On a Prime Labeling Conjecture

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Abstract: A graph with vertex set V is said to have a prime labeling if its vertices are labelled with distinct integers from $\{1,2......,|V|\}$ such that for each edge xy, the labels assigned to x and y are relatively prime. A graph that admits a prime labeling is called a prime graph. It has been conjectured [1] that when n is a prime integer and m < n, the planar grid $P_m \times P_n$ is prime. We prove the conjecture and also that $P_n x P_n$ is prime when n is a prime integer.

1. Introduction

A simple graph G = (V, E) is said to have a prime labeling if its vertices are labelled with distinct integers from $\{1,2...... |V|\}$ such that for each edge xy, the labels assigned to x and y are relatively prime. A graph with a prime labeling defined on it is called a prime graph (or) simply prime.

For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, their product $G_1 x G_2$ is defined as the graph whose vertex set is $V_1 \times V_2$ and two vertices (u_1, v_1) and (u_2, v_2) in $G_1 \times G_2$ are adjacent if $u_1 = u_2$ and v_1 is adjacent to v_2 or u_1 is adjacent to u_2 and $v_1 = v_2$.

A path of order n is denoted by P_n. For paths P_m and P_n, their product is known as a planar grid.

If n is prime and $m \leq 3$, Vilfred et al [3] proved that the planar grid $P_m \times P_n$ is prime and they conjectured that $P_m \times P_n$ is prime for prime n and m < n. We prove the conjecture. We also prove that $P_n \times P_n$ is prime if n is a prime integer.

Graph theoretic terms and notions used here are in the sense of Harary [2].

2. Planar grids

Theorem 2.1: If n is prime and $3 < m \leq n$, the planar grid $P_m \times P_n$ is prime.

Proof: Let $V(P_m \times P_n) = \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$ and

$$E(P_m \times P_n) = \{u_{ij}\, u_{i(j+1)} : 1 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{u_{ij}\, u_{(i+1)j} : 1 \leq i \leq n-1, \leq j \leq m\}.$$

Case (1) $m < n$.

We define f by $f(u_{ij}) = (j-1)\, n+i$ $(1 \leq i \leq n-1, 2 \leq j \leq m)$.

For $1 \leq i < n$, $f(u_{i1}) = \begin{cases} ni \text{ if } i=1, 3\ldots\ldots\ldots m \text{ or } m-1 \text{ according as } m \text{ is odd or even} \\ i \text{ otherwise.} \end{cases}$

$f(u_{nj}) = j$ or nj according as j is odd or even $(1 \leq j \leq m)$

Since $f(u_{ij}) \equiv i \pmod{n}$ $(1 \leq i \leq n-1, 2 \leq j \leq m)$, it is enough if we verify the 'primeness' of the labels in the first column and in the last row.

First column:

The elements at 'even junctions' and u_{11} do not pose any problem. Thus, we must establish that gcd (kn, k-1)= gcd (kn, k+1)= gcd (kn, n+k) = 1 where k=3, 5……… m or (m-1) according as m is odd or even.

Now d/kn, $d/k-1 \Rightarrow d/n$,

If d=n, $n/k-1$ which cannot happen, since $k-1<n$.

Also d/kn, $d/k+1 \Rightarrow d/n$.

If d=n, $n/k+1$ which again is not possible.

d/kn, $d/n+k \Rightarrow d/n^2$

If d=n, $n/n+k \Rightarrow n/k$, not possible.

If $d=n^2$, n^2/kn, again an impossibility (since $k<n$).

Thus in all the cases, d=1.

n^{th} row:

Here elements in the 'odd junctions' alone require 'primeness' verification. Thus we have to verify that

gcd (k, (k-1)n)= gcd (k, (k+1)n)= gcd (k, (kn-1)) = 1 where

$k = 3, 5....m$ or $(m-1)$ according as m is odd or even.
In the first two cases, if d be a common factor, then d/n.
But $d=n$ yields n/k which should not happen.

Also d/k, $d/kn-1 \Rightarrow d/1 \Rightarrow d=1$.

Case (2) $m = n$.

Assign labels to the vertices in all columns except the last one as in case (1).

For the last column, define $f(u_{in})=(n-1)\,n+i$ $(1 \leq i \leq n,\ i \neq n-2, i\neq n)$

$f(u_{nn}) = n^2-2$ and $f(u_{(n-2)n}) = n^2$.

We need to verity that

$\gcd[\, f(u_{nn}), f(u_{n(n-1)})] = \gcd[f(u_{(n-2)n}), f(u_{(n-2)(n-1)})]=1$

$\gcd[f(u_{(n-2)n}), f(u_{(n-3),n})] = 1$

That is to verify that (i) $\gcd(n^2-2, n^2-n) = 1$

(ii) $\gcd(n^2, n^2-n-2) = 1$

(iii) $\gcd(n^2, n^2-3) = 1$.

(i) d/n^2-2, $d/n^2-n \Rightarrow d/n-2 \Rightarrow d/n(n-2) \Rightarrow d/n^2-2n$.
But $d/n^2-n \Rightarrow d/n$ so that $d=n$ or 1.

If $d=n$, then $n/(n-2)$ which is not possible.
Thus $d = 1$.

(ii) Let d/n^2 and d/n^2-n-2.
Now d/n^2 and n is prime $\Rightarrow d = 1$ or n or n^2.

If $d=n$, $n/n^2-n-2 \Rightarrow n/2$ which is not possible.

If $d=n^2$, $n^2/n^2-n-2 \Rightarrow n^2/n+2$ which happens only when $n = 0$ or1 or 2.

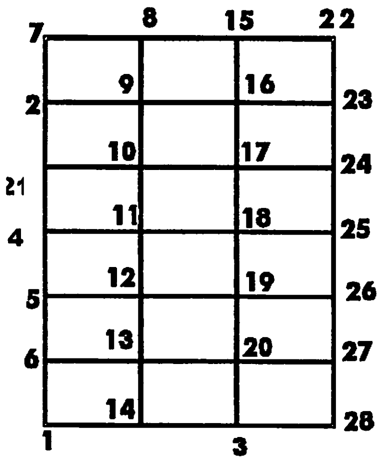
Thus $d=1$.

(iii) d/n^2, $d/n^2-3 \Rightarrow d/3$.

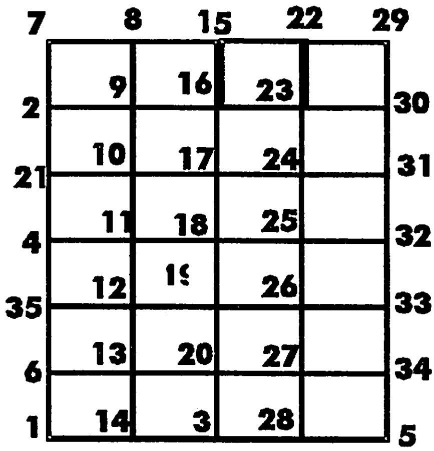
If $d=3$, $3/n^2 \Rightarrow 3/n$, an impossibility since $n \neq 3$.

Hence the proof.

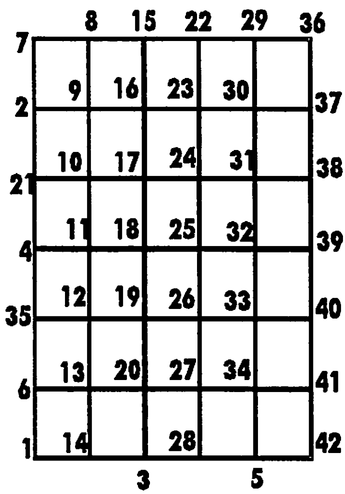
Example 2.2: For n=7, the prime labeling for $P_m \times P_n$ (as in theorem 3.1) are:



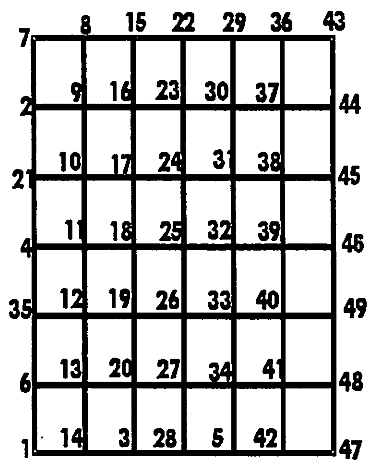
m = 4



m = 5



m=6



m= n=7

Conjecture: The planar grid P_m x P_n ($m \leq n$) is prime.

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References:

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2. F. Harary (1969): 'Graph Theory' – Addison Wesley.
3. V. Vilfred, S. Somasundaram and T. Nicholas (2002): 'Classes of prime labelled graphs' – International Journal of Management and Systems – Vol 18 No2 – P 217 – 226.