Further Ramsey Numbers for Small Complete Bipartite Graphs

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Abstract

The exact values of eleven Ramsey numbers $r(K_{l_1,n_1}, K_{l_2,n_2})$ where $3 \leq l_1 + n_1, l_2 + n_2 \leq 7$ are determined, almost completing the table of all 66 such numbers.

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1 Introduction

Considering the Ramsey number $r(K_{l_1,n_1}, K_{l_2,n_2})$ where $3 \leq l_1 + n_1, l_2 + n_2 \leq 7$, the exact values for 51 of these numbers may be gathered from various papers, while the exact results for 15 numbers have been unknown, so far. General formulas yield the exact value of $r(K_{l_1,n_1}, K_{l_2,n_2})$ for certain cases as investigated in [2, 3, 6, 7, 9, 12, 14, 15, 17], and additional single results are to be found in [5, 7, 8, 11]. Here, we almost complete the table of $r(K_{l_1,n_1}, K_{l_2,n_2})$ where $3 \leq l_1 + n_1, l_2 + n_2 \leq 7$, presenting eleven new exact results and giving upper bounds for the four still open cases.

Throughout this paper the following specialized notation will be used. A 2-coloring of a graph always means a 2-coloring of its edges with colors red and green. A $(G_1, G_2)_p$-coloring is a 2-coloring of the complete graph K_p containing neither a red copy of G_1 nor a green copy of G_2. Considering two disjoint subsets U_1 and U_2 of the vertex set of a 2-colored K_p, $q_r(U_1, U_2)$ denotes the number of red edges between U_1 and U_2. If U_1 consists of a single vertex v, then we write $q_r(v, U_2)$ instead. In a similar way the case $v \in U_2$ is handled, using the short term $q_r(v, U_2)$ meaning $q_r(v, U_2 \setminus \{v\})$.

Moreover, $r_2(U)$ refers to the overall number of common red neighbors to any pair of vertices in the vertex set U, that is

$$r_2(U) = \sum_{\{v_1,v_2\} \in \binom{U}{2}} |N_r(v_1) \cap N_r(v_2)|.$$

For the red subgraph of a 2-coloring the degree of a vertex v is denoted by $d_r(v)$, and we write Δ_r to indicate its maximum degree. Finally, $[U]_r$ is the red-edge subgraph induced by U, and $E_r(G)$ refers to the set of red edges for any graph G. Regarding the given colorings, note that they are specified as $(1,2)$-matrices where the entry "1" indicates a red edge and the entry "2" denotes a green edge joining the two vertices in question.

2 Results

From the above-mentioned papers and from the lemmas below we derive Table 1.

	$K_{1,2}$	$K_{1,3}$	$K_{1,4}$	$K_{1,5}$	$K_{1,6}$	$K_{2,2}$	$K_{2,3}$	$K_{2,4}$	$K_{2,5}$	$K_{3,3}$	$K_{3,4}$
$K_{1,2}$	**3** [2, 6]	**5** [2, 6]	**5** [2, 6]	**7** [2, 6]	**7** [2, 6]	**4** [2]	**5** [2]	**6** [2]	**7** [2]	**7** [2]	**7** [2]
$K_{1,3}$		**6** [6, 7]	**7** [6, 7]	**8** [6, 7]	**9** [6, 7]	**6** [7]	**7** [7]	**8** [7]	**9** [7]	**8** [7]	**9** [7]
$K_{1,4}$			**7** [6]	**9** [6]	**9** [6]	**7** [15]	**9** [17]	**9** [17]	**11** [17]	**11** L. 2	**11** L. 2
$K_{1,5}$				**10** [6]	**11** [6]	**8** [5]	**10** [5]	**11** [8]	**13** [17]	**12** L. 3	**13** L. 4
$K_{1,6}$					**11** [6]	**9** [5]	**11** [5]	**13** L. 1	**14** L. 1	**13** L. 4	**14** L. 4
$K_{2,2}$						**6** [3, 9]	**8** [9, 12]	**9** [9]	**11** [9]	**11** [11]	**11** [11]
$K_{2,3}$							**10** [3]	**12** [4, 12]	**13** [14]	**13** [7]	**14** L. 5
$K_{2,4}$								**14** [3]	**16** [12]	**16** L. 6	≤ 19
$K_{2,5}$									**18** [3]	**18** L. 7	≤ 21
$K_{3,3}$										**18** [7]	≤ 25 [13]
$K_{3,4}$											≤ 30 [13]

Table 1: $r(K_{l_1,n_1}, K_{l_2,n_2})$ where $3 \leq l_1 + n_1, l_2 + n_2 \leq 7$

To approach some of the exact results of this paper a combined strategy, based on both various counting arguments and the assistance of a computer, is required. In a first step the counting arguments are to offer a significant reduction of the number of potential $(K_{l_1,n_1}, K_{l_2,n_2})_p$-colorings. This is essential because any computer would easily be overstrained by considering even some major share of all 2^q possible 2-colorings of the $q = \binom{p}{2}$ edges of K_p. In the final step only the still remaining colorings are checked – and eliminated – by a computer (PA-RISC PA 8200 processor with 286 MHz) running some enhanced backtracking algorithm. Here, the decisive improvement compared to plain backtracking is that the algorithm does not need to count common red or green neighbors whenever checking for the forbidden subgraphs. The required information is automatically updated and kept in an appropriately defined array when coloring or discoloring any edge.

3 Proofs

Since all the respective lower bounds are to be established by appropriate $(K_{l_1,n_1}, K_{l_2,n_2})_{r-1}$-colorings, the subsequent lemmas focus on the corresponding upper bounds only. Our arguments are completed by the specific colorings listed at the end of this section. Furthermore, some of the proofs are just sketched in this paper. Their full versions can be found in [10], for example. For additional information on the general approach to our proofs, [11] or [14] may be consulted, too.

Lemma 1

$$\begin{aligned} r(K_{1,6}, K_{2,4}) &= 13, \\ r(K_{1,6}, K_{2,5}) &= 14. \end{aligned}$$

Proof: Both upper bounds are a direct consequence of

$$r(K_{1,n_1}, K_{2,n_2}) \leq n_1 + \frac{1}{2}\left(n_2 + \sqrt{(n_2-2)^2 + 4n_1(n_2-1)}\right) + 1$$

already given in [16]. ■

Lemma 2

$$\begin{aligned} r(K_{1,4}, K_{3,3}) &= 11, \\ r(K_{1,4}, K_{3,4}) &= 11. \end{aligned}$$

Proof: To establish $r(K_{1,4}, K_{3,3}) \leq 11$ as well as $r(K_{1,4}, K_{3,4}) \leq 11$ we may apply

$$r(K_{1,n_1}, K_{l_2,n_2}) \leq \begin{cases} n_2 + (n_1 - 2)l_2 + 2 & \text{if } n_1 \text{ is odd or } n_2 \text{ is odd,} \\ n_2 + (n_1 - 2)l_2 + 1 & \text{otherwise,} \end{cases}$$

originally proved in [1] for a strictly limited range of indices and re-proved in [8] for an extended range. ■

Lemma 3

$$r(K_{1,5}, K_{3,3}) = 12.$$

Proof: For the upper bound $r(K_{1,5}, K_{3,3}) \leq 12$, we assume that there exists a $(K_{1,5}, K_{3,3})_{12}$-coloring χ. Regarding $r(K_{1,5}, K_{2,4}) = 11$ (cf. Table 1), χ produces a green subgraph $K_{2,4}$. Hence, throughout the proof's case analysis we will consider a maximum-order green subgraph $K_{2,s}$ in χ. According to its obvious partition this subgraph's vertex set may be divided into subsets $V^* = \{v_1, v_2\}$ and $U = \{u_1, \ldots, u_s\}$. Moreover, let $W = \{w_1, \ldots, w_{10-s}\}$ denote the set of all remaining vertices.

If $s \geq 5$, then we pick an arbitrary green subgraph $K_{2,5}$, and the absence of a green subgraph $K_{3,3}$ yields $q_r(w_j, U) \geq 3$ as well as $q_r(u_i, U) \geq 2$. Therefore, we obtain $q_r(U, W) \geq 15$, and at least one vertex from U meets $q_r(u_i, W) \geq 3$, contradicting our initial assumption.

Thus, we are left with $s = 4$. Avoiding a green subgraph $K_{3,3}$, we derive $q_r(w_j, U) \geq 2$, and as the green subgraph $K_{2,4}$ is of maximum order, we additionally have $q_r(w_j, V^*) \geq 1$. Furthermore, $\Delta_r \leq 4$ yields $q_r(w_j, W) \leq 1$, and the absence of a green subgraph $K_{3,3}$ in $[W]$ forces $[W]_r = 3K_2$. Hence, the edges w_1w_2, w_3w_4, w_5w_6 may be assumed red, while all remaining edges in $[W]$ may be supposed green. However, we obtain another green subgraph $K_{2,4}$ in $[W]$, and by applying the above arguments to it we may suppose v_1v_2, u_1u_2, u_3u_4 red and all remaining edges in $[U \cup V^*]$ green, too. Then, for any pair of edges (x_1x_2, y_1y_2) where $x_1x_2 \in E_r([U \cup V^*])$ and $y_1y_2 \in E_r([W])$, $s = 4$ and $\Delta_r \leq 4$ demand both $q_r(x_\nu, W) = q_r(y_\nu, U \cup V^*) = 3$ and $q_r(x_\nu, \{y_1, y_2\}) = q_r(y_\nu, \{x_1, x_2\}) = 1$. Therefore, w.l.o.g. $v_1w_1, v_1w_3, v_1w_5, v_2w_2, v_2w_4, v_2w_6$ may be assumed red, while all remaining edges in $V^* \times W$ may be supposed green. Now, $|N_g(u_1, v_1)|, |N_g(u_1, v_2)| \leq s = 4$ yields u_1w_1, u_1w_3, u_1w_6 to be colored red, u_1w_2, u_1w_4, u_1w_5 to be colored green, and an analogous coloring of the edges in $\{u_2\} \times W$. Finally, $d_r(w_j) = 4$ assigns color red to u_3w_1, u_4w_2 and color green to u_3w_2, u_4w_1, $s = 4$ forces $u_3w_4, u_4w_3 \in E_r(K_{12})$ and $u_3w_3, u_4w_4 \in E_g(K_{12})$, and similarly we derive $u_3w_5, u_4w_6 \in E_r(K_{12})$ as well as $u_3w_6, u_4w_5 \in E_g(K_{12})$. Considering the vertex sets $\{u_1, v_1, w_5\}$ and $\{u_4, w_2, w_4\}$, we find that they produce a green subgraph $K_{3,3}$, contradicting our initial assumption and completing the proof. ■

By a similar approach as presented in the proof of Lemma 3 we investigate the properties of possible $(K_{1,5}, K_{3,4})_{13}$-, $(K_{1,6}, K_{3,3})_{13}$-, and $(K_{1,6}, K_{3,4})_{14}$-colorings. Thus, after an accurate case analysis and the application of appropriate counting arguments have yielded the colors of half the edges in any of the above cases, the aspired contradictions are achieved by running an enhanced backtracking algorithm on a computer, proving the respective upper bounds.

Lemma 4

$$\begin{aligned} r(K_{1,5}, K_{3,4}) &= 13, \\ r(K_{1,6}, K_{3,3}) &= 13, \\ r(K_{1,6}, K_{3,4}) &= 14. \end{aligned}$$

For the proof of Lemma 5 the assistance of a computer is required, too. Although we have $l_1 = 2$ instead of $l_1 = 1$ now, the procedure to fix the colors of as many edges as possible before applying our backtracking algorithm does not differ much from the one presented above. Nevertheless, about the specific differences you may confer the proof of Lemma 7 or the proofs in [11].

Lemma 5

$$r(K_{2,3}, K_{3,4}) = 14.$$

The following result is attained by applying counting arguments only.

Lemma 6

$$r(K_{2,4}, K_{3,3}) = 16.$$

Proof: While proving the upper bound $r(K_{2,4}, K_{3,3}) \leq 16$, we assume that there exists a $(K_{2,4}, K_{3,3})_{16}$-coloring χ. Regarding $r(K_{2,4}, K_{2,5}) = 16$ (cf. Table 1), χ contains a green subgraph $K_{2,5}$. According to its obvious partition this subgraph's vertex set may be divided into subsets $V^* = \{v_1, v_2\}$ and $U = \{u_1, \ldots, u_5\}$. Moreover, let $W = \{w_1, \ldots, w_9\}$ denote the set of all remaining vertices. Then, the absence of a green subgraph $K_{3,3}$ forces both $q_r(w_j, U) \geq 3$ and $q_r(u_i, U) \geq 2$. Accumulating these results for all 2-element subsets of U, we obtain $r_2(U) \geq 9\binom{3}{2} + 5\binom{2}{2}$. On the other hand, any pair of vertices selected from U may have at most three common red neighbors, yielding $r_2(U) \le 3\binom{5}{2}$, and we derive a contradiction to our initial assumption. ■

We conclude with another complex proof dealing with the Ramsey number $r(K_{2,5}, K_{3,3})$.

Lemma 7

$$r(K_{2,5}, K_{3,3}) \;=\; 18.$$

Proof: As usual the proof of the upper bound $r(K_{2,5}, K_{3,3}) \le 18$ starts with the assumption that there exists a $(K_{2,5}, K_{3,3})_{18}$-coloring χ. Regarding $r(K_{2,5}, K_{2,5}) = 18$ (cf. Table 1), χ produces a green subgraph $K_{2,5}$. Hence, throughout the proof's case analysis we will consider a maximum-order green subgraph $K_{2,s}$ in χ. According to its obvious partition this subgraph's vertex set may be divided into subsets $V^* = \{v_1, v_2\}$ and $U = \{u_1, \dots, u_s\}$. Moreover, let $W = \{w_1, \dots, w_{16-s}\}$ denote the set of all remaining vertices.

If $s \ge 6$, we choose an arbitrary green subgraph $K_{2,6}$ from χ, and due to the absence of a green subgraph $K_{3,3}$ we obtain $q_r(w_j, U) \ge 4$ as well as $q_r(u_i, U) \ge 3$. Thus, regarding these results for all 2-element subsets of U, we have the lower bound $r_2(U) \ge 10\binom{4}{2} + 6\binom{3}{2}$, contradicting the corresponding upper bound $r_2(U) \le 4\binom{6}{2}$, following from the fact that any pair of vertices selected from U may have at most four common red neighbors.

Now, the case $s = 5$ remains. Avoiding a green subgraph $K_{3,3}$, we derive both $q_r(w_j, U) \ge 3$ and $q_r(u_i, U) \ge 2$, implying $11\binom{3}{2}+5\binom{2}{2} \le r_2(U) \le 4\binom{5}{2}$. Thereof we obtain $q_r(w_j, U) = 3$ as well as $q_r(u_i, U) = 2$, especially forcing $[U]_r = [U]_g = C_5$. Furthermore, due to $q_r(U, W) = 33$ there are at least three vertices from U, e.g. u_1, u_2, u_3, matching $q_r(u_i, W) \ge 7$, where the absence of a red subgraph $K_{2,5}$ additionally demands $q_r(u_1, W) + q_r(u_2, W) + q_r(u_3, W) \le 22$. Hence, we have to face the situation that $q_r(u_1, W) \in \{7, 8\}$ and $q_r(u_2, W) = q_r(u_3, W) = 7$. Considering the red neighborhood of u_1, u_2, u_3 in W and reminding $q_r(w_j, U) = 3$, we are left with five different constellations. Then, a closer look at the possible positions of u_1, u_2, u_3 in $[U]$'s red subgraph C_5 reveals that we cannot have $|N_r(u_{i_1}, u_{i_2}) \cap W| = 4$ for all three pairs of vertices (u_1, u_2), (u_1, u_3), (u_2, u_3), which eliminates two of our originally five cases. Moreover, the constellation given by $|N_r(u_1, u_2) \cap W| = |N_r(u_1, u_3) \cap W| = 4$ and $|N_r(u_2, u_3) \cap W| \le 3$ may avoid the occurence of a red subgraph $K_{2,5}$ by $E_r([U]) = \{u_1u_2, u_1u_3, u_2u_4, u_3u_5, u_4u_5\}$ only. Therefore, the three remaining cases are specified by the following red edge subsets of $U \times W$:

$$\{u_1\} \times \{w_1, \ldots, w_8\} \cup \{u_2\} \times \{w_5, \ldots, w_{11}\}$$
$$\cup \{u_3\} \times \{w_1, w_2, w_3, w_4, w_9, w_{10}, w_{11}\},$$
$$\{u_1\} \times \{w_1, \ldots, w_7\} \cup \{u_2\} \times \{w_4, \ldots, w_{10}\}$$
$$\cup \{u_3\} \times \{w_1, w_2, w_3, w_4, w_8, w_9, w_{11}\},$$
$$\{u_1\} \times \{w_1, \ldots, w_7\} \cup \{u_2\} \times \{w_4, \ldots, w_{10}\}$$
$$\cup \{u_3\} \times \{w_1, w_2, w_3, w_8, w_9, w_{10}, w_{11}\}.$$

As the green subgraph $K_{2,5}$ is of maximum order, that is $q_r(w_j, V^*) \geq 1$, we may assume v_1w_1 and v_1w_2 red, too. Now, we complete the proof by running an enhanced backtracking algorithm on a computer, particularly considering the above restrictions on χ. ■

Since we already have $r(K_{2,5}, K_{2,5}) = 18$ and $r(K_{3,3}, K_{3,3}) = 18$ (cf. [3, 7]), Lemma 7 additionally proves the following result.

Corollary 8 *The graphs $G_1 = K_{2,5}$ and $G_2 = K_{3,3}$ are another pair of graphs matching*

$$r(G_1, G_1) = r(G_1, G_2) = r(G_2, G_2).$$

Furthermore, notice that the upper bounds not derived from [13], that is $r(K_{2,4}, K_{3,4}) \leq 19$ and $r(K_{2,5}, K_{3,4}) \leq 21$, have been verified by applying counting arguments quite similar to those presented in the proof of Lemma 6.

As notified at the beginning of this section we conclude with the colorings establishing the corresponding lower bounds for the Ramsey numbers considered in Lemmas 1 to 7:

$$\begin{pmatrix}
0&1&1&1&1&1&2&2&2&2&2&2\\
1&0&1&2&2&2&1&1&1&2&2&2\\
1&1&0&2&2&2&2&2&2&1&1&1\\
1&2&2&0&1&2&1&1&2&1&2&2\\
1&2&2&1&0&2&2&2&1&2&1&1\\
1&2&2&2&2&0&1&1&2&2&1&1\\
2&1&2&1&2&1&0&2&1&2&1&2\\
2&1&2&1&2&1&2&0&2&1&2&1\\
2&1&2&2&1&2&1&2&0&1&2&1\\
2&2&1&1&2&2&2&1&1&0&1&2\\
2&2&1&2&1&1&1&2&2&1&0&2\\
2&2&1&2&1&1&2&1&1&2&2&0
\end{pmatrix}$$

$(K_{1,6}, K_{2,4})_{12}$-coloring
$(K_{1,6}, K_{3,3})_{12}$-coloring

$$\begin{pmatrix}
0&1&1&1&1&1&2&2&2&2&2&2&2\\
1&0&1&2&2&2&1&1&1&2&2&2&2\\
1&1&0&2&2&2&2&2&2&1&1&1&2\\
1&2&2&0&1&2&1&1&2&1&2&2&2\\
1&2&2&1&0&2&2&2&1&2&1&2&1\\
1&2&2&2&2&0&1&2&1&2&2&1&1\\
2&1&2&1&2&1&0&2&2&1&1&2&2\\
2&1&2&1&2&2&2&0&2&2&2&1&1\\
2&1&2&2&1&1&2&2&0&1&2&1&2\\
2&2&1&1&2&2&1&2&1&0&2&2&1\\
2&2&1&2&1&2&1&2&2&2&0&1&1\\
2&2&1&2&2&1&2&1&1&2&1&0&2\\
2&2&2&2&1&1&2&1&2&1&1&2&0
\end{pmatrix}$$

$(K_{1,6}, K_{2,5})_{13}$-coloring
$(K_{1,6}, K_{3,4})_{13}$-coloring

$$\begin{pmatrix}
0&1&1&1&2&2&2&2&2&2\\
1&0&2&2&1&1&2&2&2&2\\
1&2&0&2&2&2&1&1&2&2\\
1&2&2&0&2&2&2&2&1&1\\
2&1&2&2&0&2&1&2&1&2\\
2&1&2&2&2&0&2&1&2&1\\
2&2&1&2&1&2&0&2&2&1\\
2&2&1&2&2&1&2&0&1&2\\
2&2&2&1&1&2&2&1&0&2\\
2&2&2&1&2&1&1&2&2&0
\end{pmatrix}$$

$(K_{1,4}, K_{3,3})_{10}$-coloring
$(K_{1,4}, K_{3,4})_{10}$-coloring

$$\begin{pmatrix}
0&1&1&1&1&2&2&2&2&2&2\\
1&0&1&2&2&1&1&2&2&2&2\\
1&1&0&2&2&2&2&1&1&2&2\\
1&2&2&0&2&1&2&1&2&1&2\\
1&2&2&2&0&2&1&2&1&2&1\\
2&1&2&1&2&0&2&2&1&2&1\\
2&1&2&2&1&2&0&1&2&1&2\\
2&2&1&1&2&2&1&0&2&2&1\\
2&2&1&2&1&1&2&2&0&1&2\\
2&2&2&1&2&2&1&2&1&0&1\\
2&2&2&2&1&1&2&1&2&1&0
\end{pmatrix}$$

$(K_{1,5}, K_{3,3})_{11}$-coloring

$$\begin{pmatrix}
0&1&1&1&1&2&2&2&2&2&2&2\\
1&0&1&2&2&1&1&2&2&2&2&2\\
1&1&0&2&2&2&2&1&1&2&2&2\\
1&2&2&0&2&1&2&1&2&1&2&2\\
1&2&2&2&0&2&1&2&1&1&2&2\\
2&1&2&1&2&0&2&1&2&2&1&2\\
2&1&2&2&1&2&0&2&1&2&1&2\\
2&2&1&1&2&1&2&0&2&2&2&1\\
2&2&1&2&1&2&1&2&0&2&2&1\\
2&2&2&1&1&2&2&2&2&0&1&1\\
2&2&2&2&2&1&1&2&2&1&0&1\\
2&2&2&2&2&2&2&1&1&1&1&0
\end{pmatrix}$$

$(K_{1,5}, K_{3,4})_{12}$-coloring

$$\begin{pmatrix}
0&1&1&1&1&1&1&2&2&2&2&2&2\\
1&0&1&1&2&2&2&1&1&2&2&2&2\\
1&1&0&2&1&2&2&1&2&1&2&2&2\\
1&1&2&0&2&1&2&2&1&2&1&2&2\\
1&2&1&2&0&2&1&2&2&2&1&1&2\\
1&2&2&1&2&0&1&2&2&1&2&2&1\\
1&2&2&2&1&1&0&2&2&2&2&1&1\\
2&1&1&2&2&2&2&0&2&1&1&2&1\\
2&1&2&1&2&2&2&2&0&2&2&1&1\\
2&2&1&2&2&1&2&1&2&0&2&1&2\\
2&2&2&1&1&2&2&1&2&2&0&1&2\\
2&2&2&2&1&2&1&2&1&1&1&0&2\\
2&2&2&2&2&1&1&1&1&2&2&2&0
\end{pmatrix}$$

$(K_{2,3}, K_{3,4})_{13}$-coloring

$$\begin{pmatrix}
0&1&2&2&2&2&1&1&1&2&2&1&1&1&2\\
1&0&2&2&2&2&1&2&2&1&1&1&1&2&1\\
2&2&0&1&1&2&1&1&1&1&1&2&2&2&2\\
2&2&1&0&1&1&1&2&2&2&2&1&1&2&2\\
2&2&1&1&0&2&2&1&2&1&2&1&2&1&1\\
2&2&2&1&2&0&2&2&1&2&1&2&1&1&1\\
1&1&1&1&2&2&0&2&1&2&1&1&2&2&2\\
1&2&1&2&1&2&2&0&1&1&2&2&1&1&2\\
1&2&1&2&2&1&1&1&0&2&2&2&2&2&1\\
2&1&1&2&1&2&2&1&2&0&1&2&1&2&1\\
2&1&1&2&2&1&1&2&2&1&0&2&2&1&2\\
1&1&2&1&1&2&1&2&2&2&2&0&2&1&1\\
1&1&2&1&2&1&2&1&2&1&2&2&0&2&2\\
1&2&2&2&1&1&2&1&2&2&1&1&2&0&2\\
2&1&2&2&1&1&2&2&1&1&2&1&2&2&0
\end{pmatrix}$$

$(K_{2,4}, K_{3,3})_{15}$-coloring

$$\begin{pmatrix}
0&1&1&2&1&2&2&2&1&1&2&2&2&1&2&1&1\\
1&0&1&1&2&1&2&2&2&1&1&2&2&2&1&2&1\\
1&1&0&1&1&2&1&2&2&2&1&1&2&2&2&1&2\\
2&1&1&0&1&1&2&1&2&2&2&1&1&2&2&2&1\\
1&2&1&1&0&1&1&2&1&2&2&2&1&1&2&2&2\\
2&1&2&1&1&0&1&1&2&1&2&2&2&1&1&2&2\\
2&2&1&2&1&1&0&1&1&2&1&2&2&2&1&1&2\\
2&2&2&1&2&1&1&0&1&1&2&1&2&2&2&1&1\\
1&2&2&2&1&2&1&1&0&1&1&2&1&2&2&2&1\\
1&1&2&2&2&1&2&1&1&0&1&1&2&1&2&2&2\\
2&1&1&2&2&2&1&2&1&1&0&1&1&2&1&2&2\\
2&2&1&1&2&2&2&1&2&1&1&0&1&1&2&1&2\\
2&2&2&1&1&2&2&2&1&2&1&1&0&1&1&2&1\\
1&2&2&2&1&1&2&2&2&1&2&1&1&0&1&1&2\\
2&1&2&2&2&1&1&2&2&2&1&2&1&1&0&1&1\\
1&2&1&2&2&2&1&1&2&2&2&1&2&1&1&0&1\\
1&1&2&1&2&2&2&1&1&2&2&2&1&2&1&1&0
\end{pmatrix}$$

$(K_{2,5}, K_{3,3})_{17}$-coloring

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