On the Supermagic Edge-splitting Extension of Graphs

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ABSTRACT. A (p,q)-graph G in which the edges are labeled 1,2,3,...q so that the vertex sums are constant, is called supermagic. If the vertex sum mod p is a constant, then G is called edge-magic. We investigate the supermagic characteristic of a simple graph G, and its edge-splitting extension SPE(G,f). The construction provides an abundance of new supermagic multigraphs.

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1. Introduction. Magic graphs were first initiated by Sedlacek (around 1963) [14,15] as the problem of labeling the edges of the graph with real numbers so that the sum of the edges label to a vertex is same as all the other vertices. Jeurissen [5], Jezney and Trenkler [6] gave characterizations of magic graphs. A characterization of regular magic graphs in term of even circuits is given by Doob [1]. Kong, Sun, Lee et al [7,24,25] provided some general constructions of magic graphs. Since Sedlacek's original article [14], literally hundred of papers have been written on magic graph labelings (see the survey article [2]).

If G is a (p,q)-graph in which the edges are labeled 1,2,3,...q so that the vertex sums defined by $f^{+}(u) = \Sigma\{f(u,v): (u,v) \text{ in } E\}$ is constant, then G is called **supermagic.** Figure 1 shows a graph with 6 vertices and 8 edges which is supermagic.

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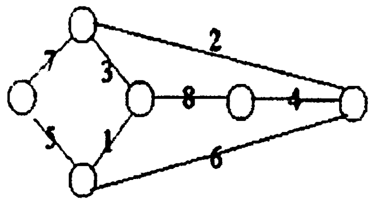
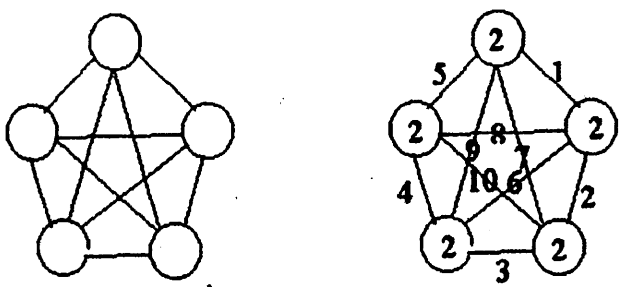


Supermagic

Figure 1

B.M. Stewart [22,23] showed that K_3, K_4, K_5 are not supermagic and when $n \equiv 0 \pmod 4$, K_n is not supermagic. For $n>5$, K_n is supermagic if and only if $n \not\equiv 0 \pmod 4$. Hartsfield and G. Ringel [2] provided some classes of supermagic graphs. Ho and Lee [4] extended the result of Stewart to regular complete k-partite graphs. Recently Shiu, Lam and Cheng [17] considered a class of supermagic graphs which are disjoint union of $K_{3,3}$.

A generalization of supermagic graphs was introduced by Lee, Seah and Tan [10]. A graph G = (V, E) with p vertices and q edges is called **edge-magic** if there is a bijection $f : E \rightarrow \{1, 2, \ldots, q\}$ such that the induced mapping $f^+ : V \rightarrow Z_p$, given by $f^+(u) = \Sigma\{f(u,v): (u,v) \text{ in } E\} \pmod p$ is a constant. A necessary condition for a (p,q)-graph to be edge-magic is $q(q+1) \equiv 0 \pmod p$. However, there are infinitely many connected graphs such as trees, cycles satisfy the necessary condition but are not edge-magic.



not super-magic edge-magic

Figure 2

The concept of edge-magic labeling of graphs is the dual concept of edge-graceful labeling [10]. In 1985, Lo Sheng-Ping [13] introduced the concept of edge-graceful graphs. A (p,q)-graph G = (V, E), of p vertices and q edges, is said to be **edge-graceful** if there exists a bijection $f : E \rightarrow \{1, 2, \ldots, q\}$ such that the induced mapping $f^+ : V\_ \{0, 1, \ldots, p-1\}$, defined by $f^+(v) = \Sigma \{f(u,v): (u,v) \in E(G)\} \pmod p$ is a bijection.

The cartesian product of two paths is frequently called the grid graph. The cartesian product of two cycles is called the torus graph. It was shown in [16] that the torus graph $C_m \times C_n$ is edge-magic for all m,n >2.

Karl Schaffer and Sin-Min Lee [16] have shown that if G and H are both odd-order, regular, edge-graceful graphs, where G is d-regular and has m vertices, and H is k-regular and has n vertices, and furthermore GCD(d,n) = GCD (k,m) =1, then G x H is edge-graceful. In particular, they showed that the torus graph $C_{2i+1} \times C_{2j+1}$ is edge-graceful.

Finding the magic labeling of graphs are related to solving system of linear Diophantine equations [21]. In general it is difficult to find an edge-magic or supermagic labeling of a graph. Several classes of graphs had been shown to be edge-magic ([8,9,10,11,12,17,18]). For more conjectures and open problems on edge-magic graphs the reader is referred to [8.9,10,11,12].

Shiu, Lam and Lee [18,19] give a general construction of supermagic graphs and edge-magic graphs. The reader should see the survey article of Gallian [2] for various labeling problems.

In this paper we introduce a construction of super magic graphs by splitting some of the edges of the graphs. We consider this construction for perfect matchings and cycles.

2. Edge-splitting extensions of graphs.

In this section, we shall introduce a general construction of multigraphs from a given simple graph. Let N={1,2,3,...} be a set of natural numbers.

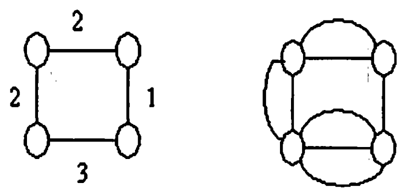
Given a pair (G, f) where G = (V ,E) is a simple graph with p vertices and q edges and $f: E(G) \rightarrow N$, we can construct a graph SPE(G,f) as follows:

For each edge e of E(G) if f(e)= k, we associate a set of parallel edges $P(e) = \{e_1,e_2,\dots,e_k\}$. We observe that V(SPE(G,f)) = V(G) and E(SPE(G,f)) = $\cup$ $\{P(e): e \in E(G)\}$

We shall call the graph SPE(G,f) as an **edge-splitting extension graph** of (G,f).

We illustrate here with one example:

Example 1. Let G= C_4 and $f:E(C_4) \rightarrow N$ as follows:



(G,f) SPE(G,f)

Figure 3

A necessary condition for a (p,q)-graph to be supermagic is $q(q+1) \equiv 0 \pmod p$. However, this is not the sufficient condition. Consider the following example:

Example 2. Let $G = C_6$ and $h: E(C_6) \to N$ be defined as follows:

	a	b	c	d	e	f
h	1	2	1	2	1	2

Then we see that the following $SPE(C_6,h)$ (Figure 4) has 6 vertices and 9 edges.

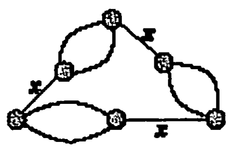


Figure 4

If the graph is supermagic then three single edges must be labeled by x for some x which is not possible.

Remark. In fact, if G is a graph with three edges a, b, and c such that a is adjacent to b and b is adjacent to c but a and c are not adjacent and $f: E(G) \to N$ with property $f(a) = f(c) = 1$, then SPE(G,f) is not supermagic.

Theorem 1. If G is a regular simple graph and $f: E(G) \to N$ is a constant map with $f(e)=2t$ for a fixed t, then SPE(G,f) is supermagic.

Proof. Suppose G is a (p,q)-graph. We see that SPE(G,f) has 2tq edges. We divide the set of integers $\{1,2,\ldots,2tq\}$ into tq pairs $\{1,2tq\},\{2,2tq-1\},\ldots,\{k,2tq-k+1\}\ldots,\{tq,tq+1\}$. We denote this set by H

For each e of G, $P(e)=\{e_1,e_2,\ldots,e_{2t}\}$ we form the t pairs $\{e_1,e_2\}$, $\{e_3,e_4\}\ldots\{e_{2t-1},e_{2t}\}$.

Then the set of edges SPE(G,f) has tq pairs. We denote the set by Q. Any bijection $F: Q \to H$ induces a super edge-magic labeling $g: E(SPE(G,f)) \to \{1,2,\ldots,2tq\}$ by letting $g(a)=i$ and $g(b)=j$ if $F(\{a,b\})=\{i,j\}$.□

There exists graph G such that SPE(G,f) is not supermagic for any $f: E(G) \to N$. For example a connected graph G has a tail of length greater than or equal to one has this property. We can formulate the above observation by the following result.

Theorem 2. Any graph G with two vertices u,v such that $\deg(u)=1$ and $\deg(v)\geq 2$ and u,v are adjacent has the property that SPE(G, f) is not supermagic for any f $E(G) \to N$.

Corollary 3. For any $n>1$, the complete bipartite graph $K_{1,n}$ has the property that $SEP(K_{1,n}, f)$ is not supermagic for any f $E(G) \to N$.

Corollary 4. If $n > 2$, then the graph SEP(P_n,f) is not supermagic for all f: E(G) → N.

Theorem 5.(a) The graph G=P_2 has the property that SPE(G, f) is supermagic for any f : E(G) → N. □

(b) P_2 is the only graph G with the property that SPE(G, f) is supermagic for any f : E(G) → N. □

3. Edge-splitting extension of perfect matchings.

Let mK_2 be the m perfect matching and f : E(mK_2) → N with f(e) = n for all e in E(G).

Shiu and Lee [20] showed that:

Theorem 6. For m,n ≥ 2, the splitting-edge extension of the m perfect matching and f : E(G) → N with f(e) = n for all e in E(G) is supermagic if and only if n is even or both m and n are odd.

Example 3. We label SPE($3K_2$,f) where f : E($3K_2$) → N with f(e) = 3 for all e of $3K_2$ as follows:

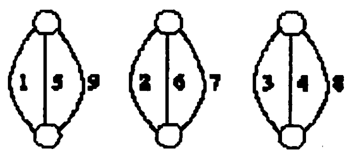


Figure 5

Theorem 7. If the graph SPE($2K_2$,f) where f : E($2K_2$) → N is supermagic, then $\Sigma f(e) \neq 1 \pmod 4$ or $\Sigma f(e) \neq 2 \pmod 4$.

Proof. If $\Sigma f(e) \equiv 1 \pmod 4$ then $\Sigma f(e)=4k+1$ for some k. As $q=4k+1$, we have $q(q+1) = (4k+1)(4k+2) = 2(4k+1)(2k+1)$.

Now p=4. It is clear that $q(q+1) \neq 0 \pmod 4$. Thus SPE($2K_2$,f) cannot be supermagic.

If $\Sigma f(e) \equiv 2 \pmod 4$ then $\Sigma f(e)=4k+2$ for some k. As $q = 4k+2$, we have $q(q+1) = (4k+2)(4k+3) = 2(4k+3)(2k+1)$. Now p=4. It is clear that $q(q+1) \neq 0 \pmod 4$. Thus SPE($2K_2$, f) cannot be supermagic. □

Theorem 8. The graph SPE($2K_2$,f) where f : E($2K_2$) → N is supermagic if f(a)=2k-1, f(b)=2k for k≥1.

Proof. We see that SPE($2K_2$,f) has 4k-1 edges. We label the edges P(a) = $\{a_1,a_2,\ldots, a_{2k-1}\}$ by A = {1, 2, 3,...,[(2k-1)/2], 4k-1, 4k-2,..., 4k-[(2k-1)/2]-1}. The other edges P(b) = $\{b_1, b_2,\ldots, b_{2k}\}$ will be labeled by the complement of A in {1, 2,..., 4k-1}.

We observe that this is a supermagic labeling. □

Example 4. We can have a supermagic labeling of $SPE(2K_2,f)$ with $f(a)=7, f(b)=8$ as follows:

$$\{1,2,3, 15,14,13,12\}, \{4,5,6,7,8,9,10,11\}.$$

Theorem 9. The graph $SPE(2K_2,f)$, where $f : E(2K_2, f) \to N$, is supermagic if $f(e_1) = n$, $f(e_2) = 2n$ for $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

Proof. We consider two cases:

Case 1. $n \equiv 0 \pmod 4$. Let $n = 4k$, $k \geq 1$.

The sum of the first 8k inters is $4k(8k+1)$, and the sum of the last 4k intergers is $2k(20k+1)$. Since they differ by $8k^2 - 2k = (k-1)8k + 6k$. We switch $\{7k + 2, 7k + 3, \ldots, 8k\}$ with $\{11k +2, 11k +3, \ldots, 12k\}$ and 10k with 7k. For the k-1 pairs of swap, the first sum increases by 4k and the second sum decreases by 4k, while the last pair increases the first sum by 3k and decreases the second sum by 3k. As a result, they both sum up to $3k(12k + 1) = 3n(3n + 1)/4$. Therefore $SPE(2K_2, f)$ is supermagic.

Case 2. $n \equiv 1 \pmod 4$. Let $n = 4k+1$, $k \geq 1$.

The sum of the first 8k +2 numbers is $(4k + 1)(8k + 3)$, and the last 4k + 1 numbers sum up to be $(4k +1)(10k + 3)$. They differ by $2k(4k + 1)$; switching the last k numbers will give both sums $(4k + 1)(9k + 3)$. We see that $SPE(2K_2, f)$ is supermagic. □

Example 5. (a) If $SPE(2K_2, f)$ has $f(e_1) = 8$ and $f(e_2) = 16$, we follow case 1 of Theorem 10, with $k = 2$ and label $P(e_1) = \{e_{1,1}, e_{1,2}, \ldots, e_{1,8}\}$ by $\{17, 18, \ldots, 24\}$ $P(e_2) = \{e_{2,1}, e_{2,2}, \ldots, e_{2,16}\}$ by $\{1, 2, \ldots, 16\}$, switching 24 with 16 and 20 with 14, we have a supermagic labeling.

(b) If $SPE(2K_2, f)$ has $f(e_1) = 13$ and $f(e_2) = 26$, we follow case 2 of Theorem 10, with $k = 3$ and label $P(e_1) = \{e_{1,1}, e_{1,2}, \ldots, e_{1,13}\}$ by $\{27, 28, \ldots, 39\}$ $P(e_2) = \{e_{2,1}, e_{2,2}, \ldots, e_{2,26}\}$ by $\{1, 2, \ldots, 26\}$, switching 39 with 26, 38 with 25, and 37 with 24, we have a supermagic labeling.

Remark. When $m = 3$ and $f : E(3K_2) \to N$ satisfies the necessary condition of supermagicness, that is $(\Sigma f(e))\,(1+\Sigma f(e)) \equiv 0 \pmod 6$, we do not guarantee that $SPE(3K_2, f)$ is supermagic.

For example if $E(3K_2) = \{a,b,c\}$ and we consider the following two mappings:

(1) $f(a)=1$, $f(b)=2$ and $f(c)=2$. We see that $5x6 \equiv 0 \pmod 6$. The graph $SPE(3K_2, f)$ is supermagic with the label $\{5\},\{1,4\}$ and $\{2,3\}$.

(2) $f(a)=1$, $f(b)=2$ and $f(c)=3$. We see that $6x7 \equiv 0 \pmod 6$. If $SPE(3K_2,f)$ is supermagic then the vertex sum should be 7 which is not possible to have such a labeling.

Theorem 10 A necessary condition for $SPE(3K_2,f)$ with $f : E(3K_2) \to N$, $f(e_1) = f(e_2) = n$, $f(e_3) = k$ to be supermagic is $q(q+1) \equiv 0 \pmod 6$ and $q(q+1)/3k \leq 2q+1 - k$.

Proof. Suppose $f(e_1) = f(e_2) = n$, $f(e_3) = k$ and $SPE(3K_2, f)$ is supermagic. Let $A = \{x_1, x_2, \ldots, x_n\}$, $B = \{y_1, y_2, \ldots, y_n\}$, $C = \{z_1, z_2, \ldots, z_k\}$ be the edge labels of $SPE(3K_2, f)$. Then we have

$$\sum x_i = \sum y_j = \sum z_m = q(q+1)/6.$$

Since the sum of $\{q, (q-1), \ldots, q-k+1\}$ is $(k/2)(2q+1-k)$, thus $q(q+1)/6 \leq (k/2)(2q+1-k)$, i.e., $q(q+1)/3k \leq 2q+1-k$. □

Remark. The above necessary condition for $SPE(3K_2, f)$ to be supermagic is not sufficient in general.

Consider $SPE(3K_2, f)$ with $f(e_1) = f(e_2) = 2$, $f(e_3) = 7$. Here $q = 11$ and $q(q+1)/6 = 22$. We observe that $q(q+1)/3k \leq 2q+1-k$.

However, it is impossible to find two numbers from {1_2_---_11} whose sum is 22; hence, $SPE(3K_2, f)$ is not supermagic.□

Theorem 11. $SPE(3K_2, f)$ is supermagic if $f : E(3K_2) \to N$ is $f(e_1) = f(e_2) = n$, $f(e_3) = 2n$, for $n \equiv 0 \pmod 3$ or $n \equiv 2 \pmod 3$.

Proof. Case 1. $n \equiv 0 \pmod 3$. Let $n = 3k$. Break the set $\{1, 2, \ldots, 12k\}$ into 3 piles: $\{1, 2, \ldots, 6k\}$, $\{6k+1, 6k+2, \ldots, 9k\}$, and $\{9k+1, 9k+2, \ldots, 12k\}$. The first pile sums up to $3k(6k+1) = 18k^2 + 3k$; the second pile sums up to $(3k/2)(15k+1) = (1/2)(45k^2+3k)$; and the third pile sums up to $(3k/2)(21k+1) = (1/2)(63k^2+3k)$.

Switch k numbers in the first pile with k numbers in the third pile with pairwise difference $6k - 1$; e.g., $\{4k+2, 4k+3, \ldots, 5k+1\}$ with $\{10k+1, 10k+2, \ldots, 11k\}$.

Next if k is odd, we switch k numbers in the second pile with k number in the third pile with pairwise difference $(3k+1)/2$; e.g., $\{7k+(k+1)/2, 7k+(k+3)/2, \ldots, 8k+(k-1)/2\}$ with $\{9k+1, 9k+2, \ldots, 10k\}$.

If k is even, we simply switch any $k/2$ numbers of the second pile with the $k/2$ numbers of the third pile which differ pairwise by $3k + 1$; e.g., $\{8k+(k/2), \ldots, 9k-1\}$ with $\{11k+(k/2)+1, \ldots, 12k\}$. Now all three piles sum up to $24k^2 + 2k$.

Case 2. For $n = 2 \pmod 3$, let $n = 3k+2$. We break the set $\{1, 2, \ldots, 12k+8\}$ into three piles as before, $\{1, 2, \ldots, 6k+4\}$, $\{6k+5, 6k+6, \ldots, 9k+6\}$, $\{9k+7, \ldots, 12k+8\}$. The first pile sums up to $(3k+2)(6k+5) = 18k^2 + 27k + 10$, the second pile sums up to $(1/2)(3k+2)(15k+11)$, and the third pile sums up to $(1/2)(3k+2)(21k+15)$.

We first switch k numbers in the first pile with the k numbers in the third pile with pairwise difference $6k$, and an extra pair with difference $7k + 2$; e.g., $\{5k+5, \ldots, 6k+4; 2k+5\}$ with $\{11k+5, \ldots, 12k+4; 9k+7\}$.

If k is odd, we switch $(k+1)/2$ numbers in the second pile with the $(k+1)/2$ numbers in the third pile with pairwise difference $3k+2$; e.g., $\{7k+1, \ldots, 7k+(k+1)/2\}$ with $\{10k+3, 10k+4, \ldots, 10k+(k+1)/2+2\}$.

If k is even, we simply switch any $k/2$ numbers in the second pile with the $k/2$ numbers in the third pile with pairwise difference $3k$ and one pair with difference $(5k+2)/2$.

Then all three piles sum up to $(6k+4)(4k+3)$.□

Example 6. If $n = 14$, $f(e_1) = f(e_2) = 14$, and $f(e_3) = 28$, we label e_1 with {29, 30, ...37,45,46,...,49}, e_2 with {1, 26, 27,28,38,39,40,41,42,44,50,51,52,53}, and e_3 with {2,3,..., 25,43, 54, 55, 56}.
Now all three piles sum up to 532.

Remark. $SPE(3K_2,f)$ is not supermagic if $f : E(3K_2) \to N$ is $f(e_1) = f(e_2) = n$, $f(e_3) = 2n$, and $n \equiv 1 \pmod 3$. For if $n = 3k+1$, then as $q = 4n = 12k+4$. We see that the $q(q+1) = 4(3k+1)(12k+5) \neq 0 \pmod 6$. Therefore it is not supermagic.

4. Edge-splitting extensions of cycles

In this section we investigate supermagicness of edge-splitting extensions of cycles.

Theorem 12. If m is odd and $k \geq 1$, then the cycle C_m, has supermagic edge splitting extension for all $f_n : E(C_m) \to N$ with $f_n(e) = n$ for all e in C_m.

For even n, Theorem 12 is a corollary of Theorem 1. For odd n, it is sufficient to prove for $n = 3$, a simple induction will prove for all odd $n > 3$.

We need the following Lemmas in the system of Diophantine equations:

Lemma 4.1. Let m be odd and let $s = 3(3m+1)/2$. A system of m Diophantine equations

$$X_{1,1} + X_{2,1} + X_{3,1} = s$$
$$X_{1,2} + X_{2,2} + X_{3,2} = s$$
$$X_{1,3} + X_{2,3} + X_{3,3} = s$$

....

$$X_{1,m} + X_{2,m} + X_{3,m} = s$$

has a solution with distinct $X_{i,j}$ in $\{1,2,\ldots 3m\}$.

Proof. Set $X_{1,1} = 1$, $X_{1,2} = 2$, $X_{1,3} = 3$, ...$X_{1,k} = k$, ... , $X_{1,m} = m$; $X_{2,m} = m+1$, $X_{2,m-2} = m+2$,..., $X_{2,m-2k} = m+1+k$, ... $X_{2,1} = m+ (m+1)/2$ and $X_{2,m-1} = m+(m+3)/2$, $X_{2,m-3} = m+(m+5)/2$,..., $X_{2,2} = 2m$.

The sums in each row are now distinct consecutive integers $m+ (m+3)/2$, $m+ (m+5)/2$,...,$2m+(m+1)/2$.

Thus setting $X_{3,m-1} = 2m+1$, $X_{3,m-3} = 2m+2$,....., $X_{3,2} = 2m+(m-1)/2$; $X_{3,m} = 2m+(m+1)/2$, $X_{3,m-2} = 2m+(m+3)/2$,..., $X_{3,1}= 3m$.

We see that the sum of each line yields $4m+(m+3)/2 = 3(3m+1)/2$. □

Lemma 4.2. Let $m \geq 4$ be even, and let $s = (9m+2)/2$. Then the system of m Diophantine equations

$$X_{1,1} + X_{2,1} + X_{3,1} = s$$
$$X_{1,2} + X_{2,2} + X_{3,2} = s+1$$
$$X_{1,3} + X_{2,3} + X_{3,3} = s$$

....

$$X_{1,m} + X_{2,m} + X_{3,m} = s+1$$

has a solution in distinct integers in $\{1,2,\ldots 3m\}$.

Proof. Set $X_{1,1} = 1$, $X_{1,2} = 2$, $X_{1,3} = 3$, ...$X_{1,k} = k$, ... , $X_{1,m} = m$; $X_{2,m-1} = m+1$, $X_{2,m-3} = m+2$,...,. $X_{2,1} = m+(m/2)$ $X_{2,m} = m+(m/2)+1$, $X_{2,m-2} = m+(m/2)+2$,...,. $X_{2,2} = 2m$.

The sums of each row are now m+ (m/2)+1, m+ (m/2)+ 2,…,,2m+(m/2). They are consecutive and distinct, except 2m+1 does not occur. By setting $X_{3,m}$ = 2m+1, $X_{3,m-2}$ = 2m+2,….., $X_{3,2}$ = 2m+(m/2); $X_{3,m-1}$ = 2m+(m/2)+1, $X_{3,m-3}$ = 2m+(m/2)+2,….., $X_{3,1}$ = 3m..
We see that the odd rows sum up to be 4m+ (m/2) +1 = s, and the even rows 4m+ (m/2) +2=s+1. □

Proof of Theorem 12. Let $E(C_m) = \{e_1,e_2,\ldots,e_m\}$ and for each k between 1 and m, let $\{e_{k,1},e_{k,2},\ldots,e_{k,n}\}$ be the parallel edges of $SPE(C_m,f)$ correspond to e_k.

We define the edge labeling g: $SPE(C_m,f) \to \{1,2,\ldots,mn\}$ by setting $g(e_{k,j}) = X_{k,j}$ for each k between 1 and m and j between 1 and 3.

Then by Lemma 4.1 and Lemma 4.2, it is a supermagic labeling.

For n $\geq$ 5, we label the edges 3m+1, 3m+ 2,…,4m consecutively in the counterclockwise manner; then label the next cycle 4m+1, 4m+2,…,5m in the clockwise manner, with the edge labels 4m and 4m+1 having the same vertices.

This process is repeated until all the edges are labeled.

It is easy to see the labeling is supermagic.

Example 7. We give a supermagic labeling of $SPE(C_7, f_3)$. We set $X_{1,1}$ = 1, $X_{1,2}$ = 2, $X_{1,3}$ = 3, $X_{1,4}$ = 4, $X_{1,5}$ = 5, $X_{1,6}$ = 6, $X_{1,7}$ = 7, $X_{2,1}$ = 11, $X_{2,3}$ = 10, $X_{2,5}$ = 9, $X_{2,7}$ = 8, $X_{2,2}$ = 14, $X_{2,4}$ = 13, $X_{2,6}$ = 12.
Then we set $X_{3,2}$ = 17, $X_{3,4}$ = 16, $X_{3,6}$ = 15, $X_{3,1}$ = 21, $X_{3,3}$ = 20, $X_{3,5}$ = 19, $X_{3,7}$ = 18, then we observe that each vertex has sum 2s = 2x33 = 66 (see Figure 6):

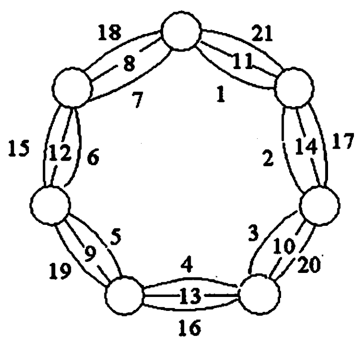


Figure 6

Example 8. We give a supermagic labeling of $SPE(C_5, f_5)$, we see that

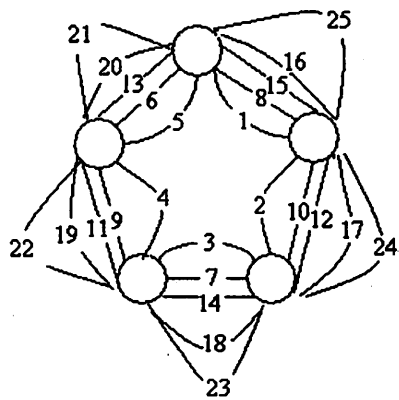
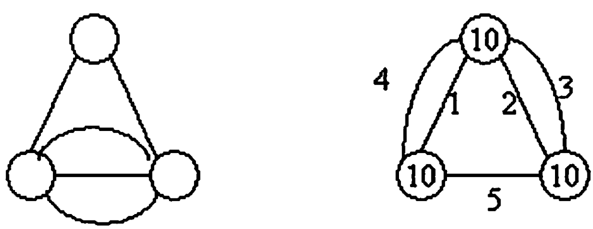


Figure 7

Example 9. Let G = C_3 and $f_i : E(C_3) \rightarrow N$ for i = 1,2 be defined as follows:

	a	b	c
f_1	1	1	3
f_2	2	2	1

Then we see that SPE(C_3,f_1) is not supermagic. However, SEP(C_3,f_2) is supermagic (see Figure 8).



SPE(C_3,f_1) is not supermagic SPE(C_3,f_2) is supermagic

Figure 8

Theorem 13. Suppose f(a) = f(b) = n and f(c) = 1 in C_3 then SPE(C_3,f) is super magic if and only if n = 2.

Theorem 14 Suppose f(a) = f(b) = n and f(c) = 2 in C_3, then SPE(C_3,f) is super magic if and only if n= 2 or 3.

Proof. If n = 2 or 3, the supermagicness of SPE(C_3,f) is shown as Figure 9

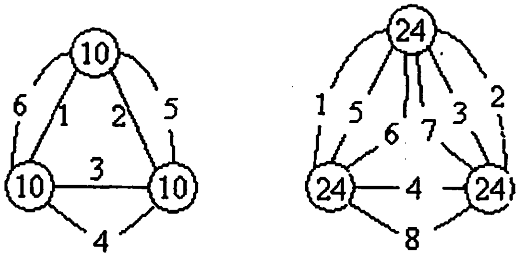


Figure 9

Theorem 15. Suppose $f(e_1) = k$, $f(e_2) = k+1$ and $f(e_3) = k+2$ in C_3, then SPE(C_3,f) is (a) not super magic for k = 1. (b) supermagic for $k \geq 2$.
Proof. (a) If k = 1, SPE(C_3,f) has the following configuration (see Figure 10):

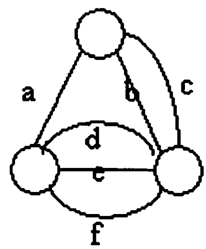


Figure 10

We see that a = b + c = d + e + f or a + b + c = a + d + e + f = b + c + d + e + f has no solution in {1,2,3,4,5,6}.
The only way to have a = d + e + f is a = 6, {d, e, f} ={1,2,3} but there is no way to obtain b and c.
(b) Assume now that $k \geq 2$, we see that $q(q+1) = 3(n+1)(3n+4) = 0 \pmod 3$. Now we set
$C_1 = \{ x: x \equiv 4 \pmod 6 \text{ or } x \equiv 3 \pmod 6, 9 < x \leq q\}$
$C_2 = \{ y: y \equiv 2 \pmod 3, 9 < x \leq q\}$
$C_3 = \{ z: z \equiv 0 \pmod 6 \text{ or } z \equiv 1 \pmod 6, 9 < x \leq q\}$
If k is even and k = 2m, $f(e_1) = k$, $f(e_2) = k + 1$ and $f(e_3) = k + 2$.
Let $A_1=\{6,9\} \cup C_1$, $A_2 = \{2,5,8\} \cup C_2$, $A_3 = \{1,3,4,7\} \cup C_3$ be the edge labels for parallel edges of e_1,e_2,e_3 respectively.
For any u in C_3 we have $f^+(u) = 2(3m+2)(2m+1)$.
If k is odd k = 2m+1, we set $B_1 = \{7,9\} \cup C_1$, $B_2 = \{2,5,8\} \cup C_2$, $B_3 = \{1,3,4,6\} \cup C_3$ as the edge labels for parallel edges of e_1,e_2,e_3 respectively.
For all u in C we have $f^+(u)=2(m+1)(6m+7)$.
Thus when $k \geq 2$, SPE(C_3,f) is supermagic. □

<u>**Example 10**</u>. We label SPE(C_3, f) for k = 2 as follows:{7,8}, {2,4,9}, {1,3,5,6};

for $k = 3$ as follows: {7,9,10}, {1,2,11,12}, {3,4,5,6,8} for $k = 4$ as follows: {1,10,14,15}, {4,5,6,12,13}, {2,3,7,8,9,10}.

If we use the method of Theorem 16 we can have another super magic labeling:

$A_1 = \{6,9,10,15\}$, $A_2 = \{2,5,8,11,14\}$ and $A_3 = \{1,3,4,7,12,13\}$

For each u in C_3, we have $f^+(u) = 80$.

For $k = 5$, $f(e_1) = k$, $f(e_2 = k+1$, $f(e_2) = k+1$, $f(e_3) = k+2$, in C, the labels of spltting edges of e_1,e_2,e_3 are

$A_1 = \{7,9,10,15,16\}$, $A_2 = \{2,5,8,11,14,17\}$ and $A_3 = \{1,3,4,6,12,13,18\}$

For each u in C_3, we have $f^+(u) = 114$.

When n is even, we observe that if we define $f_3 : C_4 \rightarrow N$ by $f_3(e) = 3$ for all e in C_4. The graph SPE(C_4, f) has 12 edges. It satisfies the necessary supermagic condition: $q(q+1) \equiv 0 \pmod p$. It is possible for us to divide {1,2,...,12} into 4 group of triples $\{x_1,x_2,x_3\}$: {1,7,11},{2,8,9} with sum 19 and {3,5,12},{4,6,10} with sum 20. So if we label the splitting edges of (v_1,v_2), (v_2,v_3), (v_3,v_4) and (v_4,v_1) consecutively by {1,7,11}, {3,5,12}, {2,8,9}, {4,6,10}. Then we see that SPE(C_4, f) is supermagic.

The result can be generalized to the following

Theorem 16. SPE(C_{2n}, f) is supermagic if for any $t \geq 2$ $f(e_i) = t$ for all e_i in $E(C_{2n})$.

Proof. (1) When $t = 2m$ by Theorem 1, we conclude that SPE(C_{2n},f) is supermagic,

(2) Assume $t = 2m+1$, in order to prove that SPE(C_{2n},f) is supermagic_it suffices to show that it is true for $t = 3$.

Since $f(e) = 3$, we have $q = \sum f(e) = 6n$, we want to partition the set {1,2,...,6n} into the following sets:

$A(k) = \{k,\ 4n + k,\ 4n - 2(k-1)\}, k = 1,2,\dots,n,$

$B(k) = \{n + k,\ 4n - (2k-1),\ 5n + 1\},\ k = 1,2,\dots,n,$

The sum of numbers in A(k) is $\sigma_1 = 8n + 2$, and the sum of numbers in B(k) is $\sigma_2 = 10n + 1$.

Assume $E(C_{2n}) = \{e_1,e_2,\dots,e_{2n}\}$ we label the splitting edges of e_{2i} by A(i) and splitting edges of e_{2i+1} by B(i) $i = 1,2,\dots, n$, then

$f^+(u) = \sigma_1+\sigma_2 = 3(6n + 1$.

Therefore SPE(C_{2n},f) is supermagic. □

Example 11. For SPE(C_{10},f) with $f(e) = 3$ for all e in C_{10}, we have $q = 30$. We partition {1, 2,..., 30} into the following 3-elements subsets

A(1) = {1, 20_21}_ A(2) = {2, 18_22}_ A(3) = {3, 16_23}_

A(4) = {4, 14_24}_ A(5) ={5, 12_25};

B(1) = {6, 19_26}_ B(2) = {7, 17_27}_ B(3) = {8, 15_28}_

B(4) = {9, 13_29}_ B(5) = {10, 11_30}.

If we label E(SPE(C_{10},f)) according the method of the method proof of Theorem 16, then we see that $\sigma_1 = 42$ _$\sigma_2 = 51$_$f^+(u) = \sigma_1+\sigma_2 = 114$.

References

[1] J. Doob, Characterization of regular magic graphs, *Journal of Combinatorial Theory,* Ser. B **25** (1978), 94-104.

[2] J.A. Gallian, A dynamic survey of graph labeling, *The Electronic J. of Combin.* (2001), # DS6, 1-79.

[3] N. Hartsfield and G. Ringel, Supermagic and Antimagic Graphs, *Journal of Recreational Mathematics*, **21** (1989), 107-115.

[4] Y.S. Ho and Sin-Min Lee, An initial result of supermagicness of complete k-partite graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing,* **39** (2001) 3-17.

[5] R. H. Jeurissen,. "Magic Graphs, a Characterization." *Europ. J. Combin.* **9**, 363-368, 1988.

[6] S. Jezney and M. Trenkler, Characterization of magic graphs, Czechoslovak Math. Journal **33** (108) (1983), 435-438.

[7] M.C. Kong, Sin-Min Lee, and Hugo Sun, On magic strength of graphs, Ars Combinatoria **45** (1997), 193-200.

[8] Sin-Min Lee, New directions in the theory of edge-graceful graphs, Proc. 6^{th} Caribbean Conference. Combinatoric and Computing (1991), 216-231.

[9] Sin-Min Lee, W.M. Pigg and T.J. Cox, On edge-magic cubic graphs conjecture, *Congressus Numerantium,* **105** (1994), 214-222.

[10] Sin Min Lee, Eric Seah and S.K. Tan, On edge-magic graphs, *Congressus Numerantium,* **86** (1992), 179-191.

[11] Sin-Min Lee, E. Seah, Siu-Ming Tong, On the edge-magic and edge-graceful total graphs conjecture, *Congressus Numerantium* **141** (1999), 37-48.

[12] Sin-Min Lee, Ling Wang and Yihui Wen, On the edge-magic cubic graphs and multigraphs, manuscript.

[13] S.P. Lo, On edge-graceful labelings of graphs, *Congressus Numerantium,* **50** (1985), 231-241.

[14].J. Sedlacek, Problem 27, "Theory of Graphs and its applications," Proc. Symposium, Smolenica, 1963, Prague, 163-164.

[15] J. Sedlacek, On magic graphs, *Math. Slov.* **26** (1976), 329-335.

[16] Karl Schaffer and Sin Min Lee, Edge-graceful and edge-magic labelings of Cartesian products of graphs, *Congressus Numerantium* **141** (1999), 119-134,

[17] W.C. Shiu, P.C.B. Lam and H.L. Cheng, Supermagic labeling of sKn,n, *Congressus Numerantium*, **146** (2000), 119-124.

[18] W.C. Shiu, P.C.B. Lam and Sin-Min Lee, On a construction of supermagic graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing* **42** (2002), 147-160.

[19] W.C. Shiu, P.C.B. Lam and Sin-Min Lee, Edge-magicness of the composition of a cycle with a null graph, *Congressus Numerantium,* **132** (1998), 9-18.

[20] W.C. Shiu and Sin-Min Lee, Some edge-magic cubic graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing* **40** (2002), 115-127.

[21] R.P.Stanley, Linear homogeneous Diophantine equations and magic labelings of graphs, *Duke Math. Journa*l **40** (1973), 607-632.

[22] B.M. Stewart, Magic graphs, *Canadian Journal of Mathematics,* **18** (1966), 1031-1059.

[23] B.M. Stewart, Supermagic complete graphs, *Canadian Journal of Mathematics,* **19** (1967), 427-438.

[24] G.C. Sun, J. Guan, and Sin-Min Lee, A labeling algorithm of magic graphs, *Congressus Numerantium*, **102** (1994), 129-137.

[25] G.C. Sun and Sin-Min Lee, Constructions of magic graphs, *Congressus Numerantium* **103** (1994), 243-251.