On Some Conjectures by Rabinowitz

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Abstract

Working on the problem of finding the numbers of lattice points inside convex lattice polygons Rabinowitz has made several conjectures dealing with convex lattice nonagons and decagons. An intensive computer search preceded a formulation of the conjectures. The main purpose of this paper is to prove some of Rabinowitz's conjectures. Moreover, we obtain an improvement of a conjectured result and give short proofs of two known results.

1 Introduction and Basic Facts

We start with some basic definitions. A *lattice point* in the plane is a point with integer coordinates. A *lattice segment* is a line segment whose endpoints are lattice points. Any line passing through two lattice points is called a *lattice line*. A *lattice polygon* is a simple polygon whose vertices are lattice points. The fine article by Scott [9] references many interesting problems dealing with lattice polygons.

By $v = v(P)$, $b = b(P)$ and $g = g(P)$ we denote the number of vertices, the number of boundary lattice points, and the number of interior lattice points, respectively, of a lattice polygon P. We will also deal with the number $G = G(P) = b(P) + g(P)$.

The problem of finding relationships between the numbers v, b, g and G is of great interest and has been investigated by many authors and in different settings (not only for the square lattice), see among others [1, 2, 3, 4, 6, 7, 9, 10, 11].

Rabinowitz [7] has obtained many relationships between the numbers g and v. He has investigated such relationships also by computer. This method of search has revealed some connections which Rabinowitz has formulated as conjectures. The purpose of this paper is to prove several conjectures from [7] which are stated below (In parenthesis we include the original names of the conjectures).

Conjecture 1.1 (The Nonagon Anomaly) *A convex lattice nonagon can have* 7 *interior lattice points or* 10 *interior lattice points, but it cannot have either* 8 *or* 9 *interior lattice points.*

The next three conjectures are closely related to the first one and simply follow from it. A lattice polygon P is *lean* if all its boundary lattice points are vertices. In other words, if $b(P) = v(P)$.

Conjecture 1.2 (The Fat Nonagon Theorem) *A non-lean convex lattice nonagon contains at least* 10 *interior lattice points.*

Conjecture 1.3 (Conjecture 7.10) *If a convex lattice polygon* K *has* 8 *interior lattice points, then* $v(K) \leq 8$.

Conjecture 1.4 (Conjecture 7.11) *If a convex lattice polygon* K *has* 9 *interior lattice points, then* $v(K) \leq 8$.

To understand the next conjecture we need a review of some definitions. An *affine transformation* is a linear transformation followed by a translation. A *unimodular transformation* is one that preserves area. If the entries of the matrix corresponding to a unimodular transformation are integers then the transformation is known as an *integral unimodular transformation.* Such a transformation has the property that it preserves convexity and the number of lattice points in a set.

Two lattice polygons are said to be *lattice equivalent* if one can be transformed into the other via an integral unimodular affine transformation. In particular, two lattice polygons are lattice equivalent if one can be transformed into the other via a *shear about line* l, that is, an integral unimodular transformation that leaves all the points on line l fixed. For example, a shear about the x-axis in the plane is given by the equations

$$\begin{aligned} x' &= x + ky \\ y' &= y, \end{aligned}$$

where k is an integer and is called its magnitude.

Conjecture 1.5 (Conjecture 6.4) *If* K *is a convex lattice decagon with* 10 *interior lattice points, then* K *is lattice equivalent to the decagon shown in Fig. 1.*

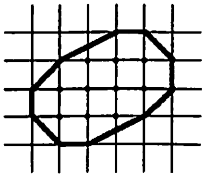

Figure 1: Conjectured unique convex lattice decagon with smallest g

By proving Conjecture 1.5 we are in a position to establish also the following strictly related conjecture.

Conjecture 1.6 (The Fat Decagon Theorem) *A non-lean convex lattice decagon contains at least* 11 *interior lattice points.*

It turns out that the result in the last conjecture can be improved. Namely, we show that a non-lean convex lattice decagon contains at least thirteen interior lattice points.

One of the key concepts used in this note is the interior hull of a lattice polygon. Let K be a convex polygon in the plane. Denote by $H(K)$ the convex hull of the lattice points in the interior of K. The set $H(K)$ is called the *interior hull* of K. Note that $H(K)$ might degenerate into a segment, a point, or even the empty set.

Let u be an edge of $H(K)$ which is a non-degenerated polygon. By $h(u)$ we denote the open halfplane bounded by the line containing u that is exterior to $H(K)$. Similarly as in [7], one can show two very useful facts. The first one says that for any edge u of $H(K)$ the open halfplane $h(u)$ contains at most two vertices of K. The other shows that if $h(u)$ contains two vertices of K, then the lattice segment joining the vertices is parallel to u. By combining the two facts we are able to find a localization of vertices of a lattice polygon K when its interior hull $H(K)$ is a given lattice polygon. Namely, the vertices of K are restricted to lie on lattice lines that are parallel to, and closest to the edges of $H(K)$. This observation leads to the following definition. If K is a convex lattice polygon, then the closed convex region bounded by lattice lines parallel to the edges of K, exterior to K, and closest to K will be called the *outer hull* of K, and is denoted by $\mathcal{O}(K)$. Let us notice that the outer hull of a lattice v-gon is a k-gon, with $k \leq v$ (Figure 6 below shows that the inequality $k < v$ is possible). Moreover, the outer hull not always is a lattice polygon (see Figure 10). One can immediately see the following fact: When the interior hull H of a lattice polygon K is given, then the vertices of K are located on the boundary of $\mathcal{O}(H(K))$. Therefore we need a procedure of "trimming" the outer hull to a polygon with desired parameters. This justifies an introduction of the following notation.

Let P be a convex polygon with edges on lattice lines and let $H(P)$ be its interior hull. By *cutting away* a vertex A (not necessarily a lattice point) of P we understand an operation that assigns to P the polygon $\mathrm{cl}\,(P \setminus \Delta_A)$, where Δ_A is the triangle with vertices at A and its closest lattice points lying on the edges adjacent to A.

We say that P *admits a trimming at its vertex* A if $\Delta_A \cap H(P) = \emptyset$, and *admits a proper trimming at its vertex* A if in addition we have $v(\mathrm{cl}\,(P \setminus \Delta_A)) = v(P) + 1$.

We notice two useful observations about trimming. First, polygon P admits a proper trimming at A only when the vertices of Δ_A, different from A, lie in the relative interiors of the edges adjacent to A. Second, if A_1 is a vertex of $\mathrm{cl}\,(P \setminus \Delta_A)$ but not of P, then $\mathrm{cl}\,(P \setminus \Delta_A)$ does not admit a proper trimming at A_1, though it still may admit a trimming at A_1.

The following lemma will be very helpful.

Lemma 1.7 *Suppose that H is a convex lattice polygon with interior lattice points. Let u be an edge of H having lattice length* 1 *and let v be an edge of $\mathcal{O}(H)$ parallel to u and lying in $h(u)$. Then v cannot contain two lattice points in its relative interior.*

Proof. Suppose to the contrary that v contains two lattice points B_1 and B_2 in its relative interior. Denote the endpoints of u by A_1 and A_2. Let l_0 be the line containing u. Draw lines: l_1 containing v, l_2 passing through points A_2 and B_1, l_3 passing through B_2 and A_2, and l_4 containing B_1 and A_1 (see Fig. 2.). Denote by u_i, $i = 1, 2$, the edge of H which is adjacent to u at A_i. If u_2 lied between lines l_0 and l_2, then the closest lattice line parallel to u_2 would cut off at least the segment B_1B_2. The placement of u_2 between l_2 and l_3 is impossible for a similar reason. Indeed, if it were placed there, then the closest lattice line parallel to u_2 would go through B_2 cutting off a part of v on one side of B_2 and contradicting the fact that B_2 was a lattice point in the relative interior of v. A similar reasoning reveals that u_1 cannot lie between lines l_0 and l_4.

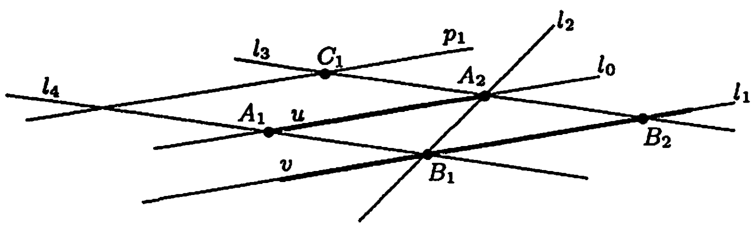

Figure 2: Illustration of Lemma 1.7

The above observations in conjunction with convexity of H would imply that H would lie in the strip, S, bounded by l_3 and l_4. Now we shall show that there is no lattice point in the interior of S. To this end consider lines p_i, $i \geq 1$, passing through consecutive lattice points on l_3 (we denote the points by C_i) and parallel to l_0. If there were a lattice point C in the interior of S then it would lie between lines p_k and p_{k+1} for some integer k. Obviously, the lattice point $B_2 + (C - C_k)$ would lie between lines l_0 and l_1 contradicting the fact that l_1 was the closest parallel lattice line to l_0.

Of course, interior lattice points of H would have to lie in the interior of S which is impossible. This establishes the lemma. □

The assumption that H contains interior lattice points is essential in Lemma 1.7. The reader can easily check that the outer hull of the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$ does not satisfy the lemma.

In this note we will intensively apply the notions of the interior and the outer hulls. We will also use the following results from [7] concerning the interior hull.

Theorem 1.8 *Let K be a convex lattice polygon with interior hull $H(K)$. If $v(K) \geq 7$, then*

$$v(H(K)) \geq \left\lceil \frac{1}{2} v(K) \right\rceil .$$

Theorem 1.9 *Let K be a convex lattice polygon with interior hull $H(K)$. If $v(K) \geq 9$, then*

$$b(H(K)) \geq \left\lceil \frac{2}{3} v(K) \right\rceil .$$

2 Proof of Conjecture 1.1

Fig. 3 illustrates that there are convex lattice nonagons with 7 and 10 interior lattice points.

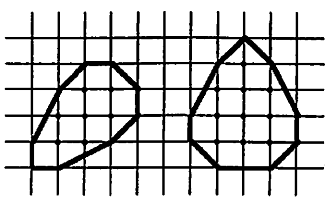

Figure 3: Nonagons with 7 and 10 interior lattice points

In order to prove the other statement of Conjecture 1.1 we first find necessary conditions for existence of convex lattice nonagons containing 8 or

9 interior lattice points. We formulate the conditions in terms of interior hulls. Next we show that the necessary conditions are not sufficient.

2.1 Convex lattice nonagon cannot have 8 interior lattice points

Let K be a convex lattice nonagon with 8 interior lattice points. Interior lattice points of K consist of all lattice points in $H(K)$. Thus $b(H) + g(H) = G(H) = g(K) = 8$. (Here and further on $H = H(K)$.) Obviously, $v(H) \leq b(H) \leq 8$. By Theorems 1.8 and 1.9 we also have $v(H) \geq 5$ and $b(H) \geq 6$. From the above conditions we get nine possible realizations of the values which could be assumed by the numbers $v(H)$, $b(H)$ and $g(H)$. We collect them in the table below.

Case	$v(H)$	$b(H)$	$g(H)$	$G(H) = g(K)$
1.1	5	6	2	8
1.2	5	7	1	8
1.3	5	8	0	8
1.4	6	6	2	8
1.5	6	7	1	8
1.6	6	8	0	8
1.7	7	7	1	8
1.8	7	8	0	8
1.9	8	8	0	8

Table 1

Since any convex lattice pentagon (hence also hexagon and so on) contains an interior lattice point we can immediately eliminate Cases 1.3, 1.6, 1.8 and 1.9. The fact that any non-lean convex lattice hexagon contains at least two interior lattice points, see [7, The Fat Hexagon Theorem], eliminates Case 1.5. Elimination of Case 1.7 follows from [7, Proposition 2.7] since any convex lattice heptagon contains at least four interior lattice points. Now we will show that in the remaining three cases it is impossible to construct any convex lattice nonagon with the interior hull having the desired parameters.

Case 1.1. In this case the interior hull H is a convex lattice pentagon with two interior lattice points and six boundary lattice points. Thus, H has four edges of lattice length 1 and one edge, call it e, of lattice length 2. We try to retrieve K from $\mathcal{O}(H(K))$. It is clear that only by cutting away vertices of $\mathcal{O}(H(K))$ we can get a convex lattice polygon having more vertices than in H. In order to get a convex lattice nonagon we would have to cut away four vertices of $\mathcal{O}(H(K))$. Since H has four edges of lattice length 1, Lemma 1.7 implies that there are four edges in $\mathcal{O}(H(K))$ with at most one lattice point in their relative interiors. Label the vertices of $\mathcal{O}(H(K))$ in such a way that B_1 and B_5 are endpoints of e. Cutting away one of the vertices B_2 or B_4 results in a possibility to cut away additionally only the other one. Clearly, this way of trimming might produce at most a heptagon. If we start with cutting away any of the remaining three vertices, then it might be possible to cut away only all three of them. In this case we might obtain at most an octagon. In either case it is impossible here to trim $\mathcal{O}(H(K))$ to a nonagon.

Case 1.2. In this case the interior hull H is a convex lattice pentagon with one interior lattice point and seven boundary lattice points. From [6, Theorem 5] it follows that there exists only one (up to lattice equivalence) lattice pentagon H satisfying the required conditions. The unique lattice pentagon and its outer hull are provided in Figure 4.

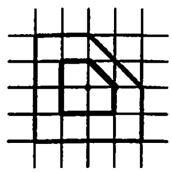

Figure 4: Unique pentagon and its outer hull

Although the outer hull admits a proper trimming at every vertex, nevertheless after cutting away any three vertices we no longer are able to increase the number of vertices. As a result we can obtain (at most) an octagon.

Case 1.4. Now the interior hull H is a convex lattice hexagon with two interior lattice points and six boundary lattice points. Clearly, every edge of H has lattice length 1.

We start with the obvious observation that the lattice segment joining the two interior lattice points of H has lattice length 1. Using similar argument to that in [6, The x-axis Lemma] we can transform H by means of an integral unimodular affine transformation in such a way that its interior lattice points are mapped into points with coordinates $(1, 0)$ and $(2, 0)$. As the two points form the interior hull of H, it is clear that such a transformation carries H into a lattice hexagon lying in the strip bounded by the lines $y = -1$ and $y = 1$. Further, applying a shear about the x-axis (if necessary), we can obtain H to be a hexagon entirely lying in the halfplane $x \geq 0$ and having $A = (0, 1)$ as its vertex. Any lattice hexagon whose all interior lattice points colline and is placed in the way described here will be referred to be in a *basic position*. It will be convenient to have the following lemma restricting possible placements of hexagons in a basic position to the rectangle R_k , $k \geq 2$, with vertices at $(0, 1), (0, -1), (k + 1, 1), (k + 1, -1)$.

Lemma 2.1 *If H_1 is a lean hexagon in a basic position having k collinear interior lattice points, then there exists a hexagon H_2 also in a basic position which is lattice equivalent to H_1 and entirely lies in R_k.*

Proof of Lemma 2.1. If H_1 entirely lies in R_k we put $H_2 = H_1$. Consider the case when $H_1 \not\subset R_k$. Clearly, the right endpoint of the edge of H_1 contained in the line $y = -1$, denote it by B, lies outside R_k. Since every edge of H_1 has lattice length 1 we see that H_1 must have vertices at the points $(0,0)$, A, $(1,1)$ and $(k+1,0)$. From convexity of H_1 it follows that $B = (t,-1)$, where $k+2 \leq t \leq 2k$. Take the shear about the x-axis

$$\begin{aligned} x' &= x + (t-k-1)y \\ y' &= y. \end{aligned}$$

It transforms B into the point $(k+1,-1)$ and the point $(1,1)$ into the point $(t-k,1)$. Since $t-k \leq k$, the image of H_1 by the shear is contained in R_k. Now take two reflections: the first one about the line $x = 0$ and the other about the line $y = 0$. Next follow them by an integral translation by $\vec{w} = (k+1,0)$. In this way we obtain a lattice hexagon in a basic position entirely lying in R_k which plays the role of H_2. It is not difficult to check that the corresponding integral unimodular affine transformation carrying H_1 into H_2 can be expressed by the 3×3 matrix in the equation

$$\begin{bmatrix} -1 & k+1-t & k+1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}.$$

From the above it is immediately seen that H_2 is lattice equivalent to H_1. This ends the proof of Lemma 2.1. $\square$

Now we return to settle Case 1.4. So, H is in a basic position. Such a placement of H forces the two edges of H adjacent to A to lie on $y = 1$ and $x = 0$. Since every edge of H has lattice length 1 we immediately see that H must have vertices at the points $(0,0)$, $(1,1)$ and $(3,0)$. By Lemma 2.1 the remaining two vertices of H are either at $(1,-1)$ and $(2,-1)$, or at $(2,-1)$ and $(3,-1)$. In this way we obtain two (up to lattice equivalence) lattice hexagons that can play the role of H. Both hexagons together with their outer hulls are provided in Figure 5.

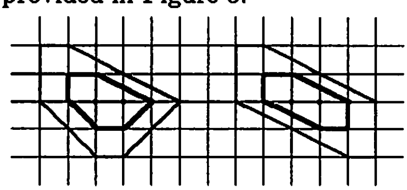

Figure 5: Lean hexagons with two interior lattice points and their outer hulls

Lemma 1.7, in connection with the fact that H is lean, implies that in order to get here a nonagon $\mathcal{O}(H(K))$ would have to admit a proper trimming at every other vertex. Clearly, the first hexagon admits a proper trimming only at one vertex but the other one does not admit a proper trimming at any vertex.

Summarizing the above three cases we can see that no convex lattice nonagon K with g(K)=8 exists. □

2.2 A convex lattice nonagon cannot have 9 interior lattice points

Assume that K is a convex lattice nonagon with the interior hull H containing 9 interior lattice points. By Theorems 1.8 and 1.9 we again have $v(H) \geq 5$ and $b(H) \geq 6$. The facts mentioned during elimination of some cases from Table 1 allow us to impose the following additional constrains: $v(H) \leq 6$ and $b(H) \leq 7$. The reader can easily check that the only possible values achieved by the numbers $v(H)$, $b(H)$ and $g(H)$ are given in Table 2.

Case	$v(H)$	$b(H)$	$g(H)$	$G(H) = g(K)$
2.1	5	6	3	9
2.2	5	7	2	9
2.3	6	6	3	9
2.4	6	7	2	9

Table 2

Case 2.1. The interior hull H is a non-lean pentagon ($b(H) = 6$) with four edges of lattice length 1. Of course we can repeat here the argument from Case 1.1 to show that no convex lattice nonagon with vertices on the boundary of $\mathcal{O}(H(K))$ exists.

Case 2.2. The interior hull is a pentagon having two interior lattice points and seven boundary lattice points. Apparently, such a pentagon has three or four edges of lattice length 1. This and Lemma 1.7 imply that its outer hull contains three or four edges with at most one lattice point in their relative interiors. Similarly as in Case 1.1 one can check that no convex lattice nonagon can be retrieved from $\mathcal{O}(H(K))$ in this case.

Case 2.3. The interior hull is a lean lattice hexagon with three interior lattice points which can be collinear or not.

Interior lattice points are collinear. We place H in a basic position. Argumentation similar to that in Case 1.4, together with Lemma 2.1, establishes the following three (up to lattice equivalence) possible interior hulls H of K with their outer hulls.

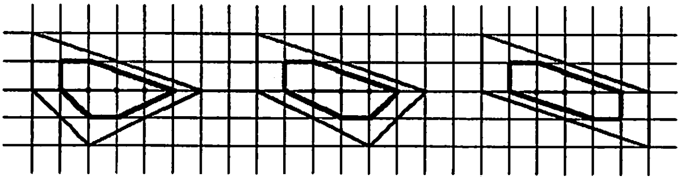

Figure 6: Lean hexagons with two interior lattice points and their outer hulls

Since all three outer hulls here are quadrilaterals, it is obviously impossible to trim $\mathcal{O}(H(K))$ to a convex lattice nonagon with the interior hull H.

Interior lattice points are not collinear. We first observe that the three interior lattice points of H, say B, C, D, can be assumed to be placed at $(1,0)$, $(2,0)$ and $(1,1)$. Indeed, by [6, The x-axis Lemma] we can always apply an integral unimodular affine transformation which transforms B and C into $(1,0)$ and $(2,0)$ and D above the x-axis. Obviously, the image of triangle BCD, denoted by T, cannot contain other lattice points than its vertices. So, by Pick's Formula the area of T is $1/2$. Since the base of T has length 1, its altitude must also have length 1. Thus, D must lie on the line $y = 1$. Applying a shear about the x-axis we can map D into the point $(1,1)$.

Triangle having vertices at $(1,0)$, $(2,0)$ and $(1,1)$ forms $H(H(K))$, that is, the interior hull of the interior hull of K. The interior hull $H(K)$ has six vertices on the boundary of $\mathcal{O}(H(H(K)))$ which is triangle Δ with vertices at $(0,-1)$, $(4,-1)$ and $(0,3)$. Clearly, to get a hexagon we have to cut away all three vertices of Δ. It is easy to check - keeping in mind that the hexagon must be lean - that the y-coordinate of exactly one vertex of it, call it E, must be 2. If $E = (0,2)$, then next vertices of $H(K)$ (going counterclockwise) must be at $(0,1)$, $(1,-1)$, $(2,-1)$, $(3,0)$ and $(2,1)$. If E lies on the hypotenuse of Δ, then $E = (1,2)$ and the remaining vertices of $H(K)$ (this time going clockwise) are at $(2,1)$, $(3,-1)$, $(2,-1)$, $(0,0)$ and $(0,1)$. One can see that a reflection about the line $y = x - 1$ carries one hexagon into the other one. Clearly, the two hexagons are lattice equivalent. The one (up to lattice equivalence) hexagon and its outer hull is shown below.

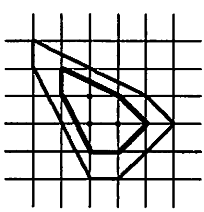

Figure 7: Unique hexagon with three non-collinear interior lattice points and its outer hull

Here every vertex of $\mathcal{O}(H(K))$ is an endpoint of an edge of lattice length 1 and therefore $\mathcal{O}(H(K))$ does not admit a proper trimming at any vertex.

Case 2.4. $v(H) = 6$, $b(H) = 7$, $g(H) = 2$. So, the interior hull H is a convex lattice hexagon with two interior lattice points and seven boundary lattice points. Place H in a basic position. We can assume that the edge of H which has lattice length 2 is adjacent to A. Besides A and $(2, 1)$, such a placement forces H to have vertices at $(0, 0)$ and $(3, 0)$. The remaining two vertices are either at $(1, -1)$ and $(2, -1)$, or at $(2, -1)$ and $(3, -1)$. A reflection about the line $x = 3/2$ followed by a shear about the x-axis maps the second hexagon into the first one. Hence, in this case it is enough to examine only one outer hull shown in Figure 8.

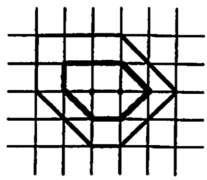

Figure 8: Non-lean hexagon with two interior lattice points and its outer hull

One can easily check that here $\mathcal{O}(H(K))$ admits a proper trimming at three vertices. However by cutting away the three vertices we obtain an octagon.

Summarizing the above four cases we can see that no convex lattice nonagon with nine interior lattice points exists. This ends the proof of Conjecture 1.1. □

As an immediate corollary from Conjecture 1.1 we obtain

Proof of Conjecture 1.2. Let K be a non-lean convex lattice nonagon. By [7, Proposition 6.1] we have $g(K) \geq 7$. Since equality $g(K) = 7$ holds only when K is the lean lattice nonagon on the left side in Figure 3 (or nonagon which is lattice equivalent to it) we infer that $g(K) \geq 8$. In the proof of Conjecture 1.1 we have just ruled out the cases $g(K) = 8$ and $g(K) = 9$. Thus $g(K) \geq 10$. Let us notice that the lattice polygon on the right side in Figure 3 illustrates that there exist non-lean convex lattice nonagons with 10 interior lattice points. □

Proof of Conjectures 1.3 and 1.4. Let K be a convex lattice polygon with eight (nine) interior lattice points. Obviously, we cannot have $v(K) \geq 9$ which immediately follows from The Nonagon Anomaly and the fact (see, [7, Proposition 6.3]) that any convex lattice decagon has at least ten interior lattice points. Thus $v(K) \leq 8$. □

3 Proof of Conjecture 1.5

We prove this conjecture using the same method as in the proof of Conjecture 1.1. We first find possible interior hulls of convex lean lattice decagons and next examine if it is possible to trim their outer hulls to a convex lattice decagon.

Let K be a convex lattice decagon. By Theorems 1.8 and 1.9 we have the inequalities $v(H) \geq 5$ and $b(H) \geq 7$ for its interior hull H. The reader can easily verify that Table 3 provides the possible values assumed by the numbers $v(H)$, $b(H)$ and $g(H)$.

Case	$v(H)$	$b(H)$	$g(H)$	$G(H) = g(K)$
3.1	5	7	3	10
3.2	5	8	2	10
3.3	6	7	3	10
3.4	6	8	2	10

Table 3

Cases 3.1 and 3.2. The interior hull H is a non-lean pentagon ($b(H) = 7$ or $b(H) = 8$) with interior lattice points. Obviously, in both cases H has at least two edges of lattice length 1. If $\mathcal{O}(H)$ has less than five vertices, then of course it is impossible to trim it to a convex lattice decagon with H as its interior hull. Suppose $\mathcal{O}(H)$ is a pentagon. By Lemma 1.7 the outer hull $\mathcal{O}(H)$ has at least one edge, say v, with at most one lattice point in its relative interior. If we cut away one endpoint of v, then by cutting away the other one we do not increase the number of vertices. This implies that no convex lattice decagon can be obtained here because in order to get a decagon we need every vertex of $\mathcal{O}(H)$ to give rise to two new vertices.

Case 3.3. H is a hexagon with one edge, say u, of lattice length 2. Let v be the corresponding edge of $\mathcal{O}(H)$. All other edges of the outer hull have, by Lemma 1.7, at most one lattice point in their relative interiors. Label the vertices of $\mathcal{O}(H)$ (we may assume that it is a hexagon) counterclockwise in such a way that B_1 and B_6 are endpoints of v. If it is possible to cut away both endpoints of v, then additionally we are able to cut away only B_3 or B_4. If only one endpoint of v is cut away, say B_1, then it may be possible to cut away B_3 and B_5. In either case we are not able to get more than nine vertices.

Case 3.4. The interior hull H is a lattice hexagon with two interior lattice points and eight boundary lattice points. If H has one edge of lattice length 3 (and remaining of lattice length 1), then the case can be ruled out in the same way as in Case 3.3.

Assume that H has two edges, say u_1 and u_2, of lattice length 2. Place H in a basic position with u_1 adjacent to A. Obviously, u_1 cannot lie on the line $x = 0$. Thus u_1 lies on $y = 1$ and u_2 must lie on the line $y = -1$. Clearly, such a placement determines H uniquely; it has vertices at A, $(0,0)$, $(1,-1)$ $(3,-1)$ $(3,0)$ and $(2,1)$. The required unique hexagon with its outer hull is shown in Fig. 9.

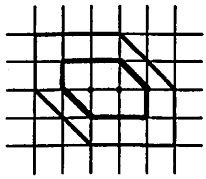

Figure 9: Lattice hexagon with $g = 2$, $b = 8$ and its outer hull

Denote by v_1 and v_2 the edges of $\mathcal{O}(H)$ that correspond to u_1 and u_2. Label the vertices of $\mathcal{O}(H)$ in such a way that B_1 and B_6 are vertices of v_1. If we cut away B_2 (or B_5) then, in connection with Lemma 1.7 and its consequences, we can additionally cut away only B_4 and B_6 (or B_3 and B_1). This way of trimming results in a nonagon.

However, in this case there exists a trimming of $\mathcal{O}(H)$ to a decagon. Indeed, when we cut away the endpoints of both v_1 and v_2 we obtain a decagon. A reflection about the line $x = 3/2$ carries it into the decagon shown in Figure 1. In this way we have shown that any convex lattice decagon with ten interior lattice points is lattice equivalent to the decagon from Figure 1. This completes the proof of Conjecture 1.5. □

Proof of Conjecture 1.6. Let K be a non-lean convex lattice decagon. By [7, Proposition 6.3] we have $g(K) \geq 10$. Since equality $g(K) = 10$ holds only when K is the lean decagon in Figure 1 (or decagon which is lattice equivalent to it) we infer that $g(K) \geq 11$. □

Conjecture 1.6 is an obvious consequence of Conjecture 1.5. It turns out that further application of the method used in this paper allows us to give an improvement of the result of Conjecture 1.6. Namely, we will show that a convex non-lean lattice decagon contains at least 13 interior lattice points. This will be obtained as an immediate consequence of the following theorem.

Theorem 3.1 (The Decagon Anomaly) *A convex lattice decagon cannot have either* 11 *or* 12 *interior lattice points.*

Proof. Let K be a convex lattice decagon with the interior hull H for which $G(H) = 11$ or $G(H) = 12$. By Theorems 1.8 and 1.9 we have the inequalities $v(H) \geq 5$ and $b(H) \geq 7$. The reader can easily verify that Table 4/5 provides the only possible values assumed by the numbers $v(H)$, $b(H)$ and $g(H)$ in both cases.

Case	$v(H)$	$b(H)$	$g(H)$	$G(H) = g(K)$
4.1/5.1	5	7	4/5	11/12
4.2/5.2	5	8	3/4	11/12
4.3/5.3	5	9	2/3	11/12
/5.4	5	10	/2	/12
4.4/5.5	6	7	4/5	11/12
4.5/5.6	6	8	3/4	11/12
4.6/5.7	6	9	2/3	11/12
/5.8	6	10	/2	/12
4.7/5.9	7	7	4/5	11/12
/5.10	7	8	/4	/12
/5.11	8	8	/4	/12

Table 4/5

Cases 4.1 - 4.3. The interior hull H has at least one edge of lattice length 1, say u. From Lemma 1.7 it follows that by cutting away both vertices of v (the edge of $\mathcal{O}(H)$ which corresponds to u) we cannot increase the number of vertices by 2. Hence no convex lattice decagon can be obtained in this case.

Cases 4.4 - 4.6. Case 4.4 can be ruled out in the same way as Case 3.3. Considerations similar to those in Case 2.3 show that no convex lattice decagon with the interior hull H described in Case 4.5 exists. We leave the details to the reader. In order to rule out Case 4.6 it is enough to notice that a convex lattice hexagon with two interior lattice points can have at most eight boundary lattice points. To see this, place such a hexagon in a basic position and the result follows immediately.

Case 4.7. By [7, Proposition 8.10] there are only two (up to lattice equivalence) convex lean lattice heptagons with four interior lattice points. We show both together with their outer hulls in Figure 10.

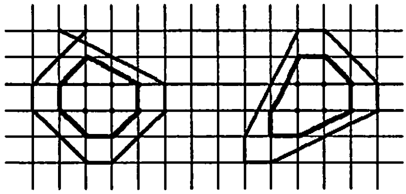

Figure 10: Two lean heptagons with $g = 4$ and their outer hulls

Clearly, none of the outer hulls admits a proper trimming at any vertex. From the above it follows that no convex lattice decagon with 11 interior lattice points exists.

Now we will show that convex lattice decagons with 12 interior lattice points do not exist either. To this end we will consider all eleven cases from Table 5.

Cases 5.1 - 5.4. The reason for ruling out Cases 5.1-5.3 is the same as in Cases 4.1-4.3. Similarly as in Case 4.6 we can show that a convex lattice pentagon with two interior lattice points can have at most nine boundary lattice points. This eliminates Case 5.4.

Cases 5.5 - 5.8. Cases 5.5 and 5.6 can be ruled out in a similar way as Case 3.3. Considerations similar to those in Case 2.3 lead in Case 5.7 to two (up to lattice equivalence) interior hulls with collinear interior lattice points and one with non-collinear interior lattice points. All three outer hulls can be trimmed to (at most) a nonagon. The fact that a convex lattice hexagon with two interior lattice points can have at most eight boundary lattice points rules out Case 5.8.

Case 5.9. The interior hull is a lean heptagon with 5 interior lattice points. Such heptagons must have vertices on the boundary of the outer hulls provided in Figure 14 below. It is easy to check that only the first outer hull can be trimmed to a lean heptagon. The reader can find two possible heptagons and check that its outer hulls (being hexagons) do not admit a proper trimming at any vertex.

Case 5.10. It is easy to check that the unique interior hull in this case is given in Figure 11.

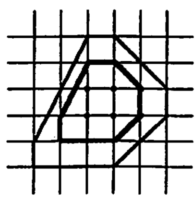

Figure 11: Convex lattice heptagon with $b = 8$ and $g = 4$ and its outer hull

The outer hull admits a proper trimming only at one vertex and therefore it is impossible to get a decagon in this case.

Case 5.11. The interior hull H is the octagon shown in Figure 13 below. One can see that $\mathcal{O}(H)$ does not admit a proper trimming at any vertex. This and the above considerations end the proof of Theorem 3.1. □

An immediate consequence of Theorem 3.1 is the following, already announced, improvement of the result conjectured as The Fat Decagon Theorem.

Theorem 3.2 (Improved Fat Decagon Theorem) *A non-lean convex lattice decagon contains at least* 13 *interior lattice points.*

Theorem 3.2 is sharp in this sense that there are non-lean convex lattice decagons with 13 interior lattice points. Such a decagon is given in Figure 12. It is worth remarking that the decagon can be trimmed to a lean convex decagon with 13 interior lattice points.

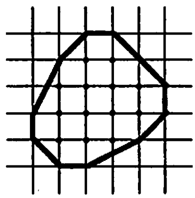

Figure 12: Non-lean convex lattice decagon with $g = 13$

4 Short proofs of two known results

Applying the method used in this paper we can give short proofs of the following two results from [7, 8] dealing with convex lattice octagons and called there The Central Octagon Theorem and The Octagon Anomaly, respectively.

Theorem 4.1 *If K is a convex lattice polygon with $v = 8$ and $g = 4$, then K is lattice equivalent to the centrally symmetric octagon shown in Figure 13.*

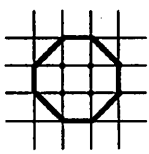

Figure 13: Unique convex lattice octagon with $g = 4$

Theorem 4.2 *A convex lattice octagon can have* 4 *interior lattice points or* 6 *interior lattice points, but it cannot have* 5 *interior lattice points.*

Proof of Theorem 4.1. Let K be a convex lattice octagon with four interior lattice points. This and Theorem 1.8 imply that $v(H(K)) = 4$. From [6, Theorem 1] it follows that $H(K)$ must be lattice equivalent to a unit square. By cutting away all the vertices of its outer hull (which is a square with edges of lattice length 3) we obtain the unique octagon. The proof is complete. □

Proof of Theorem 4.2. Let K be a convex lattice octagon. We will only prove the statement that K cannot have 5 interior lattice points. Suppose that K has 5 interior lattice points. From Theorem 1.8 and the fact that a convex lattice pentagon must contain an interior lattice point it follows that the interior hull, $H(K)$, must be a quadrilateral. The quadrilateral either has one edge of lattice length 2 or is lean and has one interior lattice point. It is easily verified that in the former case H must be lattice equivalent to the quadrilateral with vertices at $(0,0)$, $(0,1)$, $(2,1)$ and $(1,0)$. In the latter case, by [6, Theorem 5] (see also [5]), H is lattice equivalent to one of two possible quadrilaterals. All three possible interior hulls and their outer hulls are shown in Figure 14.

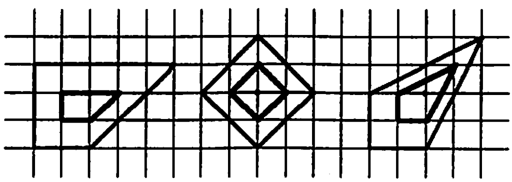

Figure 14: Possible interior hulls with their outer hulls

One can immediately see that the first outer hull can be trimmed to a heptagon, the second does not admit any trimming, and the third one admits a trimming to a pentagon. Thus, no convex lattice octagon K can be retrieved from $\mathcal{O}(H)$ and the proof is complete. □

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