# Maximum genus, Girth and Maximum non-adjacent edge set

Yichao Chen[1] Yanpei Liu[2]
Department of Mathematics, Beijing JiaoTong University, Beijing, 100044.
People's Republic of China

**Abstract**: The maximum genus, a topological invariant of graphs, was inaugurated by Nordhaus, et,al [16]. In this paper, the relations between the maximum non-adjacent edge set and the upper embeddability of a graph are discussed, and the lower bounds on maximum genus of a graph in terms of its girth and maximum non-adjacent edge set are given. Furthermore these bounds are shown to be best possible. Thus, some new results on the upper embeddability and the lower bounds on the maximum genus of graphs are given.

**Key words**: Maximum genus, Girth, Maximum non-adjacent edge set

## 1 Introduction

A *graph* is often denoted by $G = (V, E)$ with $\nu = |V|$ and $\epsilon = |E|$ which are called the *cardinality of vertex set* and *cardinality of edge set* of $G$ respectively. A graph is *simple* if it neither contain multiple edges nor self-loops. In this paper, the graphs $G$ are permitted to have multiple edges and self-loops. If a graph doesn't contain self-loops but contain multiple edges, we call it *multi-graph*. The *vertex-connectivity* $\kappa(G)$ of a graph $G$ is the minimum number vertices whose removal from $G$ results in a disconnected or trivial graph. The *edge-connectivity* $\kappa_1(G)$ of a graph $G$ is the minimum number edges whose removal from $G$ results in a disconnected or trivial graph. The length of the shortest cycle in a simple graph that

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[2] Partially supported by NNSFC under Grant No. 60373030.
[1] Partially supported by found of Beijing JiaoTong Univeristy under Grant No. 2004SM054.

contains cycle is called the *girth* of $G$ and is denoted by $g(G)$. If $G$ has no cycles, then $g(G) = \infty$. If $G$ has multiple edges (self-loops), then we define $g(G) = 2$ ($g(G) = 1$). Recall that an independent set of vertex in a graph $G$ is one whose elements are pairwise non-adjacent. The *independence number* $\alpha(G)$ of a graph $G$ is the number of vertices in a maximum independent set of $G$. We define the *line graph* of $L(G)$ of $G$ as that graph whose vertices can be put in one-to-one correspondence with the edges of $G$ in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of $G$ are adjacent. Thus the maximum non-adjacent edge set of a graph $G$ corresponds to maximum vertex independent set of its line graph $L(G)$. We denote the cardinality of the *maximum non-adjacent edge set* by $\alpha_1(G)$. It has been shown by R.Duke [5]that if a connected graph $G$ has embedding on *compact orientable 2-manifolds*, or in other words, on the surfaces of genera $m$ and $n$, then for all $k$ ($m \leq k \leq n$), $G$ has an embedding on an *orientable surface* of genus $k$. It can be shown from the *Eulerian formula* for polyhedra that

$$\left\lceil \frac{\beta(G)+1}{2} - \frac{\varepsilon}{2} \right\rceil \leq \gamma(G) \leq \left\lfloor \frac{\beta(G)}{2} \right\rfloor$$

where $\beta(G) = \varepsilon - \nu + 1$ is called the *Betti number* of $G$ and $\gamma(G)$ is the *orientable genus* of $G$.

For a graph $G$, if $\gamma_M(G) = \left\lfloor \frac{\beta(G)}{2} \right\rfloor$, then $G$ is said to be *upper embeddable* on the orientable surfaces.

The study of the maximum genus of a graph was inaugurated by Nordhaus, Stewart and White [15]. From then on, various classes of graphs have been proved upper embeddable. For example (1) 4-edge connected graphs [11,26]; (2)cyclically 4-edge connected graphs[22], (3)connected, and locally connected graphs [17]; (4)connected, and locally quasi-connected graphs [19]; (5)connected $4k+2$-regular graphs[24]; (6)a graph which has a triangulation on some surface [8]; (7) a simple graph with diameter two[23]; (8) a simple graph that has a 2-cell embedding in some surface(orientable or not) such that the length of every face does not exceed five[21]; (9)connected, $N_2$-locally connected graphs [20]. However there are examples of $k$-edge-connected ($k$-vertex-connected) graphs that are not upper embeddable [9], for $k = 1, 2, 3$. Thus, one may want to obtain the lower bounds for the maximum genus of a $k$-edge-connected graphs ($k \leq 3$). Chen et al. [4] have shown that the maximum genus of a simple graph is at least one-quarter of its Betti number. Chen et al. [3] gave tight lower bounds of the maximum genus via connectivity. Huang [6,7] obtained the lower bounds of maximum genus via girth, chromatic number and independence number. Li and Liu [12] obtained the lower bounds of maximum genus via girth and connectivity. Archdeacon et al. [1] gave shortened proofs of the known results and a new bounds of maximum genus in the non-simple case. For more details, one may see [1,3,4,6,7,12], etc. In this paper, we first discuss the relations between the maximum non-adjacent edge set and the upper embeddability of a graph, then characterize the lower bounds on the maximum genus of graphs by using maximum non-adjacent edge set and girth. It's shown that the lower bounds are best possible.

## 2 Preliminaries

There are two equivalent characterizations on the maximum genus of a graph, due to Xuong [26], Liu[13,15] and Nebesky[18], respectively.

Let $T$ be a *spanning tree* of a connected graph $G$. The edge complement $G-T$ of the spanning tree $T$ is called a *co-tree.* A component $H$ of $G-T$ is called an *odd component* if $H$ has an odd number of edges; otherwise, it is called an *even component. The deficiency* $\xi(G,T)$ *of a spanning tree* $T$ of a connected graph $G$ is defined to be the number of odd components of $G-T$. *The deficiency* $\xi(G)$ *of a graph* $G$ is the minimum of $\xi(G,T)$ over all spanning trees $T$. Now we restate the *characterization theorem.*

**Theorem 2.1**[13,26] *Let* $G$ *be a connected graph. Then*
(1) $G$ *is upper embeddable if and only if* $\xi(G) \leq 1$
(2) $\gamma_M(G) = \frac{\beta(G)-\xi(G)}{2}$.

The characterization of Nebesky is in terms of an edge cut set of the graph. The two are mutually dual. Let $A$ be a subset of $E$. let $c = c(G-A)$ be the number of components of $G-A$, and let $b = b(G-A)$ be the number of components of $G-A$ with odd Betti number.

**Theorem 2.2**[18] *Let* $G$ *be a connected graph. Then*
(1) $G$ *is upper embeddable if and only if* $c(G-A)+b(G-A)-2 \leq |A|$, *for any subset* $A$ of $E$
(2) $\xi(G) = \max_{A \subset E}\{c(G-A)+b(G-A)-|A|-1\}$

Let $F_{i_1}, \cdots, F_{i_k}$ be some connected components of $G-A$, and $E(F_{i_1}, \cdots, F_{i_k})$ be the set of edges whose one end vertex is in $V(F_{i_m})$, the other in $V(F_{i_n})(1 \leq m,n \leq k, m \neq n)$. Any set $A \subset E$ such that $\xi(G) = c(G-A)+b(G-A)-|A|-1$ will be called a *Nebesky set,* furthermore, if $A$ is minimal, then it will be called a *minimal Nebesky set.* The following Lemma can be seen in [7], etc.

**Lemma 2.3**[7] *Let* $G$ *be a connected graph. if* $G$ *is not upper embeddable, there exists a minimal Nebesky set* $A$ *such that the following properties are satisfied,*

(a) $c(G-A) \geq 2$, *and* $\beta(F) = 1$ $(mod\ 2)$ *for any connected component* $F$ *of* $G-A$;

(b) $F$ *is an vertex-induced graph of* $G$ *for any connected component* $F$ *of* $G-A$;

(c) $|E(F_{i_1}, \cdots, F_{i_k})| \leq 2k-3$ *for any* $k$ *distinct components* $F_{i_1}, \cdots, F_{i_k}$ *of* $G - A$;

(d) $\xi(F) = 1$ *for any connected component* $F$ *of* $G - A$;

(e) $\xi(G) = 2c(G - A) - |A| - 1$.

Let $A$ be a minimal Nebesky set and $F_1, \cdots, F_l$ be all connected components of $G - A$. Let $E(F_i, A)$ be the set of edges whose one end vertex is in $V(F_i)$, the other in $V(A)$, for $i = 1, \cdots, l$. The following fact is obvious.

$$|A| = \frac{1}{2}\sum_{i=1}^{l} |E(F_i, A)| \tag{1}$$

# 3 The maximum non-adjacent edge set and Upper embeddability

In this section, we will discuss the relations between the upper embeddability and maximum non-adjacent edge set of a graph.

**Theorem 3.1** *Let* $G$ *be a connected graph. If* $\alpha_1(G) \leq 1$, *then* $G$ *is upper embeddable.*

**Proof.** Suppose that the graph $G$ is not upper embeddable. By Lemma 2.1, there exists $A \subset E$ such that the properties (a)-(e) of Lemma 2.3 are satisfied. Let $F_1, F_2, \cdots, F_l (l \geq 2)$ be the all connected components of $G - A$. By (a) of Lemma 2.3, we know that $\alpha_1(F_i) \geq 1$. Since $\alpha_1(G) \geq \sum_{k=1}^{l} \alpha_1(F_k) \geq l \geq 2$, it's impossible.

**Theorem 3.2** *Let* $G$ *be a simple graph. If* $\alpha_1(G) \leq 2$, *then* $G$ *is upper embeddable.*

**Proof.** Note that if $G$ is a simple graph, by (a) of Lemma 2.3, we know that $|V(F_i)| \geq g(G) \geq 3$. If $l \geq 3$, we have $\alpha_1(G) \geq \sum_{k=1}^{l} \alpha_1(F_k) \geq l \geq 3$, otherwise $l = 2$, let $E(F_1, F_2) = \{e\}$, since $g(G) \geq 3$, we can easily find three non-adjacent edges $\{e, f, g\}$ where $f \in F_1, g \in F_2$. Thus, $\alpha_1(G) \geq 3$, it's impossible.

**Remark 1** The upper bound 1 (2) of Theorem 3.1 (Theorem 3.2) can't be replaced by 2 (3). Since the graphs $G_1$ and $G_2$ shown in Fig.1. are two counter-examples. Note that $\alpha_1(G_1) = 2$ and $\alpha_1(G_2) = 3$. It's a easy task to see that $G_1$ and $G_2$ are not upper embeddable. The *dipole* $D_p$ is the graph which consists of two vertices joined by $p$ edges. $B_n$ is the bouquet of $n$-cycles. *The bouquet of* $n$ $p-$*dipoles* $B_{n,p}$ be the graph defined as followings: The vertex set consists of $n + 1$ vertices, say $v_0, v_1, \cdots, v_n$, and for each $j = 1, 2, ..., n$, there are exactly $p$ edges, say $e_{(j-1)p+1}, \cdots, e_{jp}$, between $v_0$ and $v_j$. By Theorem 2.2, the dipole $D_p$, the bouquet of $n$-cycles $B_n$ and the bouquet of $n$ $p-$ *dipoles* $B_{n,p}$ are upper embeddable. Since $\alpha_1(C_4) = \alpha_1(K_4) = 2$, by Theorem 3.2, the cycle $c_4$ and the complete graph $K_4$ are upper embeddable. Thus, the upper bounds of Theorem 3.1 and Theorem 3.2 are best possible and can't be improved.

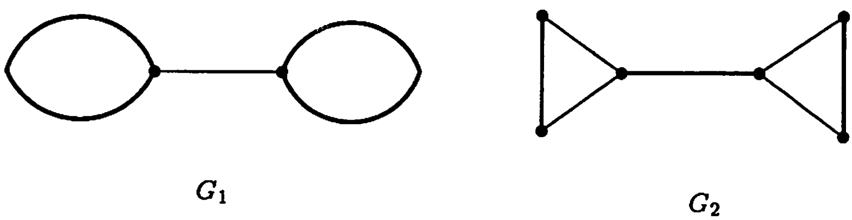



Fig. 1.

**Theorem 3.3** *Let $G$ be a graph and $\kappa_1(G) = 2$. If $\alpha_1(G) \leq 2$, then $G$ is upper embeddable.*

**Proof.** Suppose that the graph $G$ is not upper embeddable. By Lemma 2.3, there exists $A \subset E$ such that the properties (a)-(e) of Lemma 2.3 are satisfied. Let $F_1, F_2, \cdots, F_l (l \geq 2)$ be the all connected components of $G-A$. By (a) of Lemma 2.3, we know that $\alpha_1(F_i) \geq 1$. Since $G$ is 2-edge connected, we have $|E(F_i, A)| \geq 2$, for $i = 1, 2, \cdots, l$. Let $x$ be the number of such $F_i$ with the property that $|E(F_i, A)| = 2$, and $y$ be the number of such $F_i$ with the property that $|E(F_i, A)| = 3$, for $i = 1, 2, \cdots, l$. By (1), we have

$$|A| = \frac{1}{2}\sum_{i=1}^{l} |E(F_i, A)| \geq x + \frac{3}{2}y + 2(l - x - y) \qquad (2)$$

By the (e) of Lemma 2.3, we have

$$\xi(G) = 2l - |A| - 1 \geq 2 \qquad (3)$$

By (2) and (3), we can easily get

$$x + \frac{1}{2}y \geq 3.$$

Therefore, we have

$$x + y \geq x + \frac{1}{2}y \geq 3.$$

Since $\alpha_1(G) \geq \sum_{k=1}^{l} \alpha_1(F_k) \geq l \geq x + y \geq 3$, it's impossible.

**Theorem 3.4** *Let $G$ be a simple graph and $\kappa_1(G) = 2$. If $\alpha_1(G) \leq 3$, then $G$ is upper embeddable.*

**Proof.** Note that if $G$ is a simple graph, by (a) of Lemma 2.3, we know that $|V(F_i)| \geq g(G) \geq 3$. If $l \geq 4$, we have $\alpha_1(G) \geq \sum_{k=1}^{l} \alpha_1(F_k) \geq l \geq 4$, otherwise $l = 3$, let $E(F_1, F_2) = \{e\}$, since $g(G) \geq 3$, we can easily find four non-adjacent edges $\{e, f, g, h\}$ where $f \in F_1, g \in F_2, h \in F_3$. Thus, $\alpha_1(G) \geq 4$.

**Remark 2** The upper bound 2 (3) of Theorem 3.3 (Theorem 3.4) can't be replaced by 3 (4). Since the two graphs $G_3$ and $G_4$ shown in Fig.2. are two counter examples. Note that $\alpha_1(G_3) = 3$ and $\alpha_1(G_4) = 4$. However, let $A = \{c, d, e\}$. It's easy to see that

$$c(G_i - A) + b(G_i - A) - 2 = 4 > |A| = 3 \quad \text{for i} = 3, 4.$$

by Theorem 1.2, $G_3$ and $G_4$ are not upper embeddable. Let $D_p$ a dipole and $D_p'$ be a copy of $D_p$. Suppose $V(D_p) = \{v_1, v_2\}$ and $V(D_p') = \{u_1, u_2\}$. Let graph $D = D_p \cup D_p' \cup \{u_1v_1, u_2v_2\}$. It's routine task to check that $\alpha_1(D) = \kappa_1(D) = 2$, by Theorem 3.3, the graph $D$ is upper embeddable. Let $e = uv_1$, $f = uv_2$ and $g = uv_3$ be three adjacent edges of the complete graph $K_6$ (Note that $u$, $v_1, v_2$ and $v_3$ are four vertices of $K_6$) . It's easy to check that $\alpha_1(K_6 \backslash \{e, f, g\}) = 3$ and $\kappa_1(K_6 \backslash \{e, f, g\}) = 2$, by Theorem 3.4, the graph $K_6 \backslash \{e, f, g\}$ is upper embeddable. Thus, the upper bounds of Theorem 3.3 and Theorem 3.4 are best possible and can't be improved.

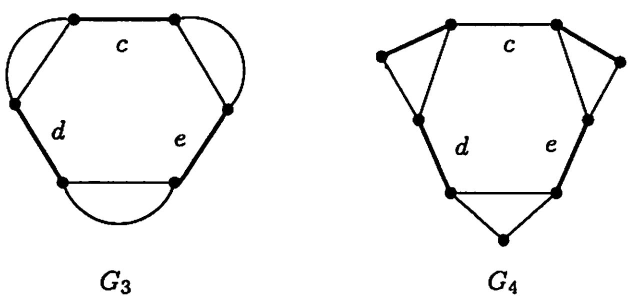



Fig. 2

**Theorem 3.5** *Let $G$ be a graph and $\kappa_1(G) = 3$. If $\alpha_1(G) \leq 5$, then*

*G is upper embeddable.*

**Proof.** By contradiction, assume that the theorem is not true. By Lemma 2.3, there exists $A \subset E$ such that the properties (a)-(e) of Lemma 2.3 are satisfied. Let $F_1, F_2, \cdots, F_l (l \geq 2)$ be the all connected components of $G - A$. Since $G$ is 3-edge connected, we have $|E(F_i, A)| \geq 3$, for $i = 1, 2, \cdots, l$. Let $x$ be the number of such $F_i$ with the property that $|E(F_i, A)| = 3$ for $i = 1, 2, \cdots, l$. By (1), we have

$$|A| = \frac{1}{2}\sum_{i=1}^{l}|E(F_i, A)| \geq \frac{3}{2}x + 2(l - x) \tag{4}$$

By the (e) of Lemma 2.3, we have

$$\xi(G) = 2l - |A| - 1 \geq 2 \tag{5}$$

By (4) and (5), we can easily get

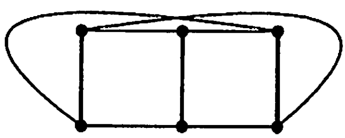
$$\frac{1}{2}x \geq 3.$$

Therefore, we have $x \geq 6$

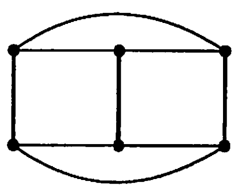
Since $\alpha_1(G) \geq \sum_{k=1}^{l}\alpha_1(F_k) \geq l \geq x \geq 6$, it's impossible

**Theorem 3.6** *Let $G$ be a simple graph and $\kappa_1(G) = 3$. If $\alpha_1(G) \leq 8$, then $G$ is upper embeddable.*

**Proof.** Note that if $G$ is a simple graph, by (a) of Lemma 2.3, we know that $|V(F_i)| \geq g(G) \geq 3$. We now construct a graph $G^*$ with respect to $A$ as follows: $V(G^*)$ is the set of components of $G - A$ and two vertices in $G^*$ are adjacent if and only if they are joined in $G$ by an edge of $A$. It's easy to know that $G^*$ are also connected and $\kappa_1(G) \leq \kappa_1(G^*)$.



$G_5$



$G_6$

Fig. 3

**Case 1** $l = 6$. By (c) of Lemma 2.3, we know that

$$|E(G^*)| = |A| \le 2 \times 6 - 3 = 9.$$

Since $G$ is 3-edge-connected, we have

$$2|E(G^*)| = \sum_{v \in V(G^*)} d_{G^*}(v) \ge 3 \times 6 = 18.$$

Thus, $|E(G^*)| = 9$ and $G^*$ is a 3–regular graph. By (c) of Lemma 2.3, $|E(F_i, F_j)| \le 1$, for $i, j = \{1, 2, \cdots, 6\}$. In other words $G^*$ is simple. Thus, $G^*$ is isomorphic to one of graphs shown in Fig. 3. It's a routine task to check that $\alpha_1(G^*) = 3$.

Let $E(F_1, F_2) = \{e\}$, $E(F_3, F_4) = \{f\}$ and $E(F_5, F_6) = \{e\}$. since $g(G) \ge 3$, we can easily find nine non-adjacent edges $\{e, f, g, h_1 \cdots, h_6\}$ where $h_i \in F_i, i = 1, \cdots, 6$. Thus, $\alpha_1(G) \ge 9$.

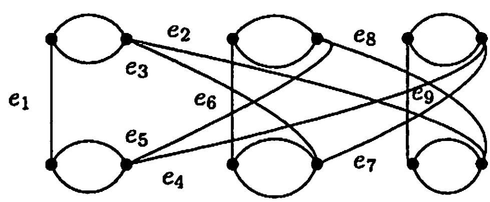
**Case 2** $l = 7$. In this case, it's a easy to see that $\alpha_1(G^*) \ge 2$. Similarly, we can find nine non-adjacent edges, Thus $\alpha_1(G) \ge 9$.

**Remark 3** Note that the upper bound 5 (8) of Theorem 3.5 (Theorem 3.6) can't be replaced by 6 (9). Since the graphs shown in Fig. 4. and Fig. 5. are two counter examples.

Note that $\alpha_1(G_7) = 6$ and $\alpha_1(G_8) = 9$. However, let $A = \{e_1, e_2, \cdots, e_9\}$. It's easy to see that

$$c(G - A) + b(G - A) - 2 = 10 > |A| = 9.$$

by Theorem 1.2, the graph $G_7$ shown in Fig. 4 and $G_8$ shown in Fig. 5. are not upper embeddable.

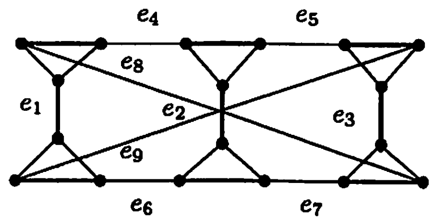



$G_7$

Fig. 4.

Let graph $H$ be obtained by replace each vertex of $K_5 \backslash \{e\}$ by the $D_p$, ($e$ is a edge of $K_5$ and $p \ge 3$). It's easy to see that $\alpha_1(H) = 5$ and $\kappa_1(G) = 3$, by Theorem 3.5, the graph $H$ is upper embeddable. Let

$uv_1, \cdots, uv_{12}$ be twelve adjacent edges of the complete graph $K_{16}$ (Note that $u$, $v_1, v_2, \cdots, v_{11}$,and $v_{12}$ are vertices of $K_{16}$) . It's easy to check that $\alpha_1(K_{16}\backslash\{uv_1, \cdots, uv_{12}\}) = 8$ and $\kappa_1(K_{16}\backslash\{uv_1, \cdots, uv_{12}\}) = 3$, by Theorem 3.4, the graph $K_{16}\backslash\{uv_1, \cdots, uv_{12}\}$ is upper embeddable. Thus, the upper bounds of Theorem 3.5 and Theorem 3.6 are best possible and can't be improved.




$G_8$

Fig. 5.

To conclude, we present a table to give a picture of the study of the maximum genus via graph's maximum non-adjacent edge set $\alpha_1(G)$ and connectivity $k$. The rows are correspondence to edge-connectivity $k = 1, 2, 3$ or $\geq 4$. Since $\kappa(G) \leq \kappa_1(G)$, we know that the same bounds also holds for vertex-connectivity.

| $\kappa_1(G)$ \ Type | Multigraph | Simple | upper embeddable |
|---|---|---|---|
| 1 | $\alpha_1(G) \leq 1$ | $\alpha_1(G) \leq 2$ | Yes |
| 2 | $\alpha_1(G) \leq 2$ | $\alpha_1(G) \leq 3$ | Yes |
| 3 | $\alpha_1(G) \leq 5$ | $\alpha_1(G) \leq 8$ | Yes |

If the graph $G$ contain self-loops, then the maximum non-adjacent edge set is not large than cardinality of vertex set. In other words, we have $\alpha_1(G) \leq \nu$. otherwise, $\alpha_1(G) \leq \frac{\nu}{2}$. By the above table, we can present a table to give a picture of the study of the maximum genus via graph's cardinality of vertex set $\nu$ and connectivity $k$. The rows are correspondence to edge-connectivity $k = 1, 2, 3$ or $\geq 4$. The same bounds also holds for vertex-connectivity.

| Type / $\kappa_1(G)$ | (Selfloops permitted) | Multigraph | Simple | upper embeddable |
|---|---|---|---|---|
| 1 | $\nu \leq 1$ | $\nu \leq 3$ | $\nu \leq 5$ | Yes |
| 2 | $\nu \leq 2$ | $\nu \leq 5$ | $\nu \leq 7$ | Yes |
| 3 | $\nu \leq 5$ | $\nu \leq 11$ | $\nu \leq 17$ | Yes |

# 4 Maximum Genus, Maximum non-adjacent edge set and Girth

In this section, we obtain the lower bounds of maximum genus in terms of maximum non-adjacent edge set and girth.

**Theorem 4.1** *Let $G$ be a connected graph with girth $g$. Then the lower bounds on the maximum genus $\gamma_M(G)$ are given in the following table. The rows are correspond to edge-connectivity $k = 1, 2, 3$ or $\geq 4$. The same bounds also holds for vertex-connectivity. Furthermore, the bounds are best possible. Note that we let $\lfloor \frac{g}{2} \rfloor = 1$ for non-simple graphs.*

| $\kappa_1(G)$ | $\gamma_M(G)$ |
|---|---|
| 1 | $\min\left\{\left\lceil \frac{1}{2}\beta(G) - \frac{\alpha_1(G)}{2\lfloor \frac{g}{2} \rfloor} \right\rceil, \left\lfloor \frac{\beta(G)}{2} \right\rfloor\right\}$ |
| 2 | $\min\left\{\left\lceil \frac{1}{2}\beta(G) - \frac{\alpha_1(G)-1}{2\lfloor \frac{g}{2} \rfloor} \right\rceil, \left\lfloor \frac{\beta(G)}{2} \right\rfloor\right\}$ |
| 3 | $\min\left\{\left\lceil \frac{1}{2}\beta(G) - \frac{\alpha_1(G)-2}{4\lfloor \frac{g}{2} \rfloor} \right\rceil, \left\lfloor \frac{\beta(G)}{2} \right\rfloor\right\}$ |
| $\geq 4$ | $\left\lfloor \frac{\beta(G)}{2} \right\rfloor$ |

**Proof.** Since the lower bounds for the k-connected graph implies those for k-edge-connected graph, we can only show the theorem is true for k-edge-connected graph. If the graph $G$ is upper embeddable, the theorem is obvious. Otherwise we can suppose that $G$ is not upper embeddable. By Lemma 2.3, there exists $A \subset E$ such that the properties (a)-(e) of Lemma 2.3 are satisfied. Let $F_1, F_2, \cdots, F_l (l \geq 2)$ be the all connected components of $G - A$.

It's easy to see that $\alpha_1(F_i) \geq \lfloor \frac{g}{2} \rfloor$, for $i = 1, 2, \cdots, l$.

Since

$$\alpha_1(G) \geq \sum_{k=1}^{l} \alpha_1(F_k) \geq l \lfloor \frac{g}{2} \rfloor$$

we have

$$\frac{\alpha_1(G)}{\lfloor \frac{g}{2} \rfloor} \geq l$$

We now construct a graph $G^*$ (with respect to $A$)as follows: $V(G^*)$ is the set of components of $G - A$ and two vertices in $G^*$ are adjacent if and only if they are joined in $G$ by an edge of $A$. It's easy to know that $G^*$ are also connected and $\kappa_1(G) \leq \kappa_1(G^*)$. Since $G^*$ is connected, we have $|A| = |E(G^*| \geq |V(G^*| - 1 = l - 1$. by Lemma 2.3, we have

$$\xi(G) = 2c(G - A) - |A| - 1 = 2l - |A| - 1 \leq l \leq \frac{\alpha_1(G)}{\lfloor \frac{g}{2} \rfloor}$$

Since

$$2|A| = 2|E(G^*| = \sum_{v \in V(G^*)} d_{G^*}(v)$$

Thus, if $\kappa_1(G) = 2$, we have $2|A| \geq 2l$. by Lemma 2.3,

$$\xi(G) = 2l - |A| - 1 \leq l - 1 \leq \frac{\alpha_1(G)}{\lfloor \frac{g}{2} \rfloor} - 1.$$

If $\kappa_1(G) = 3$, we have $2|A| \geq 3l$. by Lemma 2.3,

$$\xi(G) = 2l - |A| - 1 \leq \frac{l}{2} - 1 \leq \frac{\alpha_1(G)}{2\lfloor \frac{g}{2} \rfloor} - 1.$$

By the (2) of characterization theorem, we have $\xi(G) = \beta(G) - 2\gamma_M(G)$. Thus the upper bounds of $\xi(G)$ gives the lower bounds of $\gamma_M(G)$. It's a routine task to check that the theorem is true.

It's routine to check that the maximum genus of the graphs $G_1, \cdots, G_8$ (Fig.1-Fig.5) are attain the lower bounds of Theorem 4.1. So the bounds given in Theorem 4.1 are best possible.

**Remark 4**: We see that the graphs in Theorem 3.3 are permitted to have multiple edges and self-loops, while the graphs in [3-4,6,7,12] are simple. If a graph have multiple edges, we can't obtain a lower bound on it's maximum genus by using the results [3-4,6,7,12]. For example: the dipole $D_p$, the bouquet of $n$-cycles $B_n$ and the bouquet of $n$ $p-dipoles$ $B_{n,p}$, etc. Let $K_{1,n}$ be a complete bipartite graph. i.e., $K_{1,n}$ is a tree of vertices with one vertex $u$ of $K_{1,n}$ being adjacent to all the other vertices $v_1, v_2, \cdots, v_n$. Now we obtain a new graph from $K_{1,n}$ by replacing each vertex $v_i$ with a dipole $D_p$, for $i = 1, 2, \cdots, n$, and putting the edge formerly incident with $v_i$ to be incident with one of two vertices in $D_p$. Denote the resulting graph by $L_{n,p}$. For example, the following figure helps us to understand the graph $L_{n,p}$.

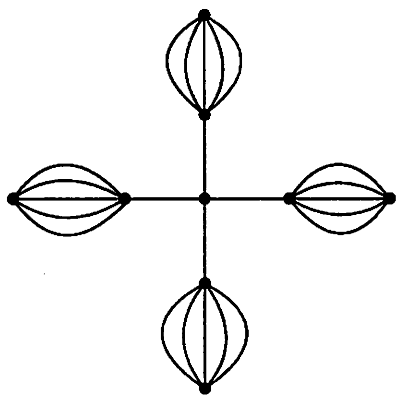


Fig. 6. The graph $L_{4,5}$

It's a routine task to check that

$$\alpha_1(L_{n,p}) = \frac{\beta(L_{n,p})}{p-1} \quad (n > 2).$$

Since the graph $L_{n,p}$ is not simple, we can't obtain a lower bound on the maximum genus of $L_{n,p}$ by using the results [3-4,6,7,12]. Note that the graph $L_{n,p}$ may not upper embeddable when $p$ is even. By theorem 3.3, we have

$$\gamma_M(L_{n,p}) \geq \frac{(p-2)\beta(L_{n,p})}{2(p-1)}.$$

This bound is tight. If we using the methods of [1], we have $\gamma_M(L_{n,p}) \geq 0$. Thus, we can obtain good lower bounds on the maximum genus of some non-simple graphs.

Let $G$ be a connected graph and $u, v \in V(G)$. Now we obtain a new graph $G'$ from $G$ by adding $n$ self-loops to $u$ or adding $n$ multiple edges to $u$ and $v$. It's easy to see that $\beta(G') = \beta(G) + n$ and $\alpha_1(G') \leq \alpha_1(G) + 1$. Without loss of generality, we suppose $\kappa_1(G') = 3$. Since

$$\lim_{n\to\infty} \frac{\gamma_M(G')}{\left\lfloor \frac{\beta(G')}{2} \right\rfloor} \geq \lim_{n\to\infty} \frac{min\left\{ \left\lceil \frac{1}{2}\beta(G') - \frac{\alpha_1(G')-2}{4} \right\rceil, \left\lfloor \frac{\beta(G')}{2} \right\rfloor \right\}}{\left\lfloor \frac{\beta(G')}{2} \right\rfloor} = 1$$

That is to say, when $n \to \infty$, $G'$ is upper embeddable. In other words, The lower bound of the maximum genus of $G'$ is very close to $\gamma_M(G')$, when $n$ is large.

**Remark 5:** Let $G$ be a simple graph with minimal degree at least 3.

Since

$$2\varepsilon = \sum_{v\in V(G)} d(v) \geq 3\nu \text{ and } \alpha_1(G) \leq \frac{\nu}{2},$$

we have

$$\beta(G) = \varepsilon - \nu + 1 \geq \frac{\nu}{2} + 1 = \alpha_1(G) + 1.$$

To conclude, we present a table to give a picture of the lower bounds of $\frac{1}{2}\beta(G) - \frac{\alpha_1(G)}{2\lfloor \frac{g}{2} \rfloor}$, $\frac{1}{2}\beta(G) - \frac{\alpha_1(G)-1}{2\lfloor \frac{g}{2} \rfloor}$, and $\frac{1}{2}\beta(G) - \frac{\alpha_1(G)-2}{4\lfloor \frac{g}{2} \rfloor}$ via girth. The rows are correspond to the values of girth $g(G)$.

| | $\frac{1}{2}\beta(G) - \frac{\alpha_1(G)}{2\lfloor \frac{g}{2} \rfloor}$ | $\frac{1}{2}\beta(G) - \frac{\alpha_1(G)-1}{2\lfloor \frac{g}{2} \rfloor}$ | $\frac{1}{2}\beta(G) - \frac{\alpha_1(G)-2}{4\lfloor \frac{g}{2} \rfloor}$ |
|---|---|---|---|
| $g = 3$ | $\frac{1}{2}$ | $1$ | $\frac{\beta(G)+3}{4}$ |
| $g = 4, 5$ | $\frac{\beta(G)+1}{4}$ | $\frac{\beta(G)+2}{4}$ | $\frac{3\beta(G)+3}{8}$ |
| $g = 6, 7$ | $\frac{2\beta(G)+1}{6}$ | $\frac{2\beta(G)+2}{6}$ | $\frac{5\beta(G)+3}{12}$ |
| $g = 8, 9$ | $\frac{3\beta(G)+1}{8}$ | $\frac{3\beta(G)+2}{8}$ | $\frac{7\beta(G)+3}{16}$ |

By the table, we can see that the lower bounds Theorem 4.1 is more better if the values of the girth is more larger. In general, the lower bound of Theorem 4.1 is better than the results of [1,3,4] (where it is proved that $\frac{\beta(G)}{4}$ (or $\frac{\beta(G)}{3}$, respectively) is a tight lower bound on the maximum genus of a simple graph G with the minimal degree at least 3 and with the edge-connectivity 1 (or, the edge-connectivity 2 or 3, respectively)) when the girth $g(G)$ is large. In other words, the results in [1,3,4] can be improved, if the number $\frac{\alpha_1(G)}{\lfloor \frac{g}{2} \rfloor}$ is small.

**Acknowledgements**

The authors wish to thank the referees for their careful reading of the paper, and for their helpful comments.

## References

[1] Dan. Archdeacon, J. Chen, Y. Huang, S.P. Kanchi, D. Li, Y.P. Liu, R. Nedela, and M. Skoviera, Maximum Genus, Connectivity, and Nebesky's Theorem. To appear in Discrete Math.
[2] M. Behzand, G. Chartrad and L. Lesniak, Graphs and Digraphs, 1979, Prindle, Weber and Schmidt, Boston.
[3] J.Chen, D.Archdeacon, and J.L.Gross, Maximum genus and connectivity, Discrete Math. 149(1996)19-29.
[4] J.Chen, S.P.Kanchi, and J.L.Gross, A tight lower bound on the maximum genus of a simplicial graph, Discrete Math. 156(1996)83-102.
[5] R. Duke, The genus, regional number, and Betti number of a graph, Canad.J.Math. 18 (1966) 817-822.
[6] Yuangqiu. Huang, Maximum genus and girth of graphs. Discrete Math. 194 (1999), 253-259.
[7] Yuangqiu. Huang, Maximum genus and chromatic number of graphs. Discrete Math. 271 (2003), 117-127
[8] F. Jaeger, C. Payan, N.H. Xuong, A class of upper embeddable graphs, J.Graph Theory, 3(1979),387-391.
[9] M. Jungerman, A characterization of upper embeddable graphs, Trans. Amer. Math. Soc. 241(1987), 401-406.
[10] N.P. Khomendo, A.D. Glukhov, On upper embeddable graphs, Graph Theory, (N.P. Khomenko, ed) Izd. Akad. Nauk Ukrain. SSR. Kiev, 1977, 85-87(In Russian).
[11] S.Kundu, Bounds on number of disjoint spanning trees, J. Combin. Theory, B 17 (1974) 199-203.
[12] Deming. Li, Yanpei. Liu, Maximum Genus, Girth and Connectivity. European Journal of Combinatorics Vol 21, (2000) 651-657.
[13] Yanpei. Liu, The maximum orientable genus of a graph, Scientia Sinica, Special Issue on Math 1979, 22 (II): 41.
[14] Yanpei. Liu, The maximum orientable genus of some typical classes of graphs, Acta Math Sinica 1981, 24 : 817.
[15] Yanpei. Liu, Embeddability in Graphs, Kluwer, Dordrecht/ Boston/ London, 1995.

[16] E.A. Nordhaus, B.M.Stewart, and A.T.White, On the maximum genus of a graph. J. Combin. Theory, 11(1971), 285-267.
[17] L. Nebesky, every connected, locally connected graph is upper embeddable,. J.Graph Theory, 3(1981), 197-199.
[18] L. Nebesky, A new characterization of the maximum genus of a graph, Czechoslovak Math. J.31(1981)604-613.
[19] L. Nebesky, On quasiconnected graph and their upper- embeddable, Czech. Math. J., 35:110(1985).
[20] L, Nebesky, $N_2$-connected graph and their upper embeddability, Czech. Math. J., 41:116(1991).
[21] R. Nedela, M. Skoviera, Topics in combinatorics and graph theory, R.Bodendiek and R. Henn (Eds), Physicaverlay, Heideberg, 1990, 519-525.
[22] C. Payan and N.H. Xoung, Upper embeddability and connectivity of graph, Discrete Math. J. 27 (1979), 71-80.
[23] M. Skoviera, On the maximum genus of graphs of diameter two. Discrete Math., 87(1991). 175-180.
[24] M. Skoviera, R. Nedela, The maximum genus of a graph and doubly Eulerian trail. Bulletin, U. M. I., 4B(1990), 541-545.
[25] A.T. White, Graphs, Groups and Surfaces, 2nd Edition, Elsevier, Amsterdam, 1984.
[26] N.H. Xuong, How to determine the maximum genus of a graph, J, Combin. Theory, B 26(1979) 217-225.