^d| = |P_*|$, on the other hand, $d(i_0, C_{P_{ij}}^d) \geq d(i_0, C_{P_{ij}}) = |P_*|$ by the definition of $C_{P_{ij}}$. Hance $d(i_0, C_{P_{ij}}^d) = |P_*|$.

Now let C_i^0 be the odd cycle such that $\gamma(i,i) = 2d(i, C_i^0) + |C_i^0| - 1$. Notice that either $C_i^0 = C_{P_{ij}}$ or $C_i^0 = C_{P_{ij}}^d$, so $\gamma(i,i) = 2d(i, C_i^0)$ and $d(i_0, C_i^0) = |P_*|$. Hence $d(i, C_i^0) \leq |iP_{ij}i_0| + d(i_0, C_i^0) = |iP_{ij}i_0| + |P_*|$.

If $d(i, C_i^0) < |iP_{ij}i_0| + |P_*|$. Then by Lemma 2.2 we obtain $\gamma(i,j) \leq d(i, C_i^0) + d(i_0, C_i^0) + |i_0P_{ij}j| < |P_{ij}| + 2|P_*| = n-1$. This contradicts the condition $\gamma(i,j) = n-1$. Hence $d(i, C_i^0) = |iP_{ij}i_0| + |P_*|$, and hence $\gamma(i,i) = 2d(i, C_i^0) = 2|iP_{ij}i_0| + 2|P_*|$.

Similarly, we have that $\gamma(j,j) = 2|i_0P_{ij}j| + 2|P_*|$.

Thus, $\gamma(i,i) + \gamma(j,j) = 2(|P_{ij}| + 2|P_*|) = 2(n-1)$. Since $\gamma(i,i) \leq n-1$ and $\gamma(j,j) \leq n-1$, it follows that $\gamma(i,i) = \gamma(j,j) = n-1$. □

We now construct two classes of graphs $\mathfrak{H}(n,l,s)$ $(0 \le l \le \lfloor \frac{n}{2} \rfloor - 1$, $l+1 \le s \le \lceil \frac{n}{2} \rceil)$ and $\mathfrak{R}(n,r,s)$ (n is odd, $0 \le r \le \frac{n-1}{2}$, $r+1 \le s \le \frac{n+1}{2}$).

Let l be an integer with $0 \le l \le \lfloor \frac{n}{2} \rfloor - 1$, and let s be an integer with $l+1 \le s \le \lceil \frac{n}{2} \rceil$, where $\lfloor a \rfloor$ denotes the largest integer not exceeding a, and $\lceil b \rceil$ denotes the smallest integer not less than b. Let $\mathfrak{H}^*(n,l,s) = (V,E)$, where $V = \{1, 2, \cdots, n\}$ and $E = \{[i,i+1] : i \in V \backslash \{l, n-l, n\}\} \cup \{[l+1, l+1], [n-l, n-l], [l,s], [n+1-l, n+1-s]\}$. Clearly, $\mathfrak{H}^*(n,0,1) = \mathfrak{H}^*(n,0,s)$ for $2 \le s \le \lceil \frac{n}{2} \rceil$. We now construct the class of graphs $\mathfrak{H}(n,l,s)$ as follows: $\mathfrak{H}(n,l,l+1)$ is obtained from $\mathfrak{H}^*(n,l,l+1)$ by putting some pairs of edges (possibly empty) such as $[i,i]$ and $[n+1-i, n+1-i]$ $(l+2 \le i \le \lceil \frac{n}{2} \rceil)$, in particular, if n is odd, then $\mathfrak{H}(n,0,1)$ is obtained from $\mathfrak{H}^*(n,0,1)$ by putting at least one pair of edges such as $[i,i]$ and $[n+1-i, n+1-i]$ $(2 \le i \le \frac{n+1}{2})$;when n is even, $\mathfrak{H}(n,l,\frac{n}{2})$ is obtained from $\mathfrak{H}^*(n,l,\frac{n}{2})$ by putting some edges (possibly empty) such as $[i, n+1-i]$ $(l+1 \le i \le \frac{n}{2} - 1)$; when n is odd, $\mathfrak{H}(n,l,\frac{n+1}{2})$ is obtained from $\mathfrak{H}^*(n,l,\frac{n+1}{2})$ by putting some pairs of edges (possibly empty) such as $[i, n-i]$ and $[i+1, n+1-i]$ $(1 \le i \le l-1$ or $l+1 \le i \le \frac{n+1}{2} - 2)$; $\mathfrak{H}(n,l,s) = \mathfrak{H}^*(n,l,s)$ for $l+2 \le s \le \lceil \frac{n}{2} \rceil - 1$.

Now suppose n is odd. Let r be an integer with $0 \le r \le \frac{n-1}{2}$, and let s be an integer with $r+1 \le s \le \frac{n+1}{2}$. Let $\mathfrak{R}^*(n,r,s) = (V,E)$, where $V = \{1, 2, \cdots, n\}$ and $E = \{[i, i+1] : i \in V \backslash \{r, n-r, n\}\} \cup \{[r+1, n-r], [r,s], [n+1-r, n+1-s]\}$. Then we construct the class of graphs $\mathfrak{R}(n,r,s)$ as follows: $\mathfrak{R}(n,r,\frac{n+1}{2})$ is obtained from $\mathfrak{R}^*(n,r,\frac{n+1}{2})$ by putting some pairs of edges (possibly empty) such as $[i, n-i]$ and $[i+1, n+1-i]$ $(1 \le i \le r-1$ or $r+1 \le i \le \frac{n+1}{2} - 2)$, or two loops $[r+1, r+1]$ and $[n-r, n-r]$; $\mathfrak{R}(n,r,r+1)$ $(0 \le r \le \frac{n-3}{2})$ is obtained from $\mathfrak{R}^*(n,r,r+1)$ $(0 \le r \le \frac{n-3}{2})$ by putting some edges (possibly empty) such as $[i, n+1-i]$ $(r+2 \le i \le \frac{n+1}{2})$; $\mathfrak{R}(n,r,s) = \mathfrak{R}^*(n,r,s)$ for $r+2 \le s \le \frac{n-1}{2}$.

Lemma 3.7 *Let G be any graph in $\mathfrak{H}(n,l,s) \cup \mathfrak{R}(n,r,s)$. Then $A(G) \in CSP(n)$ and $\gamma(A(G)) = \gamma(G) = n-1$, where $A(G)$ is the adjacency matrix of G.*

Proof. By the definitions of $\mathfrak{H}(n,l,s)$ and $\mathfrak{R}(n,r,s)$, we have that $A(G) \in CSP(n)$. So by Lemma 2.1 and Theorem 2.1 we only need to show that $\gamma(i,j) = n-1$ for some pair of vertices i and j (not necessarily distinct) of G. This can be obtained by direct verification as follows:

If $G \in \mathfrak{H}(n,l,s)$ $(0 \le l \le \lfloor \frac{n}{2} \rfloor - 1, l+1 \le s \le \lceil \frac{n}{2} \rceil - 1)$, then $\gamma(1,n) = n-1$.

If n is even and $G \in \mathfrak{H}(n,0,\frac{n}{2})$, then $\gamma(\frac{n}{2}, \frac{n+2}{2}) = n-1$.

If n is even and $G \in \mathfrak{H}(n, l, \frac{n}{2})$ $(1 \leq l \leq \frac{n}{2} - 1)$, then $\gamma(1, n) = n - 1$.

If n is odd and $G \in \mathfrak{H}(n, 0, \frac{n+1}{2})$, then $\gamma(\frac{n+1}{2}, \frac{n+1}{2}) = n - 1$.

If n is odd and $G \in \mathfrak{H}(n, l, \frac{n+1}{2})$ $(1 \leq l \leq \frac{n-3}{2})$, then $\gamma(1, 1) = n - 1$.

If $G \in \mathfrak{R}(n, r, s)$ $(1 \leq r \leq \frac{n-1}{2})$, then $\gamma(1, 1) = n - 1$.

If $G \in \mathfrak{R}(n, 0, s)$ $(1 \leq s \leq \frac{n+1}{2})$, then $\gamma(s, s) = n - 1$. □

Lemma 3.8 *Let $G = G(A)$ with $A \in CSP(n)$. Assume that $\gamma(G) = n - 1$ and $\gamma(i, i) = n - 1$ for some vertex $i \in V(G)$. Then G is isomorphic to some graph in $\mathfrak{H}(n, l, \frac{n+1}{2})$ $(0 \leq l \leq \frac{n-3}{2})$, or G is isomorphic to some graph in $\mathfrak{R}(n, r, s)$ $(0 \leq r \leq \frac{n-1}{2}, r + 1 \leq s \leq \frac{n+1}{2})$.*

Proof. Since $\gamma(i, i) = n - 1$ for some vertex $i \in V(G)$, it follows from Lemma 2.5 that $n \equiv 1 (\text{mod} 2)$. Suppose that $\gamma(w, w) = \gamma(G) = n - 1$ for a vertex $w \in V(G)$. Then $\gamma(w^d, w^d) = \gamma(w, w) = n - 1$. Let C_w be an odd cycle such that $d(w, C_w) = \min\{d(w, C) : C \textit{ is an odd cycle in } G\}$. We consider two cases.

Case 1: $V(C_w) \cap V(C_w^d) = \phi$. Then by (ii) of Lemma 3.3 we have $|C_w| = |C_w^d| = 1$. So by Lemma 3.1 we conclude that $2d(w, C_w) = \gamma(w, w) = n - 1$ and $2d(w^d, C_w^d) = \gamma(w^d, w^d) = n - 1$.

Now let $P = i_0 i_1 \cdots i_{\frac{n-1}{2}}$ be the shortest path between w and C_w, where $i_0 = w$ and $i_{\frac{n-1}{2}} \in V(C_w)$ (and thus $C_w = [i_{\frac{n-1}{2}}, i_{\frac{n-1}{2}}]$). If $V(P) \cap V(P^d) = \phi$, then $|V(P) \cup V(P^d)| = |V(P)| + |V(P^d)| = n + 1$, a contradiction. Hence $V(P) \cap V(P^d) \neq \phi$. It follows from (i) of Lemma 2.3 that $V(P^d) \cap V(P) = \{i_l\} = \{i_l^d\}$, where $0 \leq l \leq \frac{n-3}{2}$ (since $i_{\frac{n-1}{2}} \neq i_{\frac{n-1}{2}}^d$). Thus $V(P) \cup V(P^d) = V(G)$, and G contains a spanning subgraph G^* isomorphic to $\mathfrak{H}^*(n, l, \frac{n+1}{2})$ $(0 \leq l \leq \frac{n-3}{2})$.

Clearly, there is no any vertex in $V(G) \backslash \{i_{\frac{n-1}{2}}, i_{\frac{n-1}{2}}^d\}$ with loop. Let i_p and i_q be two vertices in $V(P)$. If $[i_p, i_q]$ is an edge of G, but not of G^*, then $\gamma(i_0, i_0) < n - 1$, a contradiction. If $|p - q| \neq 1$, then there is no edge joining i_p and i_q^d since $\gamma(i_0, i_0) = n - 1$. If $|p - q| = 1$, then the edge $[i_p, i_q^d] \in E(G)$ is permitted (and thus $[i_p^d, i_q] \in E(G)$), and we also have $\gamma(i_0, i_0) = n - 1$. Hence, G is isomorphic to some graph in $\mathfrak{H}(n, l, \frac{n+1}{2})$ $(0 \leq l \leq \frac{n-3}{2})$.

Case 2: $V(C_w) \cap V(C_w^d) \neq \phi$. Then by (i) of Lemma 3.3 we have $C_w = C_w^d$. So the vertex $\frac{n+1}{2} \in V(C_w)$. Let P be the shortest path between w

and C_w. We consider two subcases.

Subcase 2.1: $|P| = r \geq 1$. According to the proof in Lemma 3.3 we have that $V(P) \cup V(C_w) \cup V(P^d) = V(G)$. Note that if $V(P) \cap V(P^d) \neq \phi$, then from (i) of Lemma 2.3 we have that $V(P) \cap V(P^d) = \{\frac{n+1}{2}\} \subseteq V(C_w)$. Hence we conclude that G has a spanning subgraph G^* isomorphic to $\mathfrak{R}^*(n,r,s)$, where $1 \leq r \leq \frac{n-1}{2}$ and $r+1 \leq s \leq \frac{n+1}{2}$. We assume without loss of generality that $G^* = \mathfrak{R}^*(n,r,s)$ and the vertex $w = 1$. We consider the edges in $E(G)\backslash E(G^*)$.

If $G^* = \mathfrak{R}^*(n,r,\frac{n+1}{2})$ $(1 \leq r \leq \frac{n-1}{2})$. Then there is no any vertex in $V(G)\backslash\{r+1, n-r\}$ with loop since $\gamma(1,1) = \gamma(n,n) = n-1$. It is trivial if $E(G)\backslash E(G^*) = \phi$. Now let $E(G)\backslash E(G^*) \neq \phi$, and let $[i,j]$ be any edge in $E(G)\backslash E(G^*)$ (and thus $[i^d, j^d] \in E(G)\backslash E(G^*)$), where $i \neq j$. If $i \in V(C_1)$ and $j \in V(P) \cup V(P^d)$, then we have $\gamma(1,1) \leq n-2$, a contradiction. If $i \in V(C_1)\backslash\{\frac{n+1}{2}\}$ and $j \in V(C_1)\backslash\{\frac{n+1}{2}\}$ with $|j - i^d| = |i+j-(n+1)| \neq 1$, then we can obtain the contradiction $\gamma(1,1) \leq n-2$. If $i \in V(P)\backslash\{\frac{n+1}{2}\}$ and $j \in V(P^d)\backslash\{\frac{n+1}{2}\}$ with $|j - i^d| = |i+j-(n+1)| \neq 1$, then we also obtain the contradiction $\gamma(1,1) \leq n-2$. Therefore, according to the proof of Lemma 3.7 and the definition of $\mathfrak{R}(n,r,\frac{n+1}{2})$, we conclude that G is isomorphic to some graph in $\mathfrak{R}(n,r,\frac{n+1}{2})$ $(1 \leq r \leq \frac{n-1}{2})$.

If $G^* = \mathfrak{R}^*(n,r,r+1)$ $(1 \leq r \leq \frac{n-3}{2})$. It is trivial if $E(G)\backslash E(G^*) = \phi$. Now let $E(G)\backslash E(G^*) \neq \phi$, and let $[i,j]$ be any edge in $E(G)\backslash E(G^*)$ (and thus $[i^d, j^d] \in E(G)\backslash E(G^*)$). If $j \neq i^d$, then we can obtain the contradiction $\gamma(1,1) \leq n-2$. If $j = i^d$ and $i \in \{1, \cdots, r\}$, then i lies on an odd cycle denoted by C_i, and we obtain $d(1, C_i) \leq d(1,i) \leq r-1 < r = d(1, C_1)$. This contradicts the definition of $C_1(= C_w)$. Therefore, according to the proof of Lemma 3.7 and the definition of $\mathfrak{R}(n,r,r+1)$, we conclude that G is isomorphic to some graph in $\mathfrak{R}(n,r,r+1)$ $(1 \leq r \leq \frac{n-3}{2})$.

If $G^* = \mathfrak{R}^*(n,r,s)$ for $r+2 \leq s \leq \frac{n-1}{2}$. Notice that if $[u,v] \in E(G)\backslash E(G^*)$, then $[u^d, v^d] \in E(G)\backslash E(G^*)$. Hence $E(G)\backslash E(G^*) = \phi$ by $\gamma(1,1) = \gamma(n,n) = n-1$. Thus $G = G^* = \mathfrak{R}^*(n,r,s) = \mathfrak{R}(n,r,s)$ $(r+2 \leq s \leq \frac{n-1}{2})$.

Subcase 2.2: $|P| = 0$. Then we have $|C_w| = n$ by $\gamma(w,w) = n-1$. So G contains a subgraph G^* isomorphic to $\mathfrak{R}^*(n,0,s)$ (that is a Hamilton cycle), where $1 \leq s \leq \frac{n-1}{2}$. We assume without loss of generality that $G^* = \mathfrak{R}^*(n,0,s)$ and the vertex $w = s$. Similar to the proof in Subcase 2.1 we have that G is isomorphic to some graph in $\mathfrak{R}(n,0,s)$ $(1 \leq s \leq \frac{n+1}{2})$. □

Lemma 3.9 *Let $G = G(A)$ with $A \in CSP(n)$. Assume that $\gamma(G) = n-1$*

and $\gamma(i,i) < n-1$ for any vertex $i \in V(G)$. Then G is isomorphic to some graph in $\mathfrak{H}(n,l,s)$ $(0 \le l \le \lfloor \frac{n}{2} \rfloor - 1,\ l+1 \le s \le \lfloor \frac{n}{2} \rfloor)$.

Proof. Since $\gamma(G) = n-1$ and $\gamma(i,i) < n-1$ for any vertex $i \in V(G)$, there exists two vertices $i,\ j \in V(G)$ such that $\gamma(i,j) = \gamma(G) = n-1$. Let P_{ij} be any shortest path joining i and j. Then $1 \le |P_{ij}| \le n-1$. We consider two cases.

Case 1: $|P_{ij}| = n-1$. Then $V(P_{ij}) = V(G)$, and so $V(P_{ij}) = V(P_{ij}^d)$. By Lemma 3.2 we have $j = i^d$, and $|iP_{ij}t| = |t^dP_{ij}j|$ for each vertex $t \in V(P_{ij})$. Without loss of generality, let $P_{ij} = P_{1n} = 12\cdots(n-1)n$. Clearly, $E(G)\backslash E(P_{1n}) \ne \phi$ since G is primitive. Let $[u,v]$ be any edge in $E(G)\backslash E(P_{1n})$. If $u \ne v$, then $d(1,n) \le n-2$, contradicting the condition $|P_{1n}| = n-1$. Thus $u = v$. Now let $l+1$ be the first vertex with a loop along P_{1n}, then $0 \le l \le \lfloor \frac{n}{2} \rfloor - 1$ (since $\gamma(1,1) < n-1$), and $n-l$ is the last vertex with a loop along P_{1n}. Hence G has a spanning subgraph $\mathfrak{H}^*(n,l,l+1)$ $(0 \le l \le \lfloor \frac{n}{2} \rfloor - 1)$. Notice that if n is odd and $l = 0$, then $E(G)\backslash E(\mathfrak{H}^*(n,0,1)) \ne \phi$ since $\gamma(\frac{n+1}{2}, \frac{n+1}{2}) < n-1$. Thus, we conclude that G is isomorphic to some graph in $\mathfrak{H}(n,l,l+1)$ $(0 \le l \le \lfloor \frac{n}{2} \rfloor - 1)$.

Case 2: $1 \le |P_{ij}| < n-1$. Let $C_{P_{ij}}$ be an odd cycle such that $d(P_{ij}, C_{P_{ij}}) = \min\{d(P_{ij}, C) :\ C \textit{ is an odd cycle in } G\}$, and let P_* be the shortest path between P_{ij} and $C_{P_{ij}}$. By Lemma 3.5 we conclude $V(C_{P_{ij}}) \cap V(C_{P_{ij}}^d) = \phi$. It follows from (ii) of Lemma 3.4 that $|C_{P_{ij}}| = |C_{P_{ij}}^d| = 1$. Hence $|P_*| \ge 1$, and hence by Lemma 3.6 we conclude $V(P_*) \cap V(P_*^d) = \phi$. According to the proof in Lemma 3.4 we have that $|P_{ij}| + 2|P_*| + |C_{P_{ij}}| = n$ and $V(P_*^d) \cup V(P_{ij}) \cup V(P_*) \cup V(C_{P_{ij}}) = V(G)$, so $|P_{ij}| + 2|P_*| = n-1$, $V(P_*^d) \cup V(P_{ij}) \cup V(P_*) = V(G)$, and $V(P_{ij}) = V(P_{ij}^d)$. By Lemma 3.2 we have that $j = i^d$ and $|iP_{ij}t| = |t^dP_{ij}j|$ for each vertex $t \in V(P_{ij})$.

Now let $i_0 = V(P_*) \cap V(P_{ij})$. Then $i_0^d \in V(P_*^d) \cap V(P_{ij})$ and $i_0 \ne i_0^d$. It is not difficult to verify that $d(i_0, i_0^d)$ and n have different parity. We assume without loss of generality that $d(i, i_0) = l < d(i, i_0^d)$ and $d(i_0, i_0^d) = n+1-2s$. Then $0 \le l \le \lfloor \frac{n}{2} \rfloor - 1$, $|P_{ij}| = n+1-2s+2l$ and $|P_*| = s-l-1$. Since $d(i_0, i_0^d) \ge 1$ and $|P_*| \ge 1$, we have $l+2 \le s \le \lfloor \frac{n}{2} \rfloor$. Thus we conclude that G has a spanning subgraph G^* isomorphic to $\mathfrak{H}^*(n,l,s)$, where $0 \le l \le \lfloor \frac{n}{2} \rfloor - 1$ and $l+2 \le s \le \lfloor \frac{n}{2} \rfloor$. Without loss of generality, let $G^* = \mathfrak{H}^*(n,l,s)$ $(0 \le l \le \lfloor \frac{n}{2} \rfloor - 1,\ l+2 \le s \le \lfloor \frac{n}{2} \rfloor)$. We consider two subcases.

Subcase 2.1: $l = 0$. Then $2 \le s \le \lfloor \frac{n}{2} \rfloor$. This implies that $i = i_0 = s$, $C_{P_{ij}} = [1,1]$, $P_* = 1\cdots(s-1)$ and $P_{ij} = s\cdots(n+1-s)$.

If $2 \le s \le \lfloor \frac{n}{2} \rfloor - 1$. Then $E(G) \backslash E(G^*) = \phi$ by $\gamma(i,j) = \gamma(s, n+1-s) = n-1$, and $\mathfrak{H}(n,0,s) = \mathfrak{H}^*(n,0,s)$ by the definition of $\mathfrak{H}(n,l,s)$ for $l+2 \le s \le \lceil \frac{n}{2} \rceil - 1$. Hence $G = G^* = \mathfrak{H}(n,0,s)$ $(2 \le s \le \lfloor \frac{n}{2} \rfloor - 1)$.

If n is odd and $s = \lfloor \frac{n}{2} \rfloor$. Then $s = \frac{n-1}{2} = \lceil \frac{n}{2} \rceil - 1$. Arguing as above we have $G = G^* = \mathfrak{H}(n, 0, \lfloor \frac{n}{2} \rfloor)$.

If n is even and $s = \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$. It is trivial if $E(G) \backslash E(G^*) = \phi$. Now let $E(G) \backslash E(G^*) \neq \phi$, and let $[u, v]$ be any edge in $E(G) \backslash E(G^*)$. Then $[u^d, v^d] \in E(G) \backslash E(G^*)$ and $u, v \in V(G) \backslash \{\frac{n}{2}, \frac{n+2}{2}\}$. Suppose $v \neq u^d$, it is easy to see that $\gamma(\frac{n}{2}, \frac{n+2}{2}) < n-1$, contradicting the condition $\gamma(\frac{n}{2}, \frac{n+2}{2}) = \gamma(i,j) = n-1$. Hence $v = u^d$. Thus, according to the proof of Lemma 3.7 and the definition of $\mathfrak{H}(n, 0, \frac{n}{2})$, we conclude that G is isomorphic to some graph in $\mathfrak{H}(n, 0, \frac{n}{2})$.

Subcase 2.2: $1 \le l \le \lfloor \frac{n}{2} \rfloor - 1$.Then $i = 1$, $j = n$, $P_{ij} = 1 \cdots ls \cdots (n+1-s)(n+1-l) \cdots n$, $C_{P_{ij}} = [l+1, l+1]$ and $P_* = (l+1) \cdots s$. Notice that $l+2 \le s \le \lfloor \frac{n}{2} \rfloor$.

If $l+2 \le s \le \lceil \frac{n}{2} \rceil - 1$. Suppose $E(G) \backslash E(G^*) \neq \phi$, then it is not difficult to verify that $\gamma(1,n) < n-1$, contradicting the condition $\gamma(1,n) = n-1$. Hence $E(G) \backslash E(G^*) = \phi$, and hence $G = G^* = \mathfrak{H}^*(n,l,s) = \mathfrak{H}(n,l,s)$ $(l+2 \le s \le \lceil \frac{n}{2} \rceil - 1)$.

If n is even and $s = \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$. By a similar argument as in subcase 2.1, we have that G is isomorphic to some graph in $\mathfrak{H}(n, l, \frac{n}{2})$ $(1 \le l \le \frac{n}{2} - 1)$.□

Combining Lemma 3.7, Lemma 3.8 and Lemma 3.9, we have

Theorem 3.1 *Let $A \in CSP(n)$. Then $\gamma(A) = n-1$ if and only if $G(A)$ is isomorphic to some graph in $\mathfrak{H}(n,l,s) \cup \mathfrak{R}(n,r,s)$.*

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