Counting chains and antichains in the complete binary tree

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Abstract

Let $\mathbf{T_n}$ be the complete binary tree of height n considered as the Hasse-diagram of a poset with its root $\mathbf{1_n}$ as the maximum element. For a tree or forest T, we count the embeddings of T into $\mathbf{T_n}$ as posets by the functions $A(n;T) = |\{S \subseteq \mathbf{T_n} : \mathbf{1_n} \in S,\ S \cong T\}|$, and $B(n;T) = |\{S \subseteq \mathbf{T_n} : \mathbf{1_n} \notin S,\ S \cong T\}|$. Here we summarize what we know about the ratio $A(n;T)/B(n;T)$, in case of T being a chain or an antichain.

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1 Introduction

A *forest* is a poset $F = (X, <)$ such that every point of F has at most one immediate successor. A *tree* T is a forest with a unique maximal element called the root of T. Minimal elements of a forest are referred to as its *leaves*. In this note we call a tree *binary* if every point has at most two immediate predecessors. A binary tree is *complete* if every point different from a leaf has exactly two immediate predecessors, and all maximal chains have the same height. We use the symbol $\cong$ for the isomorphism between posets.

Let $\mathbf{T_n}$ be the complete binary tree of height n, and let $\mathbf{1_n}$ be its root. For any T, define $A(n;T) = |\{S \subseteq \mathbf{T_n} : S \cong T,\ \mathbf{1_n} \in S\}|$, and $B(n;T) = |\{S \subseteq \mathbf{T_n} : S \cong T,\ \mathbf{1_n} \notin S\}|$. These counting functions were introduced in [2] and [3] when investigating a partial order analogue of the celebrated secretary problem. In [2] we proved that $A(n;T_1)/B(n;T_1) \leq A(n;T_2)/B(n;T_2)$ holds for binary trees T_1, T_2 such that T_2 contains a subposet isomorphic to T_1. We also conjectured that the assumption that T_1 and T_2 are binary can be removed and we proved in [3] that the asymptotic version of this conjecture is true.

In section 4 we shall verify the conjecture in the simplest non-binary case where T_1 and T_2 are stars rooted at their center (Proposition 4.2). To handle the ratios A/B for k–stars, we actually count embeddings of k–antichains into $\mathbf{T_n}$. This is done in Section 3 together with the analogous problem for k–chain embeddings. In our discussions we use ordinary power series generating functions and the concept of logarithmic concavity of sequences. Related tools and the proof of a technical lemma on log-concavity (Proposition 2.1) are presented in Section 2.

2 Logarithmic concavity

A real sequence $a = \{a_n\}_0^\infty$ is *logarithmically concave*, or *log-concave* for short, if $a_i^2 \geq a_{i-1}a_{i+1}$, for all $i > 1$. A log-concave sequence is *strictly log-concave,* if $a_i^2 > a_{i-1}a_{i+1}$, for every i such that $a_i \neq 0$. Sequences of our interest enumerate combinatorial objects, thus their terms are nonnegative. Furthermore, these *combinatorial sequences* have no internal zeros, i.e., each term is positive between any two positive terms. We shall frequently refer to the following more general inequality $a_i a_j > a_{i-k}a_{j+k}$, for every $k \geq 0$ and $i \leq j$, that is actually equivalent to the definition of log-concavity in the case of our combinatorial sequences. A formal series generating function (or a polynomial) is log-concave or strictly log-concave if the sequence of its coefficients has this property. If the definition of a sequence is not otherwise restricted, we can include leading or trailing zero terms to it. For example, a linear polynomial with positive coefficients is always strictly log-concave.

A fundamental operation on sequences maintaining log-concavity is convolution. We state this well known result in terms of generating functions (as the convolution of two sequences corresponds to the multiplication of their ordinary power series generating functions). For a standard proof using the Binet–Cauchy formula for convolution matrices see Karlin [1]. The same proof also shows that the convolution of a strictly log-concave combinatorial sequence and a nonzero log-concave sequence is strictly log-concave.

Convolution Lemma. *If $F(z)$ and $G(z)$ are log-concave formal power series with nonnegative coefficients, then $F(z) \cdot G(z)$ is log-concave as well. Moreover, if $F(z)$ is strictly log-concave, then $F(z) \cdot G(z)$ is also strictly log-concave.*

The values of a combinatorial function $f(n,k)$, $n, k \geq 0$, can be considered as entries of an infinite matrix we call here an *array.* The ath row of an array $f(n,k)$ is the 'horizontal' sequence $\{f(a,k)\}_{k=0}^\infty$, and the bth column is the 'vertical' sequence $\{f(n,b)\}_{n=0}^\infty$. We always assume that the entries of an array are nonnegative and any horizontal or vertical sequence contains no internal zeros. Given a recurrence relation defining $f(n,k)$, for $n, k \geq 0$, we are interested in sufficient "boundary conditions" implying the horizontal (or vertical) logarithmic concavity of the array $f(n,k)$. The proposition below investigates an array derived as an extension of the function in Section 3 counting k–antichains in the complete binary tree poset $\mathbf{T_n}$.

Proposition 2.1 *Let $f(n,k)$ be defined by the recurrence relation*

$$f(n,0) = 1, f(n,1) = t_n \quad for \quad n \geq 0, \qquad f(0,k) = 0 \quad for \quad k \geq 2,$$

$$f(n,k) = \sum_{i=0}^{k} f(n-1,i) \cdot f(n-1,k-i) \qquad for \quad n \geq 1,\ k \geq 2.$$

If $\{t_n\}_0^\infty$ is a log-concave sequence with $t_0 = 0$, $t_1 > 0$ and $t_n \geq 2t_{n-1}$ for all $n \geq 2$, then the array $f(n,k)$ is horizontally strictly log-concave.

Proof. For fixed $n \geq 0$, let $G_n(y) = \sum_{k\geq 0} f(n,k)y^k$ be the ordinary power series generating function of $f(n,k)$, $k = 0, 1, 2, \ldots$. Using the recurrence, we get $G_0(y) = 1$, and $G_n(y) = [G_{n-1}(y)]^2 + (t_n - 2t_{n-1})y$, for $n \geq 1$. Notice that G_n is strictly log-concave for $n = 0, 1$. Furthermore, if G_{n-1} is strictly log-concave, then $[G_{n-1}]^2$ is also strictly log-concave, by the Convolution Lemma. Because $[G_{n-1}(y)]^2$ and $G_n(y) = [G_{n-1}(y)]^2 + (t_n - 2t_{n-1})y$ differ only by a linear term with nonnegative coefficient, $f(n,k)^2 > f(n,k-1)f(n,k+1)$ holds true for every $k \neq 2$. We shall verify the missing inequality:

$$(*) \qquad\qquad t_n \cdot f(n,3) < f(n,2)^2 .$$

Because $f(2,3) = 0$ and $f(2,2) = t_1^2 > 0$, (*) is true for $n = 2$. Assume that $n \geq 3$. Based on the recurrences

$$\begin{aligned} f(n,2) &= 2f(n-1,2) + t_{n-1}^2 , \quad \text{and} \\ f(n,3) &= 2f(n-1,3) + 2t_{n-1}f(n-1,2) , \end{aligned}$$

we obtain, by iteration, the following forms:

$$f(n,2) = \sum_{j=1}^{n-1} 2^{j-1} t_{n-j}^2 ,$$

$$f(n,3) = \sum_{i=1}^{n-2} t_{n-i} \sum_{j=i}^{n-2} 2^j t_{(n-1)-j}^2 .$$

The left hand side of (*) becomes

$$\begin{aligned} t_n f(n,3) &= t_n \sum_{i=1}^{n-2} t_{n-i} \sum_{j=i}^{n-2} 2^j t_{(n-1)-j}^2 \\ &= \left(\sum_{j=1}^{n-2} t_n t_{n-1} 2^j t_{(n-1)-j}^2 + \sum_{j=2}^{n-2} t_n t_{n-2} 2^j t_{(n-1)-j}^2 \right) + \\ \cdots &+ \left(\sum_{j=2k-1}^{n-2} t_n t_{n-(2k-1)} 2^j t_{(n-1)-j}^2 + \sum_{j=2k}^{n-2} t_n t_{n-2k} 2^j t_{(n-1)-j}^2 \right) + \cdots . \end{aligned}$$

We compare the sums above in parentheses with the corresponding subsums on the right hand side of (*) in the expansion

$$
\begin{aligned}
[f(n,2)]^2 &= \left(\sum_{j=1}^{n-1} 2^{j-1} t_{n-j}^2\right)^2 \\
&= \left(t_{n-1}^4 + 2t_{n-1}^2 \sum_{j=2}^{n-1} 2^{j-1} t_{n-j}^2\right) + \\
\dots &+ \left(2^{2k-2} t_{n-k}^4 + 2^k t_{n-k}^2 \sum_{j=k+1}^{n-1} 2^{j-1} t_{n-j}^2\right) + \dots \quad .
\end{aligned}
$$

The log-concavity of $\{t_n\}_0^\infty$ implies $t_a t_b \le t_{a-c} t_{b+c}$, for all positive integers a, b, c with $a > b$ and $b+c < a$. From this and from the condition $t_n \ge 2t_{n-1}$, we obtain

$$
\sum_{j=1}^{n-2} t_n t_{n-1} 2^j t_{(n-1)-j}^2 \le \sum_{j=1}^{n-2} 2^{j-1} t_{n-1}^2 t_{n-j}^2 = t_{n-1}^4 + t_{n-1}^2 \sum_{j=2}^{n-2} 2^{j-1} t_{n-j}^2 ,
$$

$$
\sum_{j=2}^{n-2} t_n t_{n-2} 2^j t_{(n-1)-j}^2 \le t_{n-1}^2 \sum_{j=2}^{n-2} 2^j t_{(n-1)-j}^2 < t_{n-1}^2 \sum_{j=2}^{n-1} 2^{j-1} t_{n-j}^2 ,
$$

where the last inequality follows since $n \ge 3$ is assumed. Therefore

$$
\sum_{j=1}^{n-2} t_n t_{n-1} 2^j t_{(n-1)-j}^2 + \sum_{j=2}^{n-2} t_n t_{n-2} 2^j t_{(n-1)-j}^2 < t_{n-1}^4 + 2t_{n-1}^2 \sum_{j=2}^{n-1} 2^{j-1} t_{n-j}^2 .
$$

Similarly, for every $1 \le k \le (n-2)/2$, we obtain

$$
\begin{aligned}
&\sum_{j=2k-1}^{n-2} t_n t_{n-(2k-1)} 2^j t_{(n-1)-j}^2 + \sum_{j=2k}^{n-2} t_n t_{n-2k} 2^j t_{(n-1)-j}^2 \\
&< t_{n-k}^4 + t_{n-k}^2 \sum_{j=2k}^{n-2} 2^{j-1} t_{n-j}^2 + t_{n-k}^2 \sum_{j=2k}^{n-1} 2^{j-1} t_{n-j}^2 \\
&< t_{n-k}^4 + 2t_{n-k}^2 \sum_{j=2k}^{n-1} 2^{j-1} t_{n-j}^2 .
\end{aligned}
$$

This bound is less than the sum of all terms in the expansion of $[f(n,2)]^2$ containing the factor $t_{n-k} t_j$ with $j \le n-k-1$. This settles (*), provided n is even. Assuming that $n \ge 3$ is odd, we have $t_n t_2 2^{n-2} t_1^2 \le t_{(n+1)/2}^2 2^{n-3} t_2^2$ for the remaining term unpaired in $t_n f(n,3)$, thus it is bounded by an unused term of the expansion of $[f(n,2)]^2$. This proves (*) and concludes the proof of the proposition. □

Given sequences $a = \{a_n\}_0^\infty$ and $b = \{b_n\}_0^\infty$, we will say that a ratio inequality $a_i/b_i \le a_j/b_j$ holds true if $a_i b_j \le a_j b_i$, for all $i, j \ge 0$.

Using ratio inequality notation, a sequence is log-concave if and only if $a_i/a_{i-1} \geq a_{i+1}/a_i$. We say that a/b is nonincreasing if the ratio inequality $a_i/b_i \geq a_{i+1}/b_{i+1}$ holds true.

3 Antichains and chains

The array of antichains. For $n, k \geq 0$, let $a(n,k)$ be the number of all k–antichains in the complete binary tree poset $\mathbf{T_n}$. Then

$$\begin{array}{lll} a(n,0) = 1 & \text{for} & n \geq 0, \\ a(n,1) = 2^n - 1 & \text{for} & n \geq 0, \\ a(0,k) = 0 & \text{for} & k \geq 2, \end{array}$$

and for every $n \geq 1$, $k \geq 2$, we have the following recurrence

$$a(n,k) = \sum_{i=0}^{k} a(n-1,i) \cdot a(n-1,k-i).$$

Consider n as a free variable, and let $G_n(y) = \sum_{k \geq 0} a(n,k)y^k$ be the ordinary power series generating function of $a(n,k)$. From the recurrence definition we obtain that the sequence $\{G_n(y)\}_0^\infty$ satisfies $G_0(y) = 1$ and $G_n(y) = [G_{n-1}(y)]^2 + y$ for $n \geq 1$.

In Table 1 we computed initial entries of the array $a(n,k)$ (using the software MATHEMATICA^{TM} Kernel version 2.2 for Windows). Note that $a(n,k) = 0$, for $k > 2^{n-1}$.

a(n, k)	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	3	1						
3	1	7	11	6	1				
4	1	15	71	166	207	146	58	12	1
5	1	31	367	2462	10435	30074	61446	91220	99919
6	1	63	1695	27678	308203	2154226	15708214	77448348	307930443
7	1	127	7359	268926	6976859	137690450	2159088758	27718243068	297791781859

TABLE 1. Fragment of the array $a(n,k)$ of antichains

An immediate corollary of Proposition 2.1 is that the array $a(n,k)$ is horizontally strictly log-concave.

Proposition 3.1 *The sequence $\{a(n,k)\}_{k=0}^{\infty}$ is horizontally strictly log-concave for any $n \geq 0$.* □

Although we were unable to prove the vertical log-concavity of $a(n,k)$, we believe that this is true even for the general array in Proposition 2.1.

Problem 1 *Let $f(n,k)$ be defined by the recurrence relation*

$$f(n,0)=1, f(n,1)=t_n \quad \textit{for} \quad n\geq 0, \qquad f(0,k)=0 \quad \textit{for} \quad k\geq 2,$$

$$f(n,k)=\sum_{i=0}^{k} f(n-1,i)\cdot f(n-1,k-i) \qquad \textit{for} \quad n\geq 1,\ k\geq 2.$$

We know that $\{t_n\}_0^\infty$ is a log-concave sequence with $t_0=0$, $t_1>0$ and $t_n\geq 2t_{n-1}$ for all $n\geq 2$. Is it true then that the sequence $\{f(n,k)\}_{n=0}^\infty$ is strictly log-concave for every fixed k?

As far as the asymptotic behavior of the sequence $\{a(n,k)\}_{n=0}^\infty$ is concerned, for k fixed, it is enough to observe that asymptotically almost all k-element sets of $\mathbf{T_n}$ are antichains. Indeed, among all $\binom{2^n-1}{k}$ possible k-element sets of $\mathbf{T_n}$ there are not more than $(2^n-1)n\binom{2^n-3}{k-2}$ containing some pair of related points. Thus we obtain easily that, for any fixed $k\geq 1$, $\lim_{n\to\infty}[a(n,k)/2^{nk}]=1/k!$.

The array of chains. Most of the results here concerning chains (totally ordered subsets) of complete binary trees are proved in [4] using different techniques. Here we shall use generating functions in order to derive log-concavity results parallel to the case of antichains.

For $n,k\geq 1$, let $p_0(n,k)$ be the number of all k–chains of the complete binary tree poset $\mathbf{T_n}$ containing its maximal element $\mathbf{1_n}$. Then $p_0(n,1)=1$ for $n\geq 1$, and $p_0(n,k)=0$ for $n<k$. Furthermore, for every $n,k\geq 2$, we have the recurrence

$$p_0(n,k)=2p_0(n-1,k-1)+2p_0(n-1,k).$$

The initial values are obvious. The first term $2p_0(n-1,k-1)$ of the recurrence counts all k–chains of $\mathbf{T_n}$ that include one of the two immediate predecessors of $\mathbf{1_n}$, and the second term $2p_0(n-1,k)$ counts the sets in $\mathbf{T_n}$ that contain neither immediate predecessor of $\mathbf{1_n}$. Indeed, an injection of the k–chains of $\mathbf{T_{n-1}}$ rooted at an immediate predecessor of $\mathbf{1_n}$ into the k–chains of $\mathbf{T_n}$ can be obtained by replacing the maximal element of each k–chain of $\mathbf{T_{n-1}}$ with $\mathbf{1_n}$.

$p_0(n,k)$	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	2								
3	1	6	4							
4	1	14	20	8						
5	1	30	68	56	16					
6	1	62	196	248	144	32				
7	1	126	516	888	784	352	64			
8	1	254	1284	2808	3344	2272	832	128		
9	1	510	3076	8184	12304	11232	6208	1920	256	
10	1	1022	7172	22520	40976	47072	34880	16256	4352	512
11	1	2046	16388	59384	126992	176096	163904	102272	41216	9728
12	1	4094	36868	151544	372752	606176	680000	532352	286976	101888
13	1	8190	81924	376824	1048592	1957856	2572352	2724704	1638656	777728

TABLE 2. Fragment of the array $p_0(n,k)$ of chains

Consider n as a free variable, and let $G_n(y) = \sum_{k\geq 1} p_0(n,k)y^k$ be the ordinary power series generating function of $p_0(n,k)$. From the recurrence, we obtain that the sequence $\{G_n(y)\}_0^\infty$ satisfies $G_1(y) = y$, and $G_n(y) = (2+2y)G_{n-1}(y) - y$ for $n \geq 2$. We computed some entries of the array $p_0(n,k)$ in Table 2 using $MATHEMATICA^{TM}$.

For $k \geq 1$, define the formal power series generating function $F_k(x) = \sum_{n\geq 1} p_0(n,k)x^n$. From the recurrence, for $p_0(n,k)$ we obtain easily that

$$F_1 = \frac{x}{1-x}, \qquad \text{and} \qquad F_k(x) = F_{k-1}(x)F_1(2x) \quad \text{for } k \geq 2,$$

and thus

$$F_k(x) = 2^{k-1}x^k \frac{1}{(1-x)(1-2x)^{k-1}}.$$

To compute the coefficient $p_0(n,k)$, divide both sides of the identity

$$1 = [-(1-2x)]^{k-1} + (2-2x)[1-(1-2x)+(1-2x)^2 - \cdots + (-1)^{k-2}(1-2x)^{k-2}]$$

by $(1-x)(1-2x)^{k-1}$. Then we have $F_k(x) =$

$$2^{k-1}x^k \left[\frac{(-1)^{k-1}}{1-x} + \frac{2}{(1-2x)^{k-1}} - \frac{2}{(1-2x)^{k-2}} + \cdots + (-1)^{k-2}\frac{2}{1-2x}\right].$$

Using the power series $\frac{1}{(1-2x)^t} = \sum_{i\geq 0} 2^i \binom{t+i-1}{i} x^i$, we obtain $p_0(n,k) =$

$$(-2)^{k-1} + 2^n \left[\binom{n-2}{k-2} - \binom{n-3}{k-3} + \binom{n-4}{k-4} - \cdots + (-1)^k \binom{n-k}{0}\right].$$

Using the identity $\binom{n-2}{k-2} - \binom{n-3}{k-3} = \binom{n-3}{k-2}$, the formula yields the approximation $p_0(n,k) \approx 2^n n^{k-2}/(k-2)!$ for any fixed $k \geq 2$. Next we discuss the log-concavity of the array $p_0(n,k)$.

Proposition 3.2 *The array $p_0(n,k)$ is horizontally strictly log-concave. Also every column different from the first one is strictly log-concave.*

Proof. (Horizontal log-concavity.) For $n = 1$, the result is trivial, so we assume $n \geq 2$. If $G_{n-1}(y)$ is strictly log-concave, then the product $(2 + 2y)G_{n-1}(y)$ is strictly log-concave, by the Convolution Lemma and because $(2 + 2y)$ is trivially strictly log-concave. In the recurrence $G_n(y) = (2 + 2y)G_{n-1}(y) - y$, the very first nonzero coefficient is to be decreased by 1, therefore the log-concave property holds true for $G_n(y)$ as well.

(Vertical log-concavity.) Observe that $p_0(n,2) = 2^n - 2$ is strictly log-concave. We have seen before that the generating function $F_k(x) = \sum_{n\geq 1} p_0(n,k)x^n$ satisfies the recurrence relation $F_1 = \frac{x}{1-x}$, and $F_k(x) = F_{k-1}(x)F_1(2x)$ for $k \geq 2$. Because $F_1(x)$ and $F_1(2x)$ are trivially log-concave, and $F_2(x)$ is strictly log-concave, the claim follows by the Convolution Lemma. □

It is straightforward to verify that $p_0(n+1,k+1) = 2p(n,k)$, where $p(n,k)$ is defined as the number of all k-chains in $\mathbf{T_n}$. Therefore, the log-concavity of the array $p(n,k)$ and the asymptotics $p(n,k) \approx 2^n n^{k-1}/(k-1)!$ follows readily from the corresponding result for $p_0(n,k)$.

4 Ratio inequalities

For $k \geq 0$, let S_k be the star with k leaves rooted at its center. Subposets of $\mathbf{T_n}$ isomorphic to S_k consist of a k-antichain with a point greater than each point of the antichain. Define

$$\begin{aligned} \mathcal{A}(S_k) &= \{S \subseteq \mathbf{T_n} : S \cong S_k, \mathbf{1_n} \in S\}, \\ \mathcal{B}(S_k) &= \{S \subseteq \mathbf{T_n} : S \cong S_k, \mathbf{1_n} \notin S\}, \end{aligned}$$

and set $A(n,k) = |\mathcal{A}(S_k)|$ and $B(n,k) = |\mathcal{B}(S_k)|$. Clearly, $A(n,0) = 1, A(n,1) = 2^n - 2 = a(n,1) - 1$ for $n \geq 1$, $A(1,k) = 0$ for $k \geq 1$, and $A(n,k) = a(n,k)$ for $n,k \geq 2$. Furthermore, $B(n,0) = 2^n - 2$ for $n \geq 1$, and $B(n,k) = 2(B(n-1,k) + A(n-1,k))$ for every $n \geq 2$, $k \geq 0$. Note that $B(n,k) = 0$ for $k > 2^{n-2}$.

Proposition 4.1 *For every $n \geq 3$ and $0 \leq k \leq 2^{n-2}$,*

$$A(n,k)/B(n,k) < A(n,k+1)/B(n,k+1).$$

Proof. The proof is by induction on n. For $n = 3$ we have

$$\frac{A(3,0)}{B(3,0)} = \frac{1}{6} < \frac{A(3,1)}{B(3,1)} = \frac{6}{4} < \frac{A(3,2)}{B(3,2)} = \frac{6}{2} < \frac{A(3,3)}{B(3,3)} = \frac{4}{0}.$$

The claim is also true for $k = 2^{n-2}$ with any $n \geq 3$, because $B(n, 2^{n-2}+1) = 0$ and $B(n, 2^{n-2}) \neq 0$.

Let $n \geq 4$, $k < 2^{n-2}$, and assume that

$$A(n-1,k)B(n-1,k+1) \leq A(n-1,k+1)B(n-1,k). \tag{1}$$

Notice that in (1) we have equality for $2^{n-3} < k < 2^{n-2}$ in which case both sides are equal to 0. In the induction step, we will use the fact that the sequence $\{A(n-1,k)\}_{k=0}^{\infty}$ is strictly log-concave. To see this recall that, by Proposition 3.1, the generating function $G_{n-2}(y)$ of $\{a(n-2,k)\}_{k=0}^{\infty}$ is strictly log-concave. The generating function of $\{A(n-1,k)\}_{k=0}^{\infty}$ is equal to $G_{n-1}(y) - y = [G_{n-2}(y)]^2$, hence the strict log-concavity of the sequence follows by the Convolution Lemma. In particular, for every $i = 1, 2, \ldots, k$, we obtain that

$$A(n-1,k-i)A(n-1,k+1) < A(n-1,k+1-i)A(n-1,k). \tag{2}$$

Combining the induction hypothesis (1) with inequality (2), we get

$$A(n-1,k-i)B(n-1,k+1) \leq A(n-1,k+1-i)B(n-1,k) \tag{3}$$

for every i, $0 \leq i \leq k$, with equality for $2^{n-3} < k < 2^{n-2}$ in which case both sides of (3) are equal to 0.

Multiplying both sides of the inequalities (2) and (3) by $A(n-1,i)$ and summing up for $i = 0, 1, \ldots, k$, we obtain

$$\begin{aligned} &\left(\sum_{i=0}^{k} A(n-1,i)A(n-1,k-i)\right)(B(n-1,k+1) + A(n-1,k+1)) \\ &< \left(\sum_{i=0}^{k} A(n-1,i)A(n-1,k+1-i)\right)(B(n-1,k) + A(n-1,k)) \\ &\leq \left(\sum_{i=0}^{k+1} A(n-1,i)A(n-1,k+1-i)\right)(B(n-1,k) + A(n-1,k)) \, . \end{aligned}$$

Using the recurrence relation, the inequality above simplifies to

$$A(n,k)B(n,k+1)/2 < A(n,k+1)B(n,k)/2 \, ,$$

and the proposition follows. □

An immediate corollary of the ratio inequality above is

Proposition 4.2 *Given $n \geq 3$, the ratio $|\mathcal{A}(S_k)|/|\mathcal{B}(S_k)|$ is strictly increasing for $k = 0, 1, \ldots, 2^{n-2}$.* □

A similar result is also true for k-paths. For $k \geq 1$, let P_k be the path with k vertices rooted at an end vertex. Set

$$\begin{aligned}\mathcal{A}(P_k) &= \{S \subseteq \mathbf{T_n} : \mathbf{1_n} \in S, S \cong P_k\}, \\ \mathcal{B}(P_k) &= \{S \subseteq \mathbf{T_n} : \mathbf{1_n} \notin S, S \cong P_k\}.\end{aligned}$$

Proposition 4.3 ([4]) *Given $n \geq 3$, the ratio $|\mathcal{A}(P_k)|/|\mathcal{B}(P_k)|$ is strictly increasing for $k = 0, 1, \ldots, n$.*

Proof. Clearly, $|\mathcal{A}(P_k)| = p_0(n, k)$ and $|\mathcal{B}(P_k)| = p_0(n, k+1)$, and the ratio inequality $|\mathcal{A}(P_k)|/|\mathcal{B}(P_k)| < |\mathcal{A}(P_{k+1})|/|\mathcal{B}(P_{k+1})|$ is equivalent to $p_0(n, k)/p_0(n, k+1) < p_0(n, k+1)/p_0(n, k+2)$. Thus the claim follows from the horizontal strict log-concavity of $p_0(n, k)$ proved in Proposition 3.2. □

The section is concluded by proving two ratio inequalities for arbitrary trees. Denote by L_n the set of leaves of the tree $\mathbf{T_n}$. Observe that $\mathbf{T_{n+1}} \setminus L_{n+1}$ is isomorphic to $\mathbf{T_n}$, and $|L_{n+1}| = 2^n > 2^n - 1 = |\mathbf{T_n}|$. We will use this observation together with its consequence as stated in the following lemma.

Lemma 4.4 *There exists a proper injection $\varphi : (\mathbf{T_{n+1}} \setminus L_{n+1}) \mapsto L_{n+1}$ such that $\varphi(x) < x$ for every $x \in \mathbf{T_{n+1}} \setminus L_{n+1}$.*

Proof. Because $|L_k| > |\mathbf{T_k}|/2$ for every $k \geq 2$, we have

$$\left| \bigcup_{x \in X} \{y \in L_{n+1} : y < x\} \right| > |X|$$

holds for every $X \subseteq \mathbf{T_{n+1}} \setminus L_{n+1}$. Then, by König–Hall's theorem (see e.g. in [5]), the required proper injection φ exists. □

Proposition 4.5 *If T is a tree with ℓ leaves, then*

$$B(n+1; T)/B(n; T) > 2^{\ell}.$$

Proof. The leaves of any $S \subseteq \mathbf{T_n}$, $S \cong T$, form an ℓ-antichain of $\mathbf{T_n}$. For any given S, we apply the injection φ obtained in Lemma 4.4 to map any subset of these ℓ leaves into L_{n+1} (by keeping all the other points of S unchanged). Repeating this for every subset of leaves we obtain 2^{ℓ} distinct sets in $\mathbf{T_{n+1}}$ each isomorphic to T. Furthermore, because φ is proper, there are sets in $\mathbf{T_{n+1}}$ isomorphic to T and including at least one leaf from L_{n+1} not present in the image of φ. Thus $B(n+1; T) > B(n; T)2^{\ell}$ follows. □

The following proposition was a basic result in [4] where it was used in the poset version of the secretary problem. We state it here with a new and shorter proof.

Proposition 4.6 ([4]). *If T is a tree different from a chain and $A(n;T) > 0$, then $B(n;T) < A(n;T)$.*

Proof. Obviously, $B(n+1;T) = 2(A(n;T) + B(n;T))$, and thus

$$A(n;T)/B(n;T) = B(n+1;T)/2B(n;T) - 1 \tag{4}$$

provided $B(n;T) \neq 0$. Because T has $k \geq 2$ leaves different from its root, inequality (4) and Proposition 4.5 imply

$$A(n;T)/B(n;T) = B(n+1;T)/2B(n;T) - 1 > 2^{k-1} - 1 \geq 1\,.$$

□

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