

DISTANCE INDEPENDENT DOMINATION IN ITERATED LINE GRAPHS

MARTIN KNOR¹⁾ AND LUDOVÍT NIEPEL²⁾

¹⁾ Slovak University of Technology, Faculty of Civil Engineering,
Department of Mathematics, Radlinského 11, 813 68 Bratislava, Slovakia,
E-mail: knor@vox.svf.stuba.sk;

²⁾ Kuwait University, Faculty of Science,
Department of Mathematics & Computer
Science, P.O. box 5969 Safat 13060, Kuwait,
E-mail: niepel@mcs.sci.kuniv.edu.kw.

ABSTRACT. Let $k \geq 1$ be an integer and let $G = (V, E)$ be a graph. A set S of vertices of G is k -independent if the distance between any two vertices of S is at least $k + 1$. We denote by $\rho_k(G)$ the maximum cardinality among all k -independent sets of G . Number $\rho_k(G)$ is called the k -packing number of G . Furthermore, S is defined to be k -dominating set in G if every vertex in $V(G) - S$ is at distance at most k from some vertex in S . A set S is k -independent dominating if it is both k -independent and k -dominating. The k -independent dominating number, $i_k(G)$, is the minimum cardinality among all k -independent dominating sets of G . We find the values $i_k(G)$ and $\rho_k(G)$ for iterated line graphs.

1. INTRODUCTION AND RESULTS

Let G be a graph. Its line graph $L(G)$ is defined as the graph whose vertices are the edges of G , with two vertices adjacent if and only if the corresponding edges are adjacent in G . Although the line graph operator is one of the most natural ones, only in recent years there is recorded a larger

1991 *Mathematics Subject Classification.* 05C69, (05C70).

Key words and phrases. Domination number, independence, iterated line graph, graph dynamics, packing claws, packing number.

¹⁾ Supported by VEGA grant 1/9176/02

²⁾ Supported by Kuwait University grant #SM 02/00

interest in studying iterated line graphs. Iterated line graphs are defined inductively as follows:

$$L^j(G) = \begin{cases} G & \text{if } j = 0, \\ L(L^{j-1}(G)) & \text{if } j > 0. \end{cases}$$

In iterated line graphs the greatest attention was devoted to Hamiltonicity. The most recent results in this area can be found in a paper by Xiong and Liu [16]. The diameter and radius of iterated line graphs are examined in [15], and [11] is devoted to the centers of these graphs. In [7] and [6], Hartke and Higgins study the growth of the minimum and the maximum degree of iterated line graphs, respectively. The connectivity of iterated line graphs is discussed in [10].

We shall use the notation of domination and independency parameters as in [4]. The closed k -neighborhood of a vertex v in a graph $G = (V(G), E(G))$ is $N_k[v] = \{w \in V; \text{dist}_G(v, w) \leq k\}$, where $\text{dist}_G(v, w)$ denotes the distance between v and w in G . A vertex set S is said to be k -dominating if $N_k[v] \cap S \neq \emptyset$ for every $v \in V(G)$. The set S of vertices is called a k -packing [14] or k -independent [9] if $\text{dist}_G(v, w) \geq k + 1$ for each pair of distinct vertices u and v in S . By $\rho_k(G)$ we denote the k -packing number of G , that is, the maximum cardinality of a k -packing. Note that $\beta(G) = \rho_1(G)$, where $\beta(G)$ is the independence number of G ; and $\rho(G) = \rho_2(G)$, where $\rho(G)$ is the packing number of G .

In general, a set S is a maximal k -packing if and only if it is k -dominating. A vertex set S is k -independent dominating set, if it is both k -packing and k -dominating set. The k -independent domination number $i_k(G)$ is the minimum cardinality of a k -independent dominating set. In particular, $i(G) = i_1(G)$ where $i(G)$ is the independent domination number. Thus, $i_k(G)$ and $\rho_k(G)$ are the minimum and maximum cardinalities, respectively, of any maximal k -packing. In [4] there are presented bounds for values of $i_k(G)$ and $\rho_k(G)$ and it is proved that the decision problem " $i_k(G) < \rho_k(G)$?" is NP-complete in general. For a survey of results concerning distance domination and independency we refer to [9] and [8].

In [2] and [3], Dutton and Brigham study the domination properties of line graphs and relationships between different parameters of domination and independence for line graphs. They proved that the domination number $\gamma(L(G))$ and independent domination number $i(L(G))$ have the same value. (Recall that $\gamma(G)$ is the minimum cardinality of a dominating set in G .) In this paper we give exact values for $\rho_k(L^j(G))$ and $i_k(L^j(G))$, providing that j is "big enough". (The proofs of our results are postponed to section 2.)

In one of the pioneering papers in graph theory [12], Kotzig proved:

Theorem A. *Let G be a connected graph with even number of edges. Then $E(G)$ can be decomposed into $\frac{1}{2}|E(G)|$ paths of length 2.*

As an immediate consequence of this result (and Lemmas B and C below) we have:

Proposition 1. *Let G be a graph and $j \geq 2$. If $L^{j-2}(G)$ is connected then*

$$\beta(L^j(G)) = \rho_1(L^j(G)) = \left\lfloor \frac{|E(L^{j-2}(G))|}{2} \right\rfloor.$$

Using Proposition 1, for independent domination number we prove:

Theorem 2. *Let G be a graph and $j \geq 3$. If $L^{j-3}(G)$ is a connected graph with $\delta(L^{j-3}(G)) \geq 3$, then*

$$i(L^j(G)) = i_1(L^j(G)) = \left\lfloor \frac{|E(L^{j-2}(G))| - \lfloor \frac{1}{2} \rfloor |V(L^{j-2}(G))|}{2} \right\rfloor.$$

Here $\delta(G)$ denotes the minimum degree of a graph G , while $\Delta(G)$ denotes its maximum degree.

It is well-known that for a general graph G the problem of finding its independence number is NP-hard, see [5]. If we consider line graphs, maximum independent sets in $L(G)$ correspond to maximum matchings in G . Hence, the problem of finding the independence number of $L(G)$ is polynomial, see [13]. By Proposition 1, the problem of finding the independence number of $L^2(G)$ is trivial, and by Theorem 2, if G has minimum degree at least 3, then the problem of finding the independent domination number of $L^3(G)$ is trivial as well.

It is interesting that analogous straightforward formulae can be given also for $\rho_k(L^j(G))$ and $i_k(L^j(G))$, $k \geq 2$, providing that j is "big enough". We prove here:

Theorem 3. *Let G be a graph, $k \geq 2$ and $j \geq k + 2$. If $L^{j-k-2}(G)$ is a connected graph with $\delta(L^{j-k-2}(G)) \geq 5$, then*

$$\rho_k(L^j(G)) = \left\lfloor \frac{|E(L^{j-k-1}(G))|}{3} \right\rfloor.$$

For the k -independent domination number we have:

Theorem 4. *Let G be a graph, $k \geq 2$ and $j \geq k + 3$. If $L^{j-k-2}(G)$ is a connected graph with $\delta(L^{j-k-2}(G)) \geq 9k - 7$ and $\delta(L^{j-k-3}(G)) \geq 3$, then*

$$i_k(L^j(G)) = \left\lfloor \frac{|E(L^{j-k-1}(G))| - \lfloor \frac{k}{k+1} \rfloor |V(L^{j-k-1}(G))|}{k+1} \right\rfloor.$$

For the number of vertices of iterated line graphs we have, see [15]:

$$|V(G)| \cdot \prod_{k=0}^{j-1} \left[2^{k-1} \cdot (\delta(G) - 2) + 1 \right] \leq |V(L^j(G))| \leq \\ |V(G)| \cdot \prod_{k=0}^{j-1} \left[2^{k-1} \cdot (\Delta(G) - 2) + 1 \right].$$

We remark that if G is distinct from a path, a cycle and a claw $K_{1,3}$, then there exists j_G such that $\delta(L^{j_G}(G)) \geq 3$. Then $\delta(L^j(G)) \geq 2^{j-j_G} \cdot (\delta(L^{j_G}(G)) - 2) + 2$, so that if j is "big enough", all the assumptions of Theorems 2, 3 and 4 are fulfilled. Hence, these theorems can be applied for any graph (distinct from a path, a cycle and a claw $K_{1,3}$), providing that j is "big enough".

Observe that $|E(L^j(G))| = |V(L^{j+1}(G))|$. Hence, if G is δ -regular graph, $\delta \geq 3$, then we can immediately write the numbers $\rho_k(L^j(G))$ and $i_k(L^j(G))$ for j "big enough", although $|V(L^j(G))|$ grows doubly exponentially as a function of j .

2. PROOFS

Let G be a graph and let v be a vertex of $L^j(G)$, $j \geq 1$. Then v corresponds to an edge of $L^{j-1}(G)$, and this edge will be called the 1-history of v . For $t \geq 2$ we define t -histories recursively. The t -history of v is a subgraph of $L^{j-t}(G)$, edges of which are induced by the vertices of $L^{j-t+1}(G)$ which are in $(t-1)$ -history of v .

Observe that the 1-history is always an edge and the 2-history is a path of length 2. The situation is more complicated for t -histories when $t \geq 3$. In [15] we have the following lemma:

Lemma B. *Let G be a graph and let $L^j(G)$ be its j -iterated line graph. Further, let $0 \leq t \leq j$ and let H be a subgraph of $L^{j-t}(G)$. Then H is a t -history of some vertex of $L^j(G)$ if and only if H is a connected graph with at most t edges, distinct from any path with less than t edges.*

Also the next lemma, which is useful for calculating the distances in iterated line graphs, can be found in [15].

Lemma C. *Let G be a connected graph, $L^j(G)$ be its iterated line graph, and let u and v be distinct vertices of $L^j(G)$. Then for any t , $0 \leq t \leq j$, if the t -histories of u and v are edge-disjoint, then the distance between u and v in $L^j(G)$ equals the minimum distance between the two vertex sets of the t -histories in $L^{j-t}(G)$, increased by t . If the t -histories of u and v are*

not edge-disjoint, then the distance in $L^j(G)$ between u and v is strictly less than t .

As can be deduced from the lemmas above, we prove our theorems for $L^j(G)$ using histories in smaller iterations. However, first we introduce two lemmas. It is worth mentioning, that their proofs (via orientations) are similar to the proof of Theorem A in [12].

Lemma 5. *Let G be a connected graph. If $\delta(G) \geq 5$, then $L(G)$ contains $\lfloor \frac{|E(L(G))|}{3} \rfloor$ edge-disjoint copies of a claw $K_{1,3}$.*

Proof. Denote $H = L(G)$. If ab is an edge of H , then its vertices correspond to pair of adjacent edges, say uv and vw , in G . In G the degree of v is at least 5, so that the edge ab lies in a copy of a complete graph K_5 in H . Hence, every edge of H lies in a copy of K_5 .

To every edge of H we assign one of the two possible orientations. In such a way we obtain from H its orientation $O(H)$. Let v be a vertex of H . Denote by $o(v)$ the number of arcs in $O(H)$ terminating at v , taken modulo 3. Further, denote by $o(H)$ the number of vertices v with $o(v) > 0$, and assume that $O(H)$ is chosen so that $o(H)$ is the minimum possible. In the following we prove $o(H) \leq 1$.

Denote by \overrightarrow{xy} an arc of $O(H)$ starting at x and terminating at y . We prove that for any edge $uv \in E(H)$, such that $o(u) > 0$, there exists an orientation $O'(H)$ such that $o'(u) = 0$ and $o'(w) = o(w)$ whenever w is distinct from u and v . There are four cases to distinguish:

- Case 1: $o(u) = 2$ and \overrightarrow{uv} is an arc in $O(H)$. Then reversing \overrightarrow{uv} we receive the required orientation $O'(H)$.
- Case 2: $o(u) = 2$ and \overrightarrow{vu} is an arc in $O(H)$. If there is a directed path from u to v in $O(H)$, then reversing its arcs we obtain the required orientation $O'(H)$. Analogously, if there is a directed path from v to u in $O(H)$, then reversing the arcs of this path and reversing \overrightarrow{vu} we receive the required orientation $O'(H)$. Hence, we can assume that there are no directed paths of these types. Since u and v lie in a copy of a complete graph K_5 , there are two vertices x and y with the arc \overrightarrow{xy} , such that either $\overrightarrow{xu}, \overrightarrow{xv}, \overrightarrow{yu}, \overrightarrow{yv}$ are in $O(H)$ or $\overrightarrow{ux}, \overrightarrow{vx}, \overrightarrow{uy}, \overrightarrow{vy}$ are in $O(H)$. In the first case reversing $\overrightarrow{xu}, \overrightarrow{xv}, \overrightarrow{xy}, \overrightarrow{yu}$ we receive the required orientation $O'(H)$, while in the second one reversing $\overrightarrow{ux}, \overrightarrow{vx}, \overrightarrow{xy}, \overrightarrow{vy}$ we get the required orientation $O'(H)$.
- Case 3: $o(u) = 1$ and \overrightarrow{vu} is an arc in $O(H)$. Then reversing \overrightarrow{vu} we receive the required orientation $O'(H)$.
- Case 4: $o(u) = 1$ and \overrightarrow{uv} is an arc in $O(H)$. Analogously as in Case 2 we can assume that there are neither directed $u - v$ paths nor directed $v - u$ paths in $O(H)$. But then there are two vertices x and y with the arc \overrightarrow{xy} , such that either $\overrightarrow{xu}, \overrightarrow{xv}, \overrightarrow{yu}, \overrightarrow{yv}$ are in $O(H)$ or $\overrightarrow{ux},$

\overrightarrow{vx} , \overrightarrow{uy} , \overrightarrow{vy} are in $O(H)$. In the first case reversing \overrightarrow{xu} , \overrightarrow{xv} , \overrightarrow{xy} , \overrightarrow{yv} we receive the required orientation $O'(H)$, while in the second one reversing \overrightarrow{uy} , \overrightarrow{vy} , \overrightarrow{xy} , \overrightarrow{xu} we obtain the required orientation $O'(H)$.

Suppose that $o(H) \geq 2$. Then there are two vertices, say u and v , such that $o(u) > 0$ and $o(v) > 0$. Since G is a connected graph, H is connected as well, so that there is a path $u=x_0, x_1, \dots, x_t=v$. Consider a sequence of orientations $O_0(H)=O(H)$, $O_1(H)$, \dots , $O_t(H)$, such that $o_i(x_{i-1}) = 0$ and $o_i(x) = o_{i-1}(x)$ whenever x is distinct from x_{i-1} and x_i . Then $o_t(x_0) = o_t(x_1) = \dots = o_t(x_{t-1}) = 0$, and for all $x \notin \{x_0, x_1, \dots, x_t\}$ we have $o_t(x) = o(x)$. Hence, $o_t(H) < o(H)$, a contradiction.

Thus, there is an orientation $O(H)$ such that $o(H) \leq 1$. That means that up to one exception, the number of arcs directed to each vertex v of $O(H)$ is a multiple of 3. Hence, these arcs can be arranged into triples to form $\lfloor \frac{|E(H)|}{3} \rfloor$ claws as required. \square

Lemma 6. *Let G be a connected graph, $k \geq 2$ and $\delta(G) \geq 9k - 7$. Moreover, let H be a graph obtained from $L(G)$ by deleting edges of vertex-disjoint paths, each of length at most k . If $|E(H)| \equiv 0 \pmod{k+1}$, then $E(H)$ can be decomposed into $\frac{|E(H)|}{k+1}$ stars $K_{1,k+1}$.*

Proof. Denote by $O(H)$ an orientation of H . Analogously as in the proof of Lemma 5, denote by \overrightarrow{uv} an arc of $O(H)$ starting at u and terminating at v . Further, denote by $o(v)$ the number of arcs in $O(H)$ terminating at v , taken modulo $k+1$. Finally, denote by $o(H)$ the number of vertices v with $o(v) > 0$, and assume that $O(H)$ is chosen so that $o(H)$ is the minimum possible.

In the next we prove $o(H) = 0$. To do this we prove that for any edge uv of H , such that $o(u) > 0$, there is an orientation $O'(H)$ such that $o'(u) = 0$ and $o'(w) = o(w)$ whenever w is distinct from u and v .

If there are $o(u)$ edge-disjoint directed $v-u$ paths in $O(H)$, then reversing all their arcs we receive the required orientation $O'(H)$. Analogously, if there are $k+1-o(u)$ edge-disjoint directed $u-v$ paths in $O(H)$, then reversing all their arcs we receive the required orientation $O'(H)$. Hence, we can assume that the number of directed paths in between u and v does not exceed $(o(u)-1) + (k+1-o(u)-1) = k-1$.

Since $\delta(G) \geq 9k-7$, in $L(G)$ the edge uv lies in a copy of a complete graph K_{9k-7} . Consider the $9k-9$ vertices of this complete graph, distinct from u and v . Since $deg_{L(G)}(x) - deg_H(x) \leq 2$ for every vertex x of H , there are at most 4 vertices out of these $9k-9$, which are not adjacent to both u and v in H . Since there are at most $k-1$ directed paths in between u and v , one of them obtained from the edge uv , there are $(9k-13) - (k-2) = 8k-11$ vertices x in the complete graph, such that either \overrightarrow{xu} , \overrightarrow{xv} are arcs of $O(H)$ or \overrightarrow{ux} , \overrightarrow{vx} are arcs of $O(H)$. Suppose that the number of vertices x with

arcs $\overrightarrow{ux}, \overrightarrow{vx}$ does not exceed the number of vertices x with arcs $\overrightarrow{xu}, \overrightarrow{xv}$. (The other case can be solved analogously.) Then there are $4k - 5$ vertices x in the complete graph, such that $\overrightarrow{xu}, \overrightarrow{xv}$ are arcs of $O(H)$. However, some edges connecting these x 's in $L(G)$ may be missing in H . But since any system of vertex-disjoint paths on n vertices contains a set of independent vertices of size greater than or equal to $\lceil \frac{n}{2} \rceil$, there are $2k - 2$ vertices x in H which are mutually adjacent and such that $\overrightarrow{xu}, \overrightarrow{xv}$ are arcs in $O(H)$.

Denote by K the complete subgraph of H , consisting of the $2k - 2$ vertices described above. Further, denote by $O(K)$ the orientation of K induced by $O(H)$. Since $(2k - 2) \cdot (k - 2) < \binom{2k - 2}{2}$, $O(K)$ contains a vertex x_0 with the out-degree at least $k - 1$. Hence, there are vertices x_0, x_1, \dots, x_{k-1} , such that $\overrightarrow{x_0 x_i}$ is an arc of $O(H)$ for $1 \leq i \leq k - 1$ and $\overrightarrow{x_j u}, \overrightarrow{x_j v}$ are arcs of $O(H)$ for $0 \leq j \leq k - 1$. Observe that $1 \leq o(u) \leq k$. Now reverse all the arcs $\overrightarrow{x_0 x_i}$, $1 \leq i \leq k - 1$, $\overrightarrow{x_0 u}, \overrightarrow{x_0 v}$, and one arc from each pair $\overrightarrow{x_i u}, \overrightarrow{x_i v}$, so that exactly $o(u)$ arcs from $\overrightarrow{x_i u}$ will be reversed. If we denote the resulting orientation by $O'(H)$, then $o'(u) = 0$ and $o'(w) = o(w)$ whenever w is distinct from u and v .

Since $k \geq 2$, we have $\delta(G) \geq 9k - 7 \geq 5$, so that each edge of $L(G)$ lies in three distinct triangles. Hence, if uv is an edge of $L(G) - H$, then there is a vertex z in H such that uz and zv are edges of H . Thus, since G is a connected graph, so is $L(G)$ and consequently also H . Now proceeding analogously as at the end of the proof of Lemma 5 it can be shown that $o(H) \leq 1$. Since $|E(H)| \equiv 0 \pmod{k+1}$, the case $o(H) = 1$ is impossible, so that $o(H) = 0$. And partitioning all the arcs terminating at v into $(k+1)$ -tuples, we obtain the required decomposition of $E(H)$ into stars $K_{1, k+1}$. \square

Now we can prove our main results. However, first we prove Theorems 3 and 4, since the proof of Theorem 2 is similar to that of Theorem 4.

Proof of Theorem 3. Let u and v be two distinct vertices of $L^j(G)$. If $\text{dist}_{L^j(G)}(u, v) > k$, then the $(k+1)$ -histories of u and v are edge-disjoint, by Lemma C. Since every $(k+1)$ -history contains at least 3 edges by Lemma B, we have $\rho_k(L^j(G)) \leq \lfloor \frac{1}{3} |E(L^{j-k-1}(G))| \rfloor$.

By Lemma 5, there are $\lfloor \frac{1}{3} |E(L^{j-k-1}(G))| \rfloor$ edge-disjoint copies of a claw $K_{1,3}$ in $L^{j-k-1}(G)$. These correspond to a k -independent set of the same size in $L^j(G)$, so that $\rho_k(L^j(G)) \geq \lfloor \frac{1}{3} |E(L^{j-k-1}(G))| \rfloor$. \square

In the proof of Theorem 4 we use the following result of Chartrand and Wall [1]:

Theorem D. *If G is a connected graph such that $\delta(G) \geq 3$, then $L^2(G)$ is Hamiltonian.*

Proof of Theorem 4. Let S be a set of vertices of $L^j(G)$ on which $i_k(L^j(G))$ is reached. Then the $(k+1)$ -histories of vertices of S are edge-disjoint, by

Lemma C. Remove the edges of these $(k+1)$ -histories from $L^{j-k-1}(G)$ and denote the resulting graph by F . Since the set S is maximal, F does not contain cycles and all vertices of F have degree less than or equal to 2 by Lemma B. Hence, F consists of vertex-disjoint paths, and by Lemma B, all these paths have lengths less than $k+1$. Thus, $|E(F)| \leq \lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \rfloor$. As a $(k+1)$ -history of a vertex contains at most $k+1$ edges, we have

$$i_k(L^j(G)) \geq \frac{|E(L^{j-k-1}(G))| - \lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \rfloor}{k+1}.$$

On the other hand, $L^{j-k-1}(G)$ contains a Hamiltonian cycle, by Theorem D. Deleting $\lfloor \frac{1}{k+1} |V(L^{j-k-1}(G))| \rfloor$ edges from this cycle we obtain a graph F^* consisting of paths of lengths less than $k+1$. Obviously, $|E(F^*)| = \lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \rfloor$. Let t be a number, $0 \leq t < k+1$, such that

$$\lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \rfloor - |E(L^{j-k-1}(G))| \equiv t \pmod{k+1}.$$

Delete from F^* exactly t edges and denote the resulting graph by F . Then $|E(L^{j-k-1}(G))| - |E(F)| \equiv 0 \pmod{k+1}$, so that $L^{j-k-1}(G) - E(F)$ can be decomposed into $\frac{1}{k+1} (|E(L^{j-k-1}(G))| - |E(F)|)$ stars $K_{1,k+1}$, by Lemma 6. These correspond to a k -independent dominating set of the same size in $L^j(G)$, so that

$$i_k(L^j(G)) \leq \frac{|E(L^{j-k-1}(G))| - |E(F)|}{k+1} = \left\lceil \frac{|E(L^{j-k-1}(G))| - \lfloor \frac{k}{k+1} |V(L^{j-k-1}(G))| \rfloor}{k+1} \right\rceil.$$

Combining the two inequalities for $i_k(L^j(G))$ we get the result. \square

Finally, we prove the result concerning $i(L^j(G))$.

Proof of Theorem 2. Substituting 1 for k , analogously as in the proof of Theorem 4 we get

$$i_1(L^j(G)) \geq \frac{|E(L^{j-2}(G))| - \lfloor \frac{1}{2} |V(L^{j-2}(G))| \rfloor}{2}.$$

By Proposition 1, $\beta(L^{j-1}(G)) = \lfloor \frac{|E(L^{j-3}(G))|}{2} \rfloor = \lfloor \frac{|V(L^{j-2}(G))|}{2} \rfloor$. Choose one set of $\beta(L^{j-1}(G))$ independent vertices of $L^{j-1}(G)$, and denote by F^* the edges of $L^{j-2}(G)$ which are 1-histories of vertices of the chosen set.

Then F^* is a collection of independent edges. Observe that $|E(F^*)| = \lfloor \frac{1}{2}|V(L^{j-2}(G))| \rfloor$. Now let t be a number, $0 \leq t < 2$, such that

$$\lfloor \frac{1}{2}|V(L^{j-2}(G))| \rfloor - |E(L^{j-2}(G))| \equiv t \pmod{2}.$$

Delete from F^* exactly t edges and denote the resulting graph by F . Since $\delta(L^{j-3}(G)) \geq 3$, each edge of F lies in a triangle in $L^{j-2}(G)$. Hence, $L^{j-2}(G) - E(F)$ is a connected graph with even number of edges. By Theorem A its edges can be decomposed into paths of length 2, so that

$$i_1(L^j(G)) \leq \frac{|E(L^{j-2}(G))| - |E(F)|}{2} = \left\lceil \frac{|E(L^{j-2}(G))| - \lfloor \frac{1}{2}|V(L^{j-2}(G))| \rfloor}{2} \right\rceil. \quad \square$$

REFERENCES

- [1] G. Chartrand, C.E. Wall, *On the hamiltonian index of a graph*, Studia Sci. Math. Hungary. 8 (1973), 43-48.
- [2] R.D. Dutton, R.C. Brigham, *A note on relations between a graph and its line graph which involve domination concepts*, J. Combin. Math. and Combin. Computing (JCMCC) 31 (1999), 161-168.
- [3] R.D. Dutton, R.C. Brigham, *Domination in claw-free graphs*, Congressus Numerantium 132 (1998), 69-75.
- [4] O. Favaron, T.W. Haynes, P.J. Slater, *Distance k -independent domination sequences*, J. Combin. Math. and Combin. Computing (JCMCC) 33 (2000), 225-237.
- [5] M.R. Garey, D.S. Johnson, *Computers and intractability, A guide to the theory of NP-completeness*, Freeman, San Francisco, 1979.
- [6] S.G. Hartke, A.W. Higgins, *Maximum degree growth of the iterated line graph*, Electron. J. Comb. 6, No. 1, Research paper R28, 9p. (1999).
- [7] S.G. Hartke, A.W. Higgins, *Minimum degree growth of the iterated line graph*, Ars Combinatoria 69 (2003), 275-283.
- [8] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, New York, 1998.
- [9] M.A. Henning, *Distance domination in graphs*, Domination in graphs: Advanced topics, Chapter 12 (T.W. Haynes, S.T. Hedetniemi and P.J. Slater, eds.), Marcel Dekker, New York, 1998, pp. 321-349.
- [10] M. Knor, L. Niepel, *Connectivity of iterated line graphs*, Discrete Applied Math. 125 (2003), 255-266.
- [11] M. Knor, L. Niepel, E. Šoltés, *Centers in iterated line graphs*, Acta Math. Univ. Comenian. LXI, 2 (1992), 237-241.
- [12] A. Kotzig, *Z teorie konečných pravidelných grafů tretieho a štvrtého stupňa*, Časopis pro pěstování matematiky 82 (1957), 76-92.
- [13] E.L. Lawler, *Combinatorial optimization: networks and matroids*, Rinehart and Winston, New York, 1976.
- [14] A. Meir, J.W. Moon, *Relations between packing and covering numbers for a tree*, Pacific J. Math. 61 (1995), 225-233.

- [15] L. Niepel, M. Knor, E. Šoltés, *Distances in iterated line graphs*, *Ars Combinatoria* **43** (1996), 193-202.
- [16] L. Xiong, Z. Liu, *Hamiltonian iterated line graphs*, *Discrete Math.* **256** (2002), 407-422.