The Dynamic Coloring Numbers of Pseudo-Halin Graphs

Xianyong Meng[a] *, Lianying Miao[b], Bentang Su[a], Rensuo Li[a]
a College of Information Science and Engineering,
Shandong Agriculture University, Taian, Shandong, 27100, China
b College of Science, China University of Mining and Technology,
Xuzhou, Jiangsu, 221008, China

Abstract

In this paper, we have discussed the dynamic coloring of an kind of planar graph. Let G be a Pseudo-Halin graph, we prove that the dynamic chromatic number of G is at most 4. Examples are given to show the bounds can be attained.

Key words: Pseudo-Halin graph; dynamic coloring; dynamic coloring number.
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1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph G, let $V(G)$,$E(G)$,$|V(G)|$,$\Delta(G)$ and δ denote, respectively, its vertex set, edge set, number of vertices, maximum degree, and minimum degree. For a vertex v, let $d(v)$and $N(v)$ denote its degree and neighbor vertex set respectively. A vertex of degree k is called a k-vertex. For two vertices $u, v \in V(G)$, let

*xym@sdau.edu.cn

$dist_G(u,v)$ denote the distance between u and v in G, the length of the shortest path connecting them.

A k-dynamic coloring of a graph G is a mapping σ from $V(G)$ to the set of colors $\{1,2,\cdots\}$, such that $\sigma(x) \neq \sigma(y)$ for every xy of G and for every $v \in V(G)$, $|C(v)| \geq min\{2, d(v)\}$, where $C(v) = \{\sigma(u)|u \in N(v)\}$. We call G k-dynamic colorable if it has a k-dynamic coloring. The dynamic chromatic number $\chi_d(G)$ of G is the smallest integer k such that G is k-dynamic colorable.

The concept was first introduced in [2] and developed by Bruce Montgomery [4]. In this paper, we proved that $\chi_d(G) \leq 4$ for every Pseudo-Halin graph.

Pseudo-Halin graph is first studied by Lin-zhong Liu and Zhong-fu Zhang in [3]. Let $G(V,E)$ be a 2-connected planar graph, f_0 be a face without chord on its boundary (a cycle) and $d(v) \geq 3$ for every $v \in V(f_0)$. When a tree T, in which all vertices $v \in V \setminus V(f_0)$ satisfies $d(v) \geq 3$, is obtained from $G(V,E)$ by deleting all edges on the boundary of f_0, then $G(V,E)$ is called a Pseudo-Halin graph; $G(V,E)$ is said to be Halin graph iff $d(v) = 3$ for every $v \in V(f_0)$. f_0 is called exterior face (the other is called interior face). The every vertex on f_0 is called exterior vertex (the other is called interior vertex), the vertex $v \in V(f_0)$ and $d(v) = 3$ is called regular vertex, the other on f_0 is irregular vertex. The regular vertex-set is denoted by $R(f_0)$, the irregular vertex-set by $IR(f_0)$.

2 Main results

Lemma 1[3]. *Let G be a Pseudo-Halin graph with outer face f_0, then:*

(1) *$v \in IR(f_0)$, all interior vertices adjacent to v aren't adjacent to each other.*

(2) *$v \in IR(f_0)$, there are at most two vertices in $N(v)$ in the same interior face.*

Lemma 2[3]. *Let G be a Pseudo-Halin graph ($G \neq W_p$) with outer face f_0, $p = u_1u_2\cdots u_k$ is a longest path in $G - E(f_0)$, $w \in \{u_2, u_{k-1}\}$, then one of the following holds:*

(1) *w is the interior vertex of G, $N(w) \subset V(f_0)$, $|N(w) \bigcap IR(f_0)| = 1$, let $N(w) = \{y_1, u_1, u_2, \cdots, u_m\} (m \geq 2)$, $xu_1, yu_m, u_iu_{i+1} \in E(f_0)$,*

$y \in IR(f_0), x \neq u_2, y \neq u_{m-1}, (i = 1, 2, \cdots, m-1)$, *then the graphes:*

$$G_1^1 = G - \{w, u_i | i = 1, 2, \cdots, k\} + \{xy\}$$

$$G_1^2 = G - \{u_i, u_{i+1}, \cdots, u_j\} + \{u_{i-1}u_{j+1}\}(2 \leq i < j \leq m-1)$$

are also Pseudo-Halin graphes.

(2) *w is the interior vertex of G and $|N(w) \bigcap (V(G) \setminus V(f_0))| = 1$, $|N(w) \bigcap R(f_0)| = d(w) - 1$. Let $N(w) = \{u, u_1, \cdots, u_m\}$, u is the interior vertex, $u_i \in R(f_0), (i = 1, 2, \cdots, m)$, xu_1, yu_m, $u_iu_{i+1} \in E(f_0), i = 1, 2, \cdots, m-1$, $x \neq u_2, y \neq u_{m-1}$, then the graphes:*

$$G_2^1 = G - \{u_1, u_2, \cdots, u_m\} + \{xw, yw\}$$

$$G_2^2 = G - \{u_i, u_{i+1}, \cdots, u_j\} + \{u_{i+1}u_{j+1}\}.(2 \leq i \leq j \leq m, m \geq 3)$$

are also Pseudo-Halin graphes.

Lemma 3[3]. *Let G be a Pseudo-Halin graph ($G \neq W_p$). Then the followings hold:*

(1) *if w, x and y are vertices such as Case* 1 *in Lemma* 2*, then $xy \notin E(G)$.*

(2) *if w, x and y are vertices such as Case* 2 *in Lemma* 2*, then $|\{x, y\} \cap IR(f_0)| \leq 1$.*

Lemma 4. *Let G be a pseudo-Halin graph, w, x and y are vertices such as Case* 2 *in Lemma* 2*, $u \in N(w) \setminus \{u_1, \cdots, u_m\}$. Then there is at least one vertex $a \in \{x, y\}$, such that $dist_T(a, u) \leq 2$ holds.*

Proof. By the former lemma, suppose that $p = v_1v_2 \cdots v_{k-3}uwv_k$ is the longest path in T, $v_k \in \{u_1, \cdots, u_m\}$. The proof is divided into the following two cases.

Case 1: $x \in IR(f_0)$. By Lemma 3, $y \in R(f_0)$. Since $p = v_1v_2 \cdots v_{k-3}uwv_k$ is the longest path, we know that $x \in \{v_1, v_2, \cdots, v_{k-3}\}$. Otherwise, p isn't the longest path in T. In fact, there is one interior vertex $b \in N(x) \setminus \{u\}$. Let $c \in N(b) \setminus \{x\}$, $p' = v_1 \cdots u \cdots xbc$. Obviously, p' is the path in T and longer than p.

Suppose that $dist_T(u, y) \geq 3$, then $dist_T(v_1, y) \geq dist_T(v_1, v_k) + 1$. This is contradiction. Let $a = y$, the conclusion is true.

Case 2: $x \in R(f_0)$. By Lemma 3, $y \in IR(f_0)$ or $y \in R(f_0)$.

Subcase 2.1: $y \in IR(f_0)$. According to the analysis above, $dist_T(u, x) \leq 2$. Let $a = x$, the conclusion is true.

Subcase 2.2: $y \in R(f_0)$. If $dist_T(u,y) \le 2$, suppose $a = y$. If $dist_T(u,y) \ge 3$. In T, let $P = v_1v_2\cdots v_{k-3}uwv_k$ is the longest path and $p_1 = u\cdots y$. If $(V(p)\cap V(p_1))\backslash\{u\} \neq \phi$, let $p_2 = v_1v_2\cdots v_{k-3}u\cdots y$. Obviously, p_2 is longer than p in T. So $(V(p)\cap V(p_1))\backslash\{u\} = \phi$. As same as the analysis above, $dist_T(u,x) \le 2$. Let $a = x$, the conclusion is true.

Lemma 5. *Let w, x, y and u be vertices such as in Lemma* 4. *If $dist_T(u,y) = 2$, then one of the followings holds:*

(1) *y is the vertex of an triangle;*

(2) *$dist_T(u,x) = 1, x \in R(f_0)$;*

(3) *$dist_T(u,x) = 2$, x is the vertex of an triangle.*

Proof. Let $p = v_1v_2\cdots v_{k-3}uwv_k, v_k \in \{u_1,\cdots,u_m\}$ be the longest path in T, $v \in N(u)\cap N(y)$, $u_1,\cdots,u_m$ referenced Lemma 2. The proof is divided into the following two cases:

Case 1: $v \in V(P)$. Similarly the analysis in Lemma 4, we obtain that $dist_T(u,x) \le 2$.

If $dist_T(u,x) = 1$, we know that $x \in R(f_0)$. Otherwise, p isn't the longest path in T. So Case 2 holds.

If $dist_T(u,x) = 2$. Let ubx is the path in T, then b is the interior vertex such as w. x is the vertex of an triangle. Case 3 holds.

Case 2: $v \notin V(p)$. According to the former analysis, y is the vertex of triangle. Case 1 holds.

Similarly, we get the Lemma 6.

Lemma 6. *Let w, x, y and u be vertices such as in Lemma* 4. *If $dist_T(u,y) = 1$, then one of the following holds:*

(1) $y \in R(f_0)$;

(2) *$dist_T(u,x) = 1, x \in R(f_0)$;*

(3) *$dist_T(u,x) = 2$, x is the vertex of an triangle.*

Theorem 1. *Let G be a pseudo-Halin graph, then $\chi_d(G) \le 4$.*

Proof. It is sufficient to prove that G has a 4-dynamic coloring. Our proof proceeds by induction on the number of vertices of G. In the following proof, let $C = \{1,2,3,4\}$; For the set S of vertices, $C_0(S) = \{\sigma_0(u)|u \in S\}$. In the following proof, the uncolored elements of $V(G)$ are colored with the same colors as in σ_0 of G_0.

The conclusion follows immediately if $|G| \le 7$. Now assume that $|G| \ge 8$ and the conclusion holds for graph H with $|H| < |G|$. When $G = W_p(p \ge 8)$, G has a 4-dynamic coloring. If $G \neq W_p$, it has one vertex w such as in Lemma 2. So the proof is divided into the following two cases.

Case 1: w is the vertex such as Case 1 in Lemma 2.

Subcase 1.1: $d(w) = 3$.

Consider the graph $G_0 = G - \{w, u_1, u_2\} + \{xy\}$. Then G_0 is a Pseudo-Halin graph and $|V(G_0)| = n - 3 < n$. By the induction hypothesis, G_0 has a 4-dynamic coloring σ_0. We now extend σ_0 to the dynamic coloring σ of G.

Let $\sigma(u_1) = \sigma_0(y); \sigma(u_2) \in C\backslash\{\sigma_0(y)\}; \sigma(w) = C\backslash\{\sigma_0(y), \sigma(u_2)\}$.

Subcase 1.2: $d(w) \geq 4$.

Consider the graph $G_0 = G - \{u_1, u_2, \cdots, u_m, w\} + \{xy\}$. Then G_0 is a pseudo-Halin graph and $|V(G_0)| = n - m - 1 < n$. By the induction hypothesis, G_0 has a 4-dynamic coloring σ_0. Now ,we construct a 4-dynamic coloring σ on σ_0.

Let $\sigma(u_1) = \sigma_0(y); \sigma(w) \in C\backslash\{\sigma_0(y)\}; u_2, u_3, \cdots, u_m$ are colored alternatively by the two colors in the set $C\backslash\{\sigma_0(y), \sigma(w)\}$ in turn.

Case 2: w is the vertex such as Case 2 in Lemma 2.

Subcase 2.1: $d(w) = 3$.

Consider the graph $G_0 = G - \{u_1, u_2\} + \{xw, yw\}$, then the graph G_0 is a pseudo-Halin graph and $|V(G_0)| = n - 2 < n$. By the induction hypotheses, G_0 has a 4-dynamic coloring σ_0. Now, we construct a 4-dynamic coloring σ on the founder of σ_0. For simplicity, let $N_1 = N(x)\backslash\{u_1\}, N_2 = N(y)\backslash\{u_2\}$ in the following proof. According to the order of the set $C_0(N_1)$ and $C_0(N_2)$, the proof is divided into the following two cases.

Subcase 2.1.1: Between $C_0(N_1)$ and $C_0(N_2)$, there is at least one set whose order is more than or equate to 2. Without loss of generality, we assume that $|C_0(N_1)| \geq 1$, $|C_0(N_2)| \geq 2$. Suppose that $a \in N(x)\backslash\{u_1\}$.

Let $\sigma(u_1) \in C\backslash\{\sigma_0(x), \sigma_0(w), \sigma_0(a)\}; \sigma(u_2) \in C\backslash\{\sigma_0(y), \sigma_0(w), \sigma(u_1)\}$.

Subcase 2.1.2: $|C_0(N_1)| = |C_0(N_2)| = 1$. By Lemma 4, without loss of generality, suppose that $dist_T(u, y) \leq 2$.

If $dist_T(u, y) = 1$. By Lemma 6, we have three cases to consider. According to hypothesis $| C_0(N_1) | = | C_0(N_2) | = 1$, it is obviously that Case 3 in Lemma 6 can't hold. Note, the vertex y in Case 1 is as similar as x in Case 2 in Lemma 6. So it is enough only to consider Case 1 in Lemma 6. Suppose that $a \in N(y)\backslash\{u, u_2\}, b \in N(a)\backslash\{y\}$.

Let $\sigma(y) \in C\backslash(\{\sigma_0(y), \sigma_0(b)\} \cup C_0(N_1))$; $\sigma(w) = \sigma_0(y)$; $\sigma(u_1) = \sigma_0(w)$,

$\sigma(u_2) \in C\backslash\{\sigma(y), \sigma(w), \sigma(u_1)\}$.

If $dist_T(u, y) = 2$. By Lemma 5 and hypothesis $| C_0(N_1) |=| C_0(N_2) |= 1$, Case 1 and 3 in Lemma 5 can't hold. So Case 2 in Lemma 5 hold. The coloring is referenced the coloring above for Case 1 in Lemma 6.

Case 2.2: $d(w) \geq 4$.

Consider the graph $G_0 = G - \{u_2\} + \{u_1, u_3\}$. Then the graph G_0 is a pseudo-Halin graph and $|V(G_0)| = n - 1 < n$. By the induction hypothesis, G_0 has a 4-dynamic coloring σ_0. Let $\sigma(u_2) \in C\backslash\{\sigma_0(u_1), \sigma_0(u_3), \sigma(w)\}$.

From all the former cases, it is not difficult to see that σ is a 4 dynamic coloring of G, and thus $\chi_d(G) \leq 4$.

Hence, by induction, we have proved theorem 1 is true.

According to the definition of pseudo-Halin graph, we get the following corollary.

Corollary 1. *Let G be a Halin graph, then $\chi_d(G) \leq 4$.*

3 Remarks

Now, we give two examples :

(a) Let $G = W_p$ (p is even number). Then $\chi_d(G) = 4$.

(b) If the graph G is isomorphic to the graphs in Figure 1, then $\chi_d(G) = 4$.

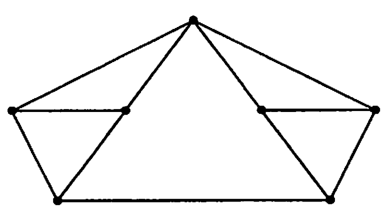

Fig.1

One can easily verify that the dynamic chromatic numbers for these graphs are 4. So for some graph, the bounds can be attained. In this meaning Theorem 1 gives sharp bounds. But on our study, we give the following conjecture:

Conjecture: Let G be pseudo-Halin graph, $|V(G)| \geq 8$ and $G \neq W_p$ (p is even number), then $\chi_d(G) = 3$.

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