A PARTIAL KITE SYSTEM OF ORDER n CAN BE EMBEDDED IN A KITE SYSTEM OF ORDER $8n + 9$

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Abstract

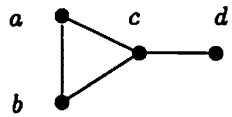
In this paper, it is shown that a partial edge-disjoint decomposition of K_n into kites (that is, into copies of K_3 with a pendant edge attached) can be embedded in a complete edge-disjoint decomposition of K_{4t+9} into kites for all even $t \geq 2n$. The proof requires first proving another interesting result, a generalization of an embeddding result on symmetric latin squares by L. D. Andersen, following a result by A. Cruse.

1 Introduction

A (partial) G-design (V, B) of order n is a partition B of (a subset of) the edges of K_n on the vertex set V into sets, each of which induces a graph isomorphic to G. A partial G-design (V, B) is said to be *embedded* in a G-design (W, C) if $V \subseteq W$ and $B \subseteq C$. There has been considerable interest over the years in the embedding problem that asks for the smallest possible order for which all (partial) G-designs (V, B) of order n can be embedded. The particular case where $G = K_3$ has been extensively studied. J. Doyen and R. M. Wilson proved the best possible result in the restricted case where the given K_3-design is in fact complete (that is, all edges of K_n occur in some copy of G). It is one of the outstanding problems in design theory to prove that every *partial* K_3-design of order n can be embedded in a complete K_3-design of order t, for all $t \geq 2n+1$ where $t \equiv 1$ or 3 (mod 6). Recently D. Bryant [3] proved the best result to date, showing such an embedding is possible if $t \geq 3n-2$ where $t \equiv 1$ or 3 (mod 6). Embedding results have also been obtained in other cases, such as where G is a cycle [4, 8, 9, 12], and when $G = K_m \setminus K_{m-2}$ [11].

A *kite* is a triangle with a tail consisting of a single edge. In this paper we address the embedding problem for kite-designs, more usually known as *kite systems*. It is known that kite systems of order n exist precisely when $n \equiv 0$ or 1 (mod 8) [2]; one can even find such designs that are 2-colorable [6]. Necessary and sufficient conditions have also been found for the embedding of a complete kite systems of order n in a complete kite system of order t [7].

In what follows we will denote the graph



by $(a, b, c) - d$ or $(b, a, c) - d$.

Given a partial kite system (X, B) of order n a natural question to ask is whether or not it can be completed; i.e., can $E(K_n) \setminus E(B)$ be partitioned into copies of kites. Clearly, this cannot be done in general, since no partial kite system of order $n \equiv 2, 3, 4, 5, 6,$ or 7 (mod 8) can be completed.

Example 1.1 (X, B) is a partial kite system of order 6, where $X = \{1, ..., 6\}$ and $B = \{(2, 3, 5) - 1, (3, 4, 6) - 1\}$.

Given that a partial system cannot necessarily be completed, the next question to ask is whether or not a partial kite system can be *embedded.* The partial kite system (X, B) is said to be *embedded* in the (partial) kite system (S, K) provided $X \subseteq S$ and $B \subseteq K$. The partial kite system of order 6 in Example 1.1 is embedded in the kite system of order 8 in Example 1.2.

Example 1.2 (S, K) is a kite system of order 8, where $S = \{1, ..., 8\}$ and $K = \{(2, 3, 5) - 1, (3, 4, 6) - 1, (4, 5, 7) - 1, (5, 6, 8) - 1, (6, 7, 2) - 1, (7, 8, 3) - 1, (8, 2, 4) - 1\}$.

Naturally, if an embedding is possible, we would like the size of the containing kite system to be as small as possible. The purpose of this paper is to show that a partial kite system of order n can be embedded in a kite system of order at most $8n+9$. To do so, we use a proof technique that requires a result that is of interest in its own right. Theorem 2.2 is a generalization of a result of L. D. Andersen [1] who generalized A. Cruse's classic result [5] that settled the embedding problem for incomplete idempotent symmetric latin squares. It is also a companion to similar generalizations of classic embedding theorems that appear in [10].

2 Preliminaries

A *partial* groupoid $(G, \circ)$ is said to be idempotent provided $x^2 = x \circ x = x$ for all $x \in G$. The quantification "partial" is only for products $x \circ y$, where $x \neq y$. A *partial* groupoid $(G, \circ)$ on order $2n$ with $G = \{1, 2, 3, \ldots, 2n\}$ is said to be half-idempotent provided $x^2 = (x+n)^2 = x$, for all $x \in \{1, 2, 3, \ldots, n\}$. Once again "partial" quantifies only products of the form $x \circ y$, where $x \neq y$. A partial groupoid is said to be *complete* if all products have been defined.

A partial groupoid $(P, \circ)$ is called a partial *embedding groupoid* provided

1. $(P, \circ)$ is idempotent,

2. if $x \neq y$ either both $x \circ y$ and $y \circ x$ are defined or neither is defined,

3. $(P, \circ)$ is row latin, and

4. each $x \in P$ occurs as a product an odd number of times.

We remark that in the case where both $x \circ y$ and $y \circ x$ are defined it is not necessary that $x \circ y = y \circ x$.

In what follows, we will sometimes abbreviate the groupoid $(Q, \circ)$ to simply Q. The context will make this approach clear.

Example 2.1 A partial embedding groupoid of order 7.

∘	1	2	3	4	5	6	7
1	1				5	6	
2		2	3		5		
3		2	3	4	5	6	
4			3	4		6	
5	1	2	3		5		7
6	1		3	4		6	7
7					5	6	7

Theorem 2.2 *A partial embedding groupoid $\mathcal{P}$ of order n can be embedded in a half-idempotent groupoid $\mathcal{R}$ of order $2n$ which satisfies*

(1) *$\mathcal{R}$ is row latin, and*

(2) *if all off-diagonal products in $\mathcal{P}$ are deleted from $\mathcal{R}$ then the resulting partial groupoid is in fact a partial quasigroup that is both commutative and half-idempotent.*

Proof Let $(P, \circ)$ be a partial embedding groupoid of order n based on $1, 2, ..., n$ and let $(L, \otimes)$ be a commutative quasigroup of order n based on $n+1, n+2, ..., 2n$. For each i, j for which $i \circ j$ is undefined in $\mathcal{P}$, let $i \circ j = j \circ i = (n+i) \otimes (n+j)$. Call the resulting partial groupoid $(P', \circ)$. We now use a proof by induction to show that $\mathcal{P}'$ can be embedded in a half-idempotent groupoid of order $2n$ which satisfies conditions (1) and (2).

For any (partial) groupoid $\mathcal{Q}$, let $N_{\mathcal{Q}}(i)$ denote the number of times symbol i occurs in $\mathcal{Q}$. For $0 \leq z \leq n$, we will embed $\mathcal{P}'$ in a groupoid $\mathcal{P}_z$ of order $n+z$ in which:

(a) if all off-diagonal entries in the partial embedding groupoid $\mathcal{P}$ are deleted then the result is a partial commutative quasigroup of order $n+z$ in which cell $(n+i, n+i)$ contains the symbol i for $1 \leq i \leq z$,

(b) each row is latin,

(c) each symbol i satisfies $N_{\mathcal{P}_z}(i) \geq 2z$, and

(d) $N_{\mathcal{P}_z}(i)$ is odd if $z+1 \leq i \leq n$, and is even otherwise.

Clearly we can let $\mathcal{P}_z = \mathcal{P}'$ if $z = 0$. So suppose for some y satisfying $0 \leq y \leq n-1$ that $\mathcal{P}_z$ satisfying (a),(b),(c), and (d) exists for $0 \leq z \leq y$. We will now show that $\mathcal{P}_{y+1}$ exists. Let B be the bipartite graph with bipartition $\{\rho_1, \rho_2, ..., \rho_{n+y}, \rho_D\}$ and $\{s_1, s_2, ..., s_{2n}\}$, where $\{\rho_D, j\} \in E(B)$ for all $j \in \{y+1, y+2, ..., n\}$ and $\{\rho_i, s_j\} \in E(B)$ if and only if the symbol j is missing from row i of $\mathcal{P}_y$. Then by (b), since each row of $\mathcal{P}_y$ contains $n+y$ distinct symbols, $d_B(\rho_i) = 2n - (n+y) = n - y$. Clearly $d_B(\rho_D) = n - y$. Also, by (c) and (d),

$$d_B(j) = n + y - N_{\mathcal{P}_y}(j) \leq n + y - 2y = n - y$$

for $1 \leq j \leq y$ and $n+1 \leq j \leq 2n$, and

$$d_B(j) = n + y - N_{\mathcal{P}_y}(j) + 1 \leq n + y - (2y+1) + 1 = n - y$$

for $y+1 \leq j \leq n$.

Notice that for $y+1 \leq j \leq n$, $N_{\mathcal{P}_y}(j) \geq 2y+1$ since by (d) $N_{\mathcal{P}_y}(j)$ is odd and by (c) $N_{\mathcal{P}_y}(j) \geq 2y$. Now give B a proper $(n-y)$-edge-coloring with the colors $1, 2, ..., n-y$, with edge $\{\rho_D, y+1\}$ being colored 1. Form the groupoid $\mathcal{Q}$ by placing the symbol $y+1$ into the cell $(n+y+1, n+y+1)$, and placing the symbol j in both row i of column $n+y+1$ and in column i of row $n+y+1$ if and only if $\{\rho_i, j\}$ is colored by 1. We now show that $\mathcal{Q}$ satisfies conditions (a), (b), (c) and (d), where $z = y+1$.

Clearly (a) is satisfied. $\mathcal{Q}$ is latin in rows 1 to $n+y$ by the definition of the edges in B, and row $n+y+1$ is latin both because the edge-coloring of B is proper and because the edge $\{\rho_D, y+1\}$ being colored 1 means that symbol $y+1$ only occurs in the diagonal cell. So (b) is satisfied. To see that

(c) is satisfied, we consider two cases. If $N_{P_y}(j) = 2y$, then $d_B(j) = n - y$, so j occurs in the added row and column and so $N_Q(j) = 2y+2 = 2(y+1)$. Secondly, if $N_{P_y}(j) = 2y+1$, then again $d_B(j) = n - y$. In this case, if $j = y+1$ then j is placed in the added diagonal cell of Q, so $N_Q(j) = 2y+2$. Otherwise, $y+2 \leq j \leq n$ and so j occurs in the added row and column, so $N_Q(j) = 2y+3$. Therefore $N_Q(j) \geq 2(y+1)$ for all symbols j, so (c) is satisfied. Since all symbols other than $y+1$ are placed an even number of times in the added row and column, it is clear that (d) is satisfied. The result follows. □

3 The $8n+9$ embedding

Let (N, B) be a partial kite system of order n, where $N = \{1, 2, \ldots, n\}$, and let O be the set of vertices of odd degree. Define a binary operation "$\circ$" on $N' = N \cup \{n+1\}$ by

(1) $x \circ x = x$ for all $x \in N'$.

(2) If $x \neq y$, $x \circ y$ and $y \circ x$ are defined and $x \circ y = y$ and $y \circ x = x$ if and only if the edge $\{x, y\}$ belongs to a kite in B.

(3) $y \circ (n+1) = n+1$ and $(n+1) \circ y = y$ for all $y \in O$.

It is straightforward to see that $(N', \circ)$ is a partial embedding groupoid. (Example 2.1 is the partial embedding groupoid constructed from the partial kite system in Example 1.1.) By Theorem 2.2 we can embed $\mathcal{N}' = (N', \circ)$ into a half-idempotent groupoid $\mathcal{Q} = (Q, \circ)$ of order $2n+2$ which is (1) row latin and (2) deleting all off-diagonal products defined in the embedding groupoid results in a partial half-idempotent commutative quasigroup.

Set $S = \{\infty\} \cup (Q \times \{1, 2, 3, 4\})$. Define a set of kites K as follows:

(1) Let $(\{1, 2, 3, 4\}, \otimes)$ be the idempotent quasigroup below (any idempotent quasigroup will suffice).

⊗	1	2	3	4
1	1	4	2	3
2	3	2	4	1
3	4	1	3	2
4	2	3	1	4

For each $(a,b,c)-d \in B$ and for each $i,j \in \{1,2,3,4\}$, place $((a,j),(b,i\otimes j),(c,i))-(d,j)$ in K.

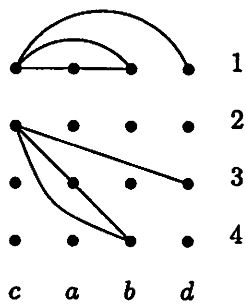


Figure 1 Type 1 kites.

(2) For each $\{a,b\} \subset \{1,2,...,2n+2\}$ that is neither in a kite in B nor satisfies $a \in O$ and $b = n'$, place in K the four kites $((b,1),(a \circ b,2),(a,1))-(b,3)$, $((b,2),(a \circ b,3),(a,2))-(b,4)$, $((b,3),(a \circ b,4),(a,3))-(b,1)$, and $((b,4),(a \circ b,1),(a,4))-(b,2)$.

(3) For each $a \in O$ place in K the 9 kites $((a,2),\infty,(n'+a,1))-(n'+a,3)$, $((a,3),\infty,(n'+a,2))-(n'+a,4)$, $((n'+a,3),(a,4),\infty)-(n'+a,4)$, $((n',1),(a,2),(a,1))-(n'+a,4)$, $((a,4),(n',1),(a,3))-(n',4)$, $((a,3),(n',3),(a,2))-(n',4)$, $((a,4),(n',3),(a,1))-(n',4)$, $((n',2),(a,3),(a,1))-\infty$, $((n',2),(a,2),(a,4))-(n',4)$, in K, where $n' = n+1$.

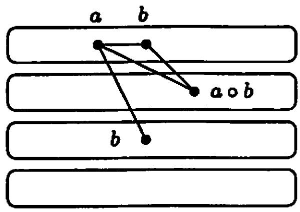


Figure 2 Type 2 kites.

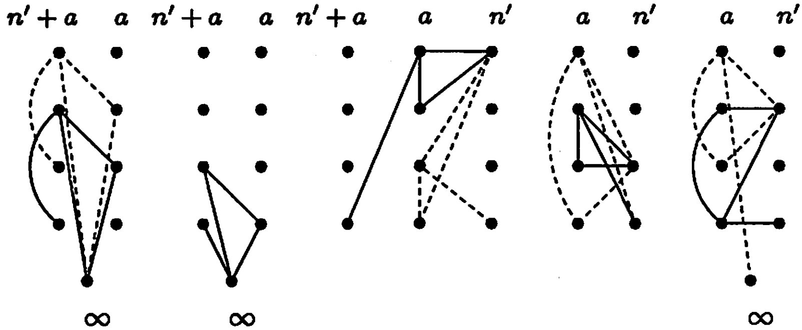


Figure 3 Type 3 kites.

(4) For each $a \in N$ with even degree and for $a = n'$, place in K the 5 kites $((a,2),\infty,(n'+a,1)) - (n'+a,3)$, $((a,3),\infty,(n'+a,2)) - (n'+a,4)$, $((n'+a,3),\infty,(a,4)) - (a,1)$, $((n'+a,4),\infty,(a,1),) - (a,2)$, and $((a,2),(a,4),(a,3)) - (a,1)$.

It is straightforward to show that each edge of K_{8n+9} with vertex set S belongs to a kite of type (1), (2), (3), or (4). Next, we will show that the number of kites in K is indeed $\binom{8n+9}{2}/4$.

Let E be the set of vertices of even degree in the partial kite system (N, B); so $|E| + |O| = n$.

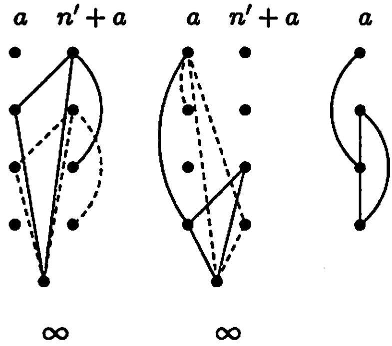


Figure 4   Type 4 kites.

Since $\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\sum_{v_i\in N} d(v_i)$ is the number of kites in (N, B), the number of type (1) kites is

$$16\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\sum_{v_i\in N} d(v_i) = 2\sum_{v_i\in N} d(v_i),$$

To count the number of type (2) kites, we first count the number of pairs $\{a, b\}$ that are used, then we can simply multiply this by 4 to get the desired number. To do this, we actually count the number of ordered pairs (a, b) and then halve this number. Counting the number of ordered pairs is a natural procedure, because each such ordered pair corresponds to a non-diagonal cell in Q that is not filled in $\mathcal{N}'$. For $1 \le i \le n$, row i of Q contains $2n + 2 - d(v_i) - 1$ such cells if $d(v_i)$ is even, and one more than this if $d(v_i)$ is odd (because cell $(i, n+1)$ is filled in $\mathcal{N}'$). Clearly row $n+1$ contains $2n + 2 - (|O| + 1)$ such cells, and for $n+2 \le i \le 2n+2$, row i of Q contains $2n + 2 - 1$ such cells. So altogether, the number of type 2 kites is:

$$4[\sum_{v_i\in E}(2n+1-d(v_i))+\sum_{v_i\in O}(2n-d(v_i))+2n+2-(|O|+1)+(n+1)(2n+1)]/2$$

$$= 2[(n+1)(2n+1) + 2n(n+1) + |E| + 1 - \sum_{v_i \in N} d(v_i) - |O|$$

$$= 2(n+1)(2n+1) + 4n(n+1) + 2|E| + 2 - 2|O| - 2 \sum_{v_i \in N} d(v_i).$$

The number of type (3) kites is clearly $9|O|$, and the number of type (4) kites is $5(|E|+1)$.

Adding the number of kites of types (1), (2), (3), and (4) gives a total of $\binom{8n+9}{2}/4$ kites as expected.

This proves that (S, K) is a kite system of order $8n+9$. □

We can now obtain the following result.

Theorem 3.1 *A partial kite system of order n can be embedded in a kite system of order at most* $8n+9$.

Proof Since the quasigroup $(\{1,2,3,4\}, \otimes)$ in (1) is idempotent, and since $(a,b,c)-d \in B$, K contains the 4 kites $((a,i),(b,i),(c,i))-(d,i)$, for all $i \in \{1,2,3,4\}$, and hence 4 disjoint copies of the partial kite system (N, B) occur in K. □

4 Concluding remarks

The $8n+9$ embedding given in this paper is probably not the best possible embedding. The problem of finding the best possible embedding remains open.

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