# Placing monitoring devices in electric power networks modelled by block graphs

[1]David Atkins, [1]Teresa W. Haynes,
and [2]Michael A. Henning*

[1]Department of Mathematics
East Tennessee State University
Johnson City, TN 37614-0002 USA

[2]School of Mathematical Sciences
University of KwaZulu-Natal
Pietermaritzburg, 3209 South Africa

**Abstract**

The problem of monitoring an electric power system by placing as few measurement devices in the system as possible is closely related to the well known vertex covering and dominating set problems in graphs (see SIAM J. Discrete Math. 15(4) (2002), 519–529). A set $S$ of vertices is defined to be a power dominating set of a graph if every vertex and every edge in the system is monitored by the set $S$ (following a set of rules for power system monitoring). The minimum cardinality of a power dominating set of a graph is its power domination number. We investigate the power domination number of a block graph.

**Keywords:** block graph, phase measurement units (PMU's), power domination;

**AMS subject classification: 05C69**

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# 1 Introduction

In this paper we continue the study of the power domination number of graphs started in [5]. The notion of power domination in graphs was inspired by a problem in the electric power system industry. Electric power companies need to continually monitor their system's state as defined by a set of state variables, for example, the voltage magnitude at loads and the machine phase angle at generators [8]. One method of monitoring these variables is to place *Phasor Measurement Units*, called PMU's, at selected locations in the system. Because of the high cost of a PMU, it is desirable to minimize their number while maintaining the ability to monitor (observe) the entire system. A system is said to be *observed* if all of the state variables of the system can be determined from a set of measurements (e.g., voltages and currents).

Let $G = (V, E)$ be a graph representing an electric power system, where a vertex represents an electrical node (a substation **bus** where transmission lines, loads, and generators are connected) and an edge represents a transmission line joining two electrical nodes. The problem of locating a smallest set of PMU's to monitor the entire system is a graph theory problem closely related to the well-known vertex covering and domination problems.

A phasor is a complex vector in polar form that has a magnitude and a phase angle. This stems from the fact that a power system operates in the steady-state as an Alternate Current (AC) network where the voltages and currents are sinusoidal functions of time with constant frequency (60 Hz). As a result, a nodal voltage at a bus and a current through a line are expressed, respectively, as

$$v(t) = V\cos(2pft + q) \qquad \text{and} \qquad i(t) = I\cos(2pft + y),$$

where $f$ is the frequency (60 Hz in the US), $V$ and $I$ are the magnitudes, and $q$ and $y$ are the phase angles of $v(t)$ and $i(t)$, respectively.

A PMU measures the state variable (voltage and phase angle) for the vertex at which it is placed and its incident edges and their end-vertices (these vertices and edges are said to be **observed**). Using Ohm's Law and Kirchoff's Law, we can state the other rules to show observability.

**Rules for Direct Current network:**

1. *Ohm's Law*, P=IR: Any bus (vertex) that is incident to an observed line (edge) connected to an observed bus is observed (the known cur-

rent in the line, the known voltage at the observed bus, and the known resistance of the line determines the voltage at the bus).

2. *Ohm's Law*, I=P/R: Any line joining two observed buses is observed (the known voltage at both observed buses and the known resistance of the line determines the current on the line).

3. *Kirchoff's Law*: If all the lines incident to an observed bus are observed, except one, then all of the lines incident to that bus are observed (the net current flowing through a bus is zero).

**Rules for Alternate Current network:**

1. *Ohm's Law*, $V_i = V_j + Z_{ij} I_{ij}$. Here, $Z_{ij}$ is the impedance of the line connecting buses numbered $i$ and $j$. It is a complex quantity expressed as $Z = R + jX$, where $R$ is the resistance and $X$ is the reactance of the line.

2. *Ohm's Law*, $I_{ij} = (V_i - V_j)/Z_{ij}$: Any line joining two observed buses is observed (the known voltages at both observed buses and the known impedance of the line determines the current of the line).

3. *Kirchoff's Law*: If the current injection at an observed bus is known and if the lines incident to that bus are observed, except one, then all the lines incident at that bus are observed (the net current flowing through the bus is zero).

For a given set of vertices $P \subseteq V$ representing the nodes where the PMU's are placed, the following algorithm determines the sets of (observed) vertices $C$ and edges $F$.

1. Initialize $C = P$ and $F = \{e \in E \mid e \text{ is incident to a vertex in } P\}$.
2. Add to $C$ any vertex not already in $C$ which is incident to an edge in $F$.
3. Add to $F$ any edge not already in $F$ such that
    a. both of its end-vertices are in $C$ or
    b. it is incident to a vertex $v$ of degree greater than one for which all the other edges incident to $v$ are in $F$.
4. If steps 2 and 3 fail to locate any new edges or vertices for inclusion, stop. Otherwise, go to step 2.

Therefore, to solve the power system monitoring problem, we want $C = V$, $F = E$, and to minimize $|P|$. This monitoring problem was introduced and studied in [1, 2, 4, 8].

In [5], the power system monitoring problem was first studied as a variation of the well-known dominating set problem (see [6, 7]). Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a **power dominating set** (PDS) in $G$ if every vertex and every edge in $G$ is observed by $S$ and the **power domination number** $\gamma_P(G)$ is the minimum cardinality of a PDS of $G$. A PDS of cardinality $\gamma_P(G)$ we call a $\gamma_P(G)$-set. The following observation is proven in [5].

**Observation 1** ([5]) *If $G$ is a connected graph with maximum degree at least 3, then $G$ contains a $\gamma_P(G)$-set in which every vertex has degree at least 3.*

Boisen, Baldwin, and Mili [2] investigated approximation algorithms to find a solution to the power system monitoring problem. Haynes et. al. [5] showed that the PDS problem is NP-complete even when restricted to bipartite graphs or chordal graphs. On the other hand, they give a linear algorithm to solve the problem for trees and study theoretical properties of the power domination number in trees. In [3], the power domination number of an $n \times m$ grid graph is determined. In this paper, we investigate the power domination number of a block graph.

# 2 Notation

We shall follow the notation in [6]. In particular, $G = (V, E)$ denotes a graph with vertex set $V$ and edge set $E$. An *end-vertex* is a vertex of degree 1. A *spider* is defined in [5] as a tree with at most one vertex of degree 3 or more. The *spider number* of a tree $T$, denoted by $\mathrm{sp}(T)$, is the minimum number of subsets into which $V(T)$ can be partitioned so that each subset induces a spider.

A *block* of a graph $G$ is a maximal, 2-connected subgraph of $G$. A graph $G$ is a *block graph* if and only if every block of $G$ is a complete graph. In particular, a tree is a block graph in which every block is a $K_2$. We call a block of $G$ that is a complete graph $K_r$, a $K_r$-block of $G$. We define the *block-degree* of a vertex $v$ in $G$ as the number of blocks in $G$ that contain $v$. An *end-block* of $G$ is a block that contains only one cut-vertex of $G$. The number of blocks in $G$ is denoted by $b(G)$.

Let $G$ be a block graph. If $G$ itself is a block or if every block of $G$ is an end-block, then we call $G$ a *block-star*. In particular, a star $K_{1,n}$ where $n \geq 1$ is a block-star in which every block is a $K_2$.

A block graph formed from a block-star by attaching a path to all or to some (including the possibility of none) of its vertices so that the resulting paths are vertex-disjoint, we call a *block-spider*. In particular, if every block is a $K_2$, then a block-spider is a spider. If the block-star of a block-spider has a cut-vertex, then we call it the *head* of the block-spider; otherwise, we designate any vertex of the block-star as the *head* of the block-spider. Thus every vertex of a block-spider, except for possibly its head, belongs to at most two blocks in the block-spider. If any vertex except the head belongs to two blocks, then at least one of these blocks is a $K_2$.

We define the **block-spider number** of a block graph $G$, denoted by $\mathrm{sp}_b(G)$, to be the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a block-spider. We call such a partition a *block-spider partition* and each set of the partition a *block-spider subset*.

Given a connected block graph $G$, we *root* $G$ as follows. We first identify a vertex $r$ of $G$ which we call the *root* of $G$. For each vertex $v \in V(G)$, define the *level number* of $v$, which we denote by $\ell(v)$, to be its distance $d(v,r)$ from $r$. If a vertex $u$ of $G$ is adjacent to $v$ and $\ell(u) > \ell(v)$, then we call $u$ a *block-child* of $v$, and $v$ its *block-parent*. A vertex $w$ is a *block-descendant* of $v$ (and $v$ is a *block-ancestor* of $w$) if the level numbers of the vertices on the $v$–$w$ path are monotonically increasing. We let $D_b(v)$ denote the set of block-descendants of $v$ in the rooted block graph $G$, and we define $D_b[v] = D_b(v) \cup \{v\}$. We define the *maximal block subgraph of $G$ rooted at $v$* to be the block subgraph of $G$ induced by $D_b[v]$, and we denote it by $G_v$.

# 3 Main Results

The following result which states that the power domination number of a tree is precisely its spider number is established in [5].

**Theorem 2** ([5]) *For any tree $T$, $\gamma_P(T) = \mathrm{sp}(T)$.*

Our aim in this paper is to extend the result of Theorem 2 by establishing a relationship between the power domination number of a block graph and its block-spider number.

We prove first that the power domination number of a block graph $G$ equals one if and only if $G$ is a block-spider. A proof of Theorem 3 is given in Section 4.

**Theorem 3** *For any block graph $G$, $\gamma_P(G) = 1$ if and only if $G$ is a block-spider. Furthermore, the head of a block-spider is a PDS of $G$.*

Our main result is the following relationship between the power domination number of a block graph and its block-spider number. A proof of Theorem 4 is given in Section 5.

**Theorem 4** *If $G$ is a connected block graph, then* $\mathrm{sp}_b(G) \leq \gamma_P(G) \leq 2\mathrm{sp}_b(G) - 1$.

The next result shows that the result of Theorem 4 is sharp. A proof of Theorem 5 is given in Section 6.

**Theorem 5** *Given any integers $k$ and $\ell$ with $1 \leq \ell \leq k \leq 2\ell - 1$, there exists a connected block graph $G$ satisfying $\mathrm{sp}_b(G) = \ell$ and $\gamma_P(G) = k$.*

As a consequence of Theorem 4, we can determine a lower bound on the power domination number of a connected block graph in terms of the number of vertices of block-degree at least 3. A proof of Theorem 6 is given in Section 7.

**Theorem 6** *If $G$ is a connected block graph having $k$ vertices of block-degree at least three, then*

$$\gamma_P(G) \geq \frac{k+2}{3},$$

*with equality if and only if $G$ has a block-spider partition such that the blocks of $G$ containing vertices from different block-spiders form a disjoint union of $K_2$s.*

We show finally that if every block of a connected block graph is $K_2$ or $K_3$, then its power domination number is precisely its block-spider number. A proof of Theorem 7 is given in Section 8.

**Theorem 7** *If $G$ is a connected block graph in which every block is $K_2$ or $K_3$, then the heads of the block-spiders induced by a block-spider partition of $V(G)$ form a PDS of $G$. Consequently, $\gamma_P(G) = \mathrm{sp}_b(G)$.*

Theorem 2 is a special case of Theorem 7. As illustrated in Section 6, Theorem 7 is not true if we allow our block graph to contain complete graphs of order greater than 3. We close with the following observation, a proof of which is given in Section 9.

**Observation 8** *For any integer $t \geq 4$, there exists a connected block graph $G$ with one $K_t$-block and all other blocks either $K_2$ or $K_3$ that satisfies $\gamma_P(G) > \text{sp}_b(G)$.*

# 4 Proof of Theorem 3

Suppose $G$ is a block-spider. Then its head is a PDS in $G$, and so $\gamma_P(G) = 1$. To prove the necessity, suppose that $G$ is not a block-spider. Then $G$ contains at least two vertices, $u$ and $v$ say, that both have block-degree at least 3 or that both belong to at least two blocks of order at least 3. We now root the block graph $G$ at any vertex of $G$. Let $S$ be any PDS of $G$. If $|S| = 1$, then, renaming $u$ and $v$ if necessary, we may assume that no vertex in the maximal block subgraph $G_u$ rooted at $u$ belongs to $S$. Since there at least two edges of $G_u$ incident with $u$, no edge in $G_u$ is observed, a contradiction. Therefore, $|S| \geq 2$, and so $\gamma_P(G) \geq 2$.

# 5 Proof of Theorem 4

To prove Theorem 4, we consider the lower bound and the upper bound cases in turn.

## 5.1 Lower Bound

To prove that $\text{sp}_b(G) \leq \gamma_P(G)$, we proceed by induction on $m = \gamma_P(G)$. Suppose $m = 1$. Then $G$ is a block-spider, and so, by Theorem 3, $\text{sp}_b(G) = 1 = \gamma_P(G)$. Suppose, then, that for all connected block graphs $G'$ with $\gamma_P(G') = m$, where $m \geq 1$, that $\text{sp}_b(G') \leq \gamma_P(G')$. Let $G$ be a connected block graph with $\gamma_P(G) = m+1$. By Observation 1, $G$ contains a $\gamma_P(G)$-set $S = \{h_1, h_2, \ldots, h_{m+1}\}$ in which every vertex has degree at least 3.

We now root the block graph $G$ at the vertex $h_{m+1}$. Renaming the vertices of $S$ if necessary, we may assume that among all the vertices of $S$, $h_1$ has the largest level number, i.e., among all vertices in $S$, $h_1$ is at maximum distance from $h_{m+1}$ in $G$. Let $w$ be the block-parent of $h_1$ and let $B_1$ be the block containing $h_1$ and $w$. We now consider two possibilities.

**Case 1.** $w$ has block-degree two.

Let $u$ be the block-ancestor of $w$ of degree at least 3 that is at minimum distance from $w$. Then every internal vertex on the $u$–$w$ path has degree 2 in $G$. Let $v$ be the block-child of $u$ on the $u$–$w$ path (possibly, $v = w$). Note that no internal-vertex on the $u$–$h_1$ path (including the vertex $w$) belongs to $S$.

We now define a set $V_1$ as follows. If $S$ contains two vertices of $B_1$, let $V_1 = D_b[h_1]$. If $h_1$ is the only vertex of $S$ in $B_1$ and if $u \in S$, let $V_1 = D_b[v]$. If $h_1$ is the only vertex of $S$ in $B_1$ and if $u \notin S$, let $V_1' = D_b[v] \cup \{u\}$. If now $G - V_1'$ contains a path-component $P$ that contains no vertex of $S$, then let $V_1 = V_1' \cup V(P)$ (notice that since $S$ is a PDS of $G$, there is at most one such path-component $P$ and $u$ is adjacent to an end-vertex of $P$ and to no other vertex of $P$). On the other hand, if every component of $G - V_1'$ contains a vertex of $S$, then let $V_1 = V_1'$. In all the above cases, let $G' = G - V_1$.

By construction, $G[V_1]$ is a block-spider with head $h_1$, and $G'$ is a block graph (possibly disconnected) in which $S - \{h_1\}$ is a PDS of $G'$. Thus, $\gamma_P(G') \leq m$. Applying the inductive hypothesis to each component of $G'$, we have $\text{sp}_b(G') \leq \gamma_P(G')$. Thus there exists a block-spider partition of $V(G')$ with $m$ or fewer subsets. Adding the subset $V_1$ to the block-spider partition of $V(G')$ produces a block-spider partition of $V(G)$ with at most $m + 1$ subsets. Thus, $\text{sp}_b(G) \leq \gamma_P(G)$.

**Case 2.** $w$ has block-degree at least 3.

We now define a set $V_1$ as follows. If $S$ contains at least two vertices of $B_1$, let $V_1 = D_b[h_1]$. If $h_1 \in S$ and $w \in S$, let $V_1 = D_b[h_1]$. If $h_1$ is the only vertex of $S$ in $B_1$ and if $w \notin S$, let

$$V_1' = \left( \bigcup_{x \in N[h_1] - \{w\}} D_b[x] \right) \cup \{w\}.$$

If now $G - V_1'$ contains a path-component $P$ that contains no vertex of $S$, then let $V_1 = V_1' \cup V(P)$. On the other hand, if every component of $G - V_1'$ contains a vertex of $S$, then let $V_1 = V_1'$. In all the above cases, let $G' = G - V_1$. Proceeding now exactly as in paragraph three of Case 1 above, we have $\text{sp}_b(G) \leq \gamma_P(G)$.

## 5.2 Upper bound

Next we prove that $\gamma_P(G) \le 2\mathrm{sp}_b(G) - 1$. Suppose $\mathrm{sp}_b(G) = m$. If $m = 1$, then $G$ is a block-spider, and so, by Theorem 3, its head is a PDS of $G$ and $\gamma_P(G) = 1 = \mathrm{sp}_b(G)$. Suppose, then, that $m \ge 2$. Let $G$ be a block graph with $\mathrm{sp}_b(G) = m$. Let $\{V_1, V_2, \ldots, V_m\}$ be a block-spider partition of $V(G)$. For $i = 1, 2, ..., m$, let $G_i$ be the block-spider induced by $V_i$, and so $G_i = G[V_i]$, and let $h_i$ be the head of $G_i$. Then, $\{h_i\}$ is a PDS of $G_i$. Let $F$ be the graph with vertex set $\{V_1, V_2, \ldots, V_m\}$ where two vertices $V_i$ and $V_j$ are adjacent in $F$ if and only if there is an edge of $G$ joining a vertex of $V_i$ and a vertex of $V_j$. Since $G$ is a block graph, so too is $F$. Further, every block in $F$ corresponds to a block in $G$. For each block in $F$, we select one vertex from the corresponding block in $G$ and we let $S_F$ denote the resulting set of selected vertices. Since $F$ has order $m$, there are at most $m - 1$ blocks in $F$, and so $|S_F| = b(F) \le m - 1$ with equality if and only if $F$ is a tree. Then, $S_F \cup \{h_1, h_2, \ldots, h_m\}$ is a PDS of $G$, and so $\gamma_P(G) \le |S_F| + m \le 2m - 1 = 2\mathrm{sp}_b(G) - 1$.

# 6 Proof of Theorem 5

Let $t \ge 3$ be a fixed integer. Consider two complete graphs $K_t$ that have exactly one vertex $h$ in common. Let $u_1$ be a vertex from one of these complete graphs and $v_1$ from the other where $u_1 \ne h$ and $v_1 \ne h$. Attach to $u_1$ a path $u_1, u_2, u_3, u_4$ and to $v_1$ a path $v_1, v_2, v_3, v_4$ (so that the resulting paths are vertex disjoint). Let $F$ denote the resulting graph. Then, $F$ is a block-spider with head $h$. Let $F_1, \ldots, F_\ell$ be $\ell$ disjoint copies of $F$. For $i = 1, \ldots, \ell$, we label the vertices $u_j$ and $v_j$, $1 \le j \le 4$, of $F$ by $u_{i,j}$ and $v_{i,j}$, respectively, in $F_i$ and the vertex $h$ of $F$ by $h_i$ in $F_i$. Let $F'$ be the disjoint union $\cup_{i=1}^{\ell} F_i$ of the graphs $F_i$.

Suppose that $k = \ell + r$ where $r \in \{0, \ldots, \ell - 1\}$. If $r = 0$, let $E_1 = \emptyset$; otherwise, let $E_1 = \{v_{i,2}u_{i+1,j}, v_{i,3}u_{i+1,j} \mid 1 \le i \le r \text{ and } 2 \le j \le 3\}$. If $r = \ell - 1$, let $E_2 = \emptyset$; otherwise, let $E_2 = \{v_{i,3}u_{i+1,3} \mid i = r + 1, \ldots, \ell - 1\}$. Let $G$ be obtained from $F$ by adding the set of edges $E_1 \cup E_2$. (The graph $G$ when $t = 3$, $\ell = 4$ and $k = 6$ is illustrated in Figure 1 where the six large darkened vertices from a $\gamma_P(G)$-set.)

Then, $G$ is a connected block graph in which every block is $K_2$, $K_4$ or $K_t$. The partition $\{V(F_1), \ldots, V(F_\ell)\}$ of $V(G)$ is a minimum block-spider partition of $G$, and so $\mathrm{sp}_b(G) = \ell$. If $r = 0$, let $S = \emptyset$; otherwise, let $S = \cup_{i=1}^{r} \{v_{i,3}\}$. Then, the set $\{h_1, \ldots, h_\ell\} \cup S$ is a minimum PDS of $G$,

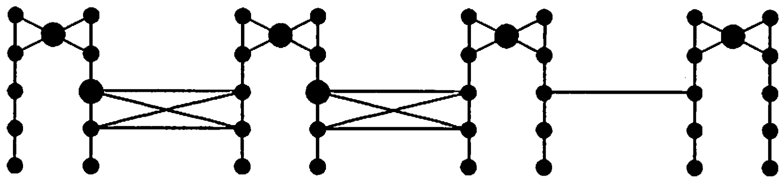


Figure 1: A connected block graph $G$ with $\mathrm{sp}_b(G) = 4$ and $\gamma_P(G) = 6$.

and so $\gamma_P(G) = \ell + r = k$.

# 7 Proof of Theorem 6

Let $\mathrm{sp}_b(G) = m$. Then by Theorem 3, $\gamma_P(G) \geq m$. Let $\{V_1, V_2, \dots, V_m\}$ be a block-spider partition of $V(G)$. For $i = 1, 2, \dots, m$, let $G_i$ be the block-spider induced by $V_i$; that is, $G_i = G[V_i]$. Further, let $h_i$ be the head of $G_i$. Since each $G_i$ is a block-spider, every vertex of $G_i$, except for possibly its head, belongs to at most two blocks in the block-spider.

Let $F$ be the graph with vertex set $\{V_1, V_2, \dots, V_m\}$ where two vertices $V_i$ and $V_j$ are adjacent in $F$ if and only if there is an edge of $G$ joining a vertex of $V_i$ and a vertex of $V_j$. Since $G$ is a block graph, so too is $F$. Suppose that $K_t$ is the largest block in $F$. For $\ell = 2, \dots, t$, let $b_\ell$ denote the number of $K_\ell$-blocks in $F$. Let $T_F$ be the tree obtained from $F$ by replacing every $K_\ell$-block in $F$ where $\ell \geq 3$ by a spanning tree of the $K_\ell$-block (of order $\ell$ and size $\ell - 1$). Then,

$$m - 1 = |E(T_F)| = \sum_{\ell=2}^{t} (\ell - 1) b_\ell \geq \sum_{\ell=2}^{t} b_\ell. \tag{1}$$

Each vertex of $V_i - \{h_i\}$ that is adjacent in $G$ to only vertices of $V_i$ has block-degree at most 2 in $G$. Let $B$ be a block in $G$ corresponding to a $K_\ell$-block of $F$. Then, $|V(B) \cap V_j| \leq 2$ for all $j = 1, 2, \dots, m$ unless $h_i \in V(B)$ in which case possibly $|V(B) \cap V_j| > 2$. Let $E_B$ denote the set of all edges of $B$ that do not belong to any of the block-spiders $G_i$. Suppose $|V(B) \cap V_i| \geq 2$ for some $i$. Then the block-degree of each vertex of $V(B) \cap V_i$ in $G$ is the same as its block-degree in $G - E_B$. On the other hand, if $V(B) \cap V_i = \{v\}$ for some $i$, then the block-degree of $v$ in $G$ is one larger than its block-degree in $G - E_B$. Since $|V(B) \cap V_i| \geq 1$ for exactly $\ell$ values of $i$, at most $\ell$ vertices in $G$ (each from different sets $V_i$) have block-degrees in $G$ one more than their block-degrees in $G - E_B$. This implies that $G$ contains at most $m + \sum_{\ell=2}^{t} \ell b_\ell$ vertices of block-degree at least 3. Hence,

$$k \le m + \left( \sum_{\ell=2}^{t} b_\ell \right) + \sum_{\ell=2}^{t} (\ell - 1) b_\ell. \tag{2}$$

Thus, by Equations (1) and (2), $k \le 3m - 2$. Hence, $\gamma_P(T) \ge m \ge (k+2)/3$. Suppose $\gamma_P(T) = (k+2)/3$. Then we must have equality throughout Equations (1) and (2). In particular, $F$ is a tree and each block $B$ of $G$ corresponding to an edge of $F$ is a $K_2$-block the two vertices of which are from different block-spiders and are not the heads of the block-spiders. Further, no two such blocks $B$ have any vertex in common. The desired characterization follows.

That this bound is sharp, may be seen as follows. Let $n \ge 1$ and $t \ge 2$ be integers. Let $G'$ be the corona of a path $P$ on $(2t-1)n$ vertices; that is, $G' = P_{(2t-1)n} \circ K_1$ (the corona of a path is also called a *comb*). Let the path be denoted by $P: v_1, v_2, \ldots, v_{(2t-1)n}$. For each $i = 0, \ldots, n-1$, let $E_i'$ and $E_i''$ be the set of edges defined by $E_i' = \{v_{(2t-1)i+j}\, v_{(2t-1)i+\ell} \mid 1 \le j < \ell \le t\}$ and $E_i'' = \{v_{(2t-1)i+j}\, v_{(2t-1)i+\ell} \mid t \le j < \ell \le 2t-1\}$. Each of the sets $E_i'$ and $E_i''$ induce a complete graph $K_t$ and have only the vertex $v_{(2t-1)i+t}$ in common. For $i = 0, \ldots, n-1$, let $E_i = (E_i' \cup E_i'') - E(P)$. Let $G$ be the graph obtained from $G'$ by adding the edges $\cup_{i=1}^{n} E_i$. Then, $G$ is a connected block graph in which every block is $K_2$ or $K_t$. (The graph $G$ when $n = t = 3$ is illustrated in Figure 2 where the three large darkened vertices from a $\gamma_P(G)$-set.) Let $D = \cup_{i=0}^{n-1} \{v_{(2t-1)i+t}\}$. Then, $D$ is a PDS of $G$, and so $\gamma_P(G) \le |D| = n$. Let $S = \cup_{i=1}^{n-1} \{v_{(2t-1)i}, v_{(2t-1)i+1}\}$. Then the set of vertices of $G$ of block-degree at least three is the set $D \cup S$. Hence, $G$ has $k = |D| + |S| = 3n - 2$ vertices of block-degree at least three, and so as shown earlier, $\gamma_P(G) \ge (k+2)/3 = n$. Consequently, $\gamma_P(G) = (k+2)/3 = n$.

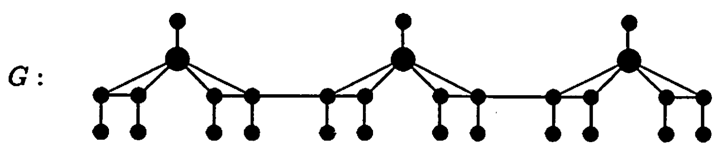


Figure 2: A connected block graph $G$ with $k = 7$ vertices of block-degree at least 3 and $\gamma_P(G) = (k+2)/3$.

## 8 Proof of Theorem 7

We proceed by induction on $m = \mathrm{sp}_b(G)$. Suppose $m = 1$. Then $G$ is a block-spider, and so, by Theorem 3, its head is a PDS of $G$ and $\gamma_P(G) = 1 = \mathrm{sp}_b(G)$. Suppose, then, that for all connected block graphs $G'$ in which every block is $K_2$ or $K_3$ with $\mathrm{sp}_b(G') = m$, where $m \geq 1$, the heads of the block-spiders induced by a block-spider partition of $V(G')$ power dominate $G'$. Let $G$ be a block graph in which every block is $K_2$ or $K_3$ with $\mathrm{sp}_b(G) = m+1$. Let $\{V_1, V_2, \ldots, V_{m+1}\}$ be a block-spider partition of $V(G)$. For $i = 1, 2, ..., m+1$, let $G_i$ be the block-spider induced by $V_i$, and so $G_i = G[V_i]$, and let $h_i$ be the head of $G_i$.

Let $F$ be the graph with vertex set $\{V_1, V_2, \ldots, V_{m+1}\}$ where two vertices $V_i$ and $V_j$ are adjacent in $F$ if and only if there is an edge of $G$ joining a vertex of $V_i$ and a vertex of $V_j$. Since $G$ is a block graph in which every block is $K_2$ or $K_3$, so too is $F$. We must now consider two cases, depending on whether $F$ contains an end-block $K_2$ or whether every end-block of $F$ is a $K_3$.

**Case 1.** $F$ contains an end-block $K_2$.

Without loss of generality, we may assume that $V_1$ is an end-vertex of $F$ and that $V_1V_2 \in E(F)$. The edge $V_1V_2$ in $F$ corresponds to a block $B$ in $G$ that contains at least one vertex of each of $V_1$ and $V_2$, but no vertex in $V(G) - V_1 - V_2$. Let $G' = G - V_1$; that is, $G'$ is the connected block graph (in which every block is $K_2$ or $K_3$) obtained from $G$ by deleting the vertices in the subset $V_1$. If $\mathrm{sp}_b(G') < m$, then we can add the subset $V_1$ to a minimum block-spider partition of $V(G')$ to produce a block-spider partition of $V(G)$ of cardinality $\mathrm{sp}_b(G') + 1 < m + 1 = \mathrm{sp}_b(G)$, which is impossible. Hence, $\mathrm{sp}_b(G') \geq m$. Since $\{V_2, \ldots, V_{m+1}\}$ is a block-spider partition of $V(G')$, $\mathrm{sp}_b(G') \leq m$. Consequently, $\mathrm{sp}_b(G') = m$. Applying the inductive hypothesis to $G'$, $S' = \{h_2, \ldots, h_{m+1}\}$ is a PDS of $G'$. Thus all vertices and edges of $G'$ are observed by $S'$. Let $S = S' \cup \{h_1\}$. We show that $S$ is a PDS of $G$, and so $\gamma_P(G) \leq m + 1 = \mathrm{sp}_b(G)$. We consider two possibilities depending on whether $B = K_2$ or $B = K_3$.

Suppose that $B = K_2$. Let $V(B) = \{v_1, v_2\}$ where $v_i \in V_i$ for $i = 1, 2$. Thus, $v_1v_2 \in E(G)$ and this is the only edge joining a vertex in $V_1$ and $V(G) - V_1$. Since $S'$ is a PDS of $G'$, the vertex $v_2$ is observed by the set $S'$ in $G$. The vertex $v_1$ is observed by the vertex $h_1$ in $G$. Hence the edge $v_1v_2$ is observed by the set $S$. It follows that $S$ is a PDS of $G$.

Suppose that $B = K_3$. Suppose, first, that $V(B) = \{u_1, v_1, v_2\}$ where

$\{u_1, v_1\} \subset V_1$ and $v_2 \in V_2$. Let $G_1$ be rooted at $h_1$. We may assume that $u_1$ is a block-child of $v_1$ (possibly, $h_1 = v_1$). The vertex $v_1$ and all its incident edges except for the edge $u_1v_1$ are observed by the vertex $h_1$ in $G$, while $v_2$ is observed by the set $S'$ in $G$. Hence the edge $v_1v_2$ is observed by $S$ in $G$. Thus, the edges $v_1u_1$ and $u_1v_2$ in turn become observed by $S$ in $G$. It follows that $S$ is a PDS of $G$.

Suppose, secondly, that $V(B) = \{v_1, v_2, u_2\}$ where $v_1 \in V_1$ and $\{u_2, v_2\} \subset V_2$. The vertex $v_1$ is observed by $h_1$ in $G$. If $u_2$ and $v_2$ are both observed by $S'$ in $G'$ before the edge $u_2v_2$ is observed, then $S$ observes both $v_1u_2$ and $v_1v_2$ in $G$. Consequently, the edge $u_2v_2$ is observed by $S$ in $G$. It follows that $S$ is a PDS of $G$. On the other hand, if $u_2$ or $v_2$, say $u_2$, is only observed after the edge $u_2v_2$ is observed by $S'$ in $G'$, then $S$ observes the vertices $v_1$ and $v_2$ in $G$, and therefore the edge $v_1v_2$. Thus, all edges incident with $v_2$ are observed by $S$ in $G$, and so the vertex $u_2$ becomes observed by $S$ in $G$. It follows that $S$ is a PDS of $G$.

**Case 2.** Every end-block of $F$ is $K_3$.

Without loss of generality, we may assume that $F[\{V_1, V_2, V_3\}]$ is an end-block of $F$. The block $F[\{V_1, V_2, V_3\}]$ in $F$ corresponds to a block $B = G[\{v_1, v_2, v_3\}] = K_3$ in $G$ where $v_i \in V_i$ for $i = 1, 2, 3$. If $m = 3$, then the set $\{h_1, h_2, h_3\}$ observes each of $v_1$, $v_2$ and $v_3$ in $G$, and therefore observes each of the edges $v_1v_2$, $v_1v_3$ and $v_2v_3$. It follows that if $m = 3$, then $\{h_1, h_2, h_3\}$ is a PDS of $G$. Hence we may assume that $m \geq 4$ and that $V_3$ is a cut-vertex of $F$. Thus each of $V_1$ and $V_2$ has degree 2 in $F$, while $V_3$ has degree at least 3 in $F$.

Let $G' = G - V_1 - V_2$; that is, $G'$ is the connected block graph (in which every block is $K_2$ or $K_3$) obtained from $G$ by deleting the vertices in the subsets $V_1$ and $V_2$. If $\text{sp}_b(G') < m - 1$, then we can add the subsets $V_1$ and $V_2$ to a minimum block-spider partition of $V(G')$ to produce a block-spider partition of $V(G)$ of cardinality $\text{sp}_b(G') + 1 < m + 1 = \text{sp}_b(G)$, which is impossible. Hence, $\text{sp}_b(G') \geq m - 1$. Since $\{V_3, \ldots, V_{m+1}\}$ is a block-spider partition of $V(G')$, $\text{sp}_b(G') \leq m - 1$. Consequently, $\text{sp}_b(G') = m - 1$.

Applying the inductive hypothesis to $G'$, $S' = \{h_3, \ldots, h_{m+1}\}$ is a PDS of $G'$. Therefore all vertices and edges of $G'$ are observed by $S'$. Let $S = S' \cup \{h_1, h_2\}$. Then the vertex $v_1$ is observed by $h_1$, the vertex $v_2$ by $h_2$, and the vertex $v_3$ by $S'$ in $G$. Thus each of the edges $v_1v_2$, $v_1v_3$ and $v_2v_3$ is observed by $S$ in $G$. It follows that since $h_i$ is a PDS of $G_i$ for $i = 1, 2$ and since $S'$ is a PDS of $G'$, $S$ is a PDS of $G$, and so $\gamma_P(G) \leq m + 1 = \text{sp}_b(G)$. However by Theorem 4, $\gamma_P(G) \geq \text{sp}_b(G)$. Consequently, $\gamma_P(G) = \text{sp}_b(G)$.

# 9 Proof of Theorem 8

Suppose first that $t = 2k$ for some integer $k \geq 2$. For $i = 1, \ldots, k$, let $G_i$ be the graph obtained from the path $u_{i,1}, u_{i,2}, \ldots, u_{i,6}$ by adding the edge $u_{i,4}u_{i,6}$. Then, $G_i$ is a block-spider. Let $h_i = u_{i,4}$. Let $G$ be obtained from the disjoint union $\cup_{i=1}^{k} G_i$ of the graphs $G_i$ by forming a clique on the set $S = \cup_{i=1}^{k} \{u_{i,2}, u_{i,3}\}$. Then, $G[S] = K_t$ and $G$ is a connected block graph in which every block is $K_2$ or $K_3$, except for the block induced by the set $S$. (The graph $G$ when $t = 6$ is illustrated in Figure 3 where the four large darkened vertices from a $\gamma_P(G)$-set.) The partition $\{V(G_1), \ldots, V(G_k)\}$ of $V(G)$ is a minimum block-spider partition of $G$, and so $\text{sp}_b(G) = k$. On the other hand, the set $\{h_1, \ldots, h_k\} \cup \{u_{1,3}\}$ is a minimum PDS of $G$, and so $\gamma_P(G) = k + 1$. Thus, $\gamma_P(G) > \text{sp}_b(G)$.

$G:$

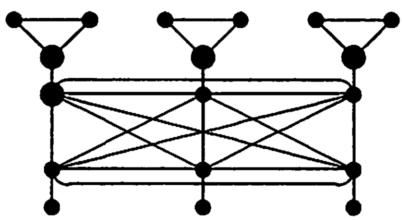

Figure 3: A connected block graph $G$ with $\gamma_P(G) > \text{sp}_b(G)$ in which all but one block is $K_2$ or $K_3$.

Suppose secondly that $t = 2k + 1$ where $k \geq 2$. Let $G_{k+1}$ be the graph obtained from the path $u_{k+1,1}, u_{k+1,2}, \ldots, u_{k+1,6}$ and adding the edge $u_{k+1,4}u_{k+1,6}$. Then, $G_i$ is a block-spider. Let $H$ be the graph obtained from the disjoint union of graph $G$ constructed earlier and the graph $G_{k+1}$ by adding all edges joining $u_{k+1,3}$ to vertices in the set $S$. Then, $H[S \cup \{u_{k+1,3}\}] = K_t$ and $H$ is a connected block graph in which every block is $K_2$ or $K_3$, except for the block induced by the set $S \cup \{u_{k+1,3}\}$, with $\gamma_P(G) = k + 2 > k + 1 = \text{sp}_b(G)$.

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