# HYPERHAMILTONICITY OF THE CARTESIAN PRODUCT OF TWO DIRECTED CYCLES

MICAH MILLER

ABSTRACT. Let  be the product of two directed cycles, let $\mathbf{Z}_c$ be a subgroup of $\mathbf{Z}_a$, and let $\mathbf{Z}_d$ be a subgroup of $\mathbf{Z}_b$. Also, let $A = \frac{a}{c}$ and $B = \frac{b}{d}$. We say that  is $(\mathbf{Z}_c \times \mathbf{Z}_d)$-hyperhamiltonian if there is a spanning connected subgraph of  that has degree $(2,2)$ at the vertices of $\mathbf{Z}_c \times \mathbf{Z}_d$ and degree $(1,1)$ everywhere else. We show that the graph  is $(\mathbf{Z}_c \times \mathbf{Z}_d)$-hyperhamiltonian if and only if there exist positive integers $m$ and $n$ such that $Am+Bn = AB+1$, $\gcd(m,n) = 1$ or 2, and when $\gcd(m,n) = 2$, then $\gcd(dm, cn) = 2$.

## 1. INTRODUCTION

Curran and Witte [3] proved using the theory of torus knots that the Cartesian product $\mathbb{Z}_a \times \mathbb{Z}_b$ of two directed cycles is hamiltonian if and only if there exists a pair of relatively prime positive integers $m$ and $n$ such that $am + bn = ab$. Gallian and Witte [4] defined a digraph  to be hyperhamiltonian if there is a spanning connected subgraph of  which passes through one vertex exactly twice and all others exactly once. They showed that the digraph  is hyperhamiltonian if and only if there exist positive integers $m$ and $n$ such that $am + bn = ab + 1$ and $\gcd(m,n) = 1$ or 2.

Note that passing through one vertex from $\mathbb{Z}_a \times \mathbb{Z}_b$ twice as in [4] is equivalent to passing through the vertices of the subgroup $\mathbb{Z}_1 \times \mathbb{Z}_1$ twice. In this paper, we define the graph  to be $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian if there is a spanning connected subgraph in which the vertices of $\mathbb{Z}_c \times \mathbb{Z}_d$ have two in-edges and two out-edges and all other vertices have one in-edge and one out-edge. Here, $\mathbb{Z}_c$ is a subgroup of $\mathbb{Z}_a$ and $\mathbb{Z}_d$is a subgroup of $\mathbb{Z}_b$. Hence, our result is a natural generalization of Gallian and Witte's result.

The methods used in this paper are similar to the ones used in [1]. That paper generalized results for hamiltonicity of vertex-deleted digraphs [5] to subgroup-deleted digraphs.

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## 2. Background

We recall some definitions and results that will be useful. We refer to [2] for the basic language of digraphs, but we remind the reader of two useful definitions. First, a vertex of a digraph has degree $(r, s)$ if it has $r$ in-edges and $s$ out-edges. Second, a digraph is connected if there is a directed path from any vertex to any other vertex.

**Definition 2.1.** Let $G$ be a digraph, and let $V$ be a set of vertices of $G$. Then $G$ is $V$-hyperhamiltonian if there is a connected spanning subgraph that has degree $(2,2)$ at the vertices of $V$ and degree $(1,1)$ at all other vertices. Such a subgraph is called a $V$-hyperhamiltonian circuit of $G$.

The idea is that a digraph $G$ is $V$-hyperhamiltonian if there exists a closed directed walk that passes through each vertex of $V$ exactly twice and passes through the other vertices exactly once. See Figure 1 for a picture of the digraph $\mathbb{Z}_9 \times \mathbb{Z}_4$ with a $(\mathbb{Z}_3 \times \mathbb{Z}_2)$-hyperhamiltonian circuit.
In this paper, we only consider the case when $G$ is the digraph  and $V$ consists of the vertices belonging to the subgroup $\mathbb{Z}_c \times \mathbb{Z}_d$.

**Definition 2.2.** Let $H$ be a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit on . Then a vertex $(x, y)$ **travels by** $(1,0)$ if $H$ contains the directed edge from $(x, y)$ to $(x+1, y)$. Similarly, a vertex $(x, y)$ **travels by** $(0,1)$ if $H$ contains the directed edge from $(x, y)$ to $(x, y+1)$.

Note that in a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit, a vertex $(x, y)$ that does not belong to $\mathbb{Z}_c \times \mathbb{Z}_d$ travels by either $(1,0)$ or by $(0,1)$. A vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$ travels by both $(1,0)$ and $(0,1)$.

**Definition 2.3.** Let $g = \gcd(a, b)$. For any integer $p$, let $\langle p \rangle$ be the subset of $\mathbb{Z}_a \times \mathbb{Z}_b$ consisting of pairs $(x, y)$ such that $x + y = p$ modulo $g$.

The subset $\langle 0 \rangle$ is the subgroup of $\mathbb{Z}_a \times \mathbb{Z}_b$ generated by $(1, -1)$, and $\langle p \rangle$ is the coset $(p, 0) + \langle 0 \rangle$. The subgroup $\langle 0 \rangle$ has index $g$, so there are exactly $g$ distinct cosets.

The following lemma shows why such cosets are useful.

**Lemma 2.4.** *Let $H$ be a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit of .*

(1) *If $(x+1, y-1)$ does not belong to $\mathbb{Z}_c \times \mathbb{Z}_d$ and $(x+1, y-1)$ travels by $(1,0)$, then $(x, y)$ also travels by $(1,0)$.*
(2) *If $(x+1, y)$ does not belong to $\mathbb{Z}_c \times \mathbb{Z}_d$ and $(x+1, y-1)$ travels by $(0,1)$, then $(x, y)$ also travels by $(0,1)$.*

*Proof.* In part (1), the vertex $(x+1, y)$ must have at least one in-edge. By assumption, this in-edge does not come from $(x+1, y-1)$, so it must come from $(x, y)$.

In part (2), the vertex $(x+1, y)$ can have at most one in-edge. By assumption, it has an in-edge from $(x+1, y-1)$, so it cannot have an in-edge from $(x, y)$. Thus $(x, y)$ does not travel by $(1, 0)$, so it must travel by $(0, 1)$. □

We recall the following result from [4].

**Theorem 2.5** (Gallian-Witte). *The digraph* $\mathbb{Z}_a \times \mathbb{Z}_b$ *is hyperhamiltonian if and only if there exist positive integers* $m$ *and* $n$ *such that* $am + bn = ab + 1$ *and* $\gcd(m, n) = 1$ *or* $2$.

In Theorem 2.5, the numbers $m$ and $n$ have useful geometric interpretations. If we embed $\mathbb{Z}_a \times \mathbb{Z}_b$ in the torus in the obvious way, then a hyperhamiltonian circuit consists of two embedded directed loops that meet at a single vertex, and the total knot class of these two loops is equal to $(m, n)$. See [6] for more details on knot classes.

Using intersection numbers, Gallian and Witte [4] showed that when $\gcd(m, n) = 1$, these two directed loops have knot classes $(m_1, n_1)$ and $(m_2, n_2)$, where $m_1 n_2 - m_2 n_1$ equals 1 or $-1$. When $\gcd(m, n) = 2$, the two directed loops both have knot class $(\frac{m}{2}, \frac{n}{2})$.

## 3. The Main Theorem

We now come to our main result.

**Theorem 3.1.** *Let* $A = \frac{a}{c}$ *and* $B = \frac{b}{d}$. *Then is* $(\mathbb{Z}_c \times \mathbb{Z}_d)$*-hyperhamiltonian if and only if there exist positive integers* $m$ *and* $n$ *such that*

(1) $Am + Bn = AB + 1$,
(2) $\gcd(m, n) = 1$ *or* $2$, *and*
(3) *when* $\gcd(m, n) = 2$, *then* $\gcd(dm, cn) = 2$.

When $A = a$, $B = b$, then $c = d = 1$. In this case, the conditions in Theorem 3.1 reduce to the conditions in Theorem 2.5.

**Example 3.2.** We give two examples illustrating the theorem. First, consider the digraph $\mathbb{Z}_9 \times \mathbb{Z}_4$, as shown in Figure 1. This digraph has a $(\mathbb{Z}_3 \times \mathbb{Z}_2)$-hyperhamiltonian circuit. The values $m = 1$ and $n = 2$ satisfy the three conditions of the theorem.

However, the digraph $\mathbb{Z}_{10} \times \mathbb{Z}_6$ does not have a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-hyperhamiltonian circuit. When $m = n = 2$, the first two conditions are satisfied, but the third condition is not.

In order to prove this theorem, we need the following facts about $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian graphs.

**Lemma 3.3.** *If is* $(\mathbb{Z}_c \times \mathbb{Z}_d)$*-hyperhamiltonian, then* $\gcd(A, B) = 1$ *and every coset contains at least one vertex of* $\mathbb{Z}_c \times \mathbb{Z}_d$.

*Proof.* Suppose for contradiction that $\langle -1 \rangle$ contains no vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$. Since $(0,0)$ belongs to $\mathbb{Z}_c \times \mathbb{Z}_d$, the vertex $(-1,0)$ travels by $(1,0)$. A repeated application of part (1) of Lemma 2.4 implies that every vertex of $\langle -1 \rangle$ also travels by $(1,0)$. This is a contradiction because the vertex $(0,-1)$ travels by $(0,1)$. This means that $\langle -1 \rangle$ contains at least one vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$. Let $(\alpha a, \beta B)$ be such a vertex, where $\alpha$ and $\beta$ are integers.

From the definition of a coset we find that $\alpha A + \beta B = -1 \bmod g$. There exists an integer $\gamma$ such that $\alpha A + \beta B = -1 + g\gamma$. Now $\gcd(A,B)$ divides $g$, $\alpha A$, and $\beta B$ so it must also divide $-1$. This shows that $\gcd(A,B) = 1$.

Now consider an arbitrary coset $\langle p \rangle$. Multiplying the equation of the previous paragraph by $-p$, we get $(-\alpha p)A + (-\beta p)B = p \bmod g$. From the definition of a coset, we know $((-\alpha p)A, (-\beta p)B)$ is in $\langle p \rangle$. □

**Corollary 3.4.** *If  has a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit, then it is unique.*

*Proof.* Let $(x,y)$ be an arbitrary vertex. Since every coset contains a vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$ by Lemma 3.3, $(x,y)$ can be written (uniquely) in the form $(x_0 - k, y_0 + k)$, where $(x_0, y_0)$ belongs to $\mathbb{Z}_c \times \mathbb{Z}_d$, $k$ is a non-negative integer, and $(x_0 - j, y_0 + j)$ does not belongto $\mathbb{Z}_c \times \mathbb{Z}_d$ for $0 < j \leq k$. By induction on $k$, we show that there is no choice in the directions in which $(x,y)$ must travel.

First, if $k = 0$, then $(x,y)$ belongs to $\mathbb{Z}_c \times \mathbb{Z}_d$. Thus, it must travel both by $(1,0)$ and by $(0,1)$. Now assume that $k > 0$, so $(x,y)$ travels either by $(1,0)$ or by $(0,1)$ but not by both. If $(x+1,y)$ belongs to $\mathbb{Z}_c \times \mathbb{Z}_d$, then $(x,y)$ must travel by $(1,0)$. Hence we may assume that $(x+1,y)$ does not belong to $\mathbb{Z}_c \times \mathbb{Z}_d$; this allows us to apply Lemma 2.4.

If $k = 1$, then part (2) of Lemma 2.4 tells us that $(x,y)$ must travel by $(0,1)$. Now suppose for sake of induction $k \geq 2$ and that we know the direction in which $(x_0 - (k-1), y_0 + (k-1))$ travels. Parts (1) and (2) of Lemma 2.4 tell us that $(x_0 - k, y_0 + k)$ must travel in the same direction as $(x_0 - (k-1), y_0 + (k-1))$.

□

The following corollary tells us that $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuits on $\mathbb{Z}_a \times \mathbb{Z}_b$ are suitably periodic.

**Corollary 3.5.** *Suppose  has a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit. Let $\alpha$ and $\beta$ be integers. Then $(x,y)$ and $(x + \alpha A, y + \beta B)$ travel in the same direction.*

*Proof.* Let $H$ be a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit of . Then $\phi(x,y) = (x + \alpha A, y + \beta B)$ is an automorphism of  that preserves $\mathbb{Z}_c \times \mathbb{Z}_d$. So $\phi(H)$ is a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit of  as well. But the $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit is unique (by Corollary 3.4), so $\phi(H) = H$. Thus, any vertex $(x,y)$ travels in the same direction as the vertex $\phi(x,y)$. □

Note that the proofs of Lemma 3.3, Corollary 3.4, and Corollary 3.5 are similar to the proofs found in [1].

**Lemma 3.6.** *Suppose that $H$ is a spanning subgraph of such that the vertices of $\mathbb{Z}_c \times \mathbb{Z}_d$ have degree $(2,2)$ in $H$ and all other vertices have degree $(1,1)$ in $H$ and such that every directed loop in $H$ contains a vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$. Then $H$ is connected if and only if for any two vertices $v$ and $w$ in $\mathbb{Z}_c \times \mathbb{Z}_d$, there is a directed path in $H$ from $v$ to $w$.*

*Proof.* One direction follows from the definition of connectedness. For the other direction, suppose that for any two vertices $v_1$ and $v_2$ in $\mathbb{Z}_c \times \mathbb{Z}_d$, there is a directed path in $H$ from $v_1$ to $v_2$. We need to show is that there is a directed path from $w_1$ to $w_2$ for any two vertices $w_1$ and $w_2$ of . Under our assumptions, there is a directed path from any vertex $w_1$ to some vertex $v_1$ in $\mathbb{Z}_c \times \mathbb{Z}_d$. Similarly, there is a directed path from some vertex $v_2$ in $\mathbb{Z}_c \times \mathbb{Z}_d$ to the vertex $w_2$. Finally, there is a directed path from $v_1$ to $v_2$ by assumption. Therefore, there is a path from $w_1$ to $w_2$.

□

Now we are ready to prove Theorem 3.1.

*Proof.* First suppose that $H$ is a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit of . We define a function $f$ from to $\mathbb{Z}_A \times \mathbb{Z}_B$ by $f(x,y) = (x \mod A, y \mod B)$. Define a subgraph $H'$ of $\mathbb{Z}_A \times \mathbb{Z}_B$ by requiring the vertex $(x,y)$ in $\mathbb{Z}_A \times \mathbb{Z}_B$ to travel in the same direction as the vertices $(x+\alpha A, y+\beta B)$ in where $\alpha$ and $\beta$ are integers. We can define a subgraph in this way because Corollary 3.4 states that all vertices of the form $(x+\alpha A, y+\beta B)$ travel in the same direction. Note that $H'$ is just $f(H)$.

Let $(x_1,y_1)$ and $(x_2,y_2)$ be any two vertices in . There is a directed path in $H$ from $(x_1,y_1)$ to $(x_2,y_2)$, so there is a directed path in $f(H)$ from $f(x_1,y_1)$ to $f(x_2,y_2)$. This means that $H'$ is connected, so $\mathbb{Z}_A \times \mathbb{Z}_B$ is $(\mathbb{Z}_1 \times \mathbb{Z}_1)$-hyperhamiltonian. By Theorem 2.5, conditions (1) and (2) are satisfied.

In order to show condition (3), assume that $\gcd(m,n) = 2$. The two directed loops of $H'$ both have knot class $(\frac{m}{2}, \frac{n}{2})$, so there is a directed path from any vertex $(\alpha A, \beta B)$ of $\mathbb{Z}_c \times \mathbb{Z}_d$ to the vertex $(\alpha A + (\frac{m}{2})A, \beta B + (\frac{n}{2})B)$ of $\mathbb{Z}_c \times \mathbb{Z}_d$. Since $H$ is connected, there is a directed path in $H$ from any vertex in $\mathbb{Z}_c \times \mathbb{Z}_d$ to any other vertex in $\mathbb{Z}_c \times \mathbb{Z}_d$. This means that $(\frac{m}{2}, \frac{n}{2})$ generates $\mathbb{Z}_c \times \mathbb{Z}_d$. In particular, $\mathbb{Z}_c \times \mathbb{Z}_d$ must be cyclic, so $\gcd(c,d) = 1$. This allows us to define an isomorphism $\phi : \mathbb{Z}_c \times \mathbb{Z}_d \to \mathbb{Z}_{cd}$ by the formula $\phi(x,y) = dx + cy$. Since $\phi(\frac{m}{2}, \frac{n}{2})$ must be a generator of $\mathbb{Z}_{cd}$, $\gcd(cd, d(\frac{m}{2}) + c(\frac{n}{2})) = 1$. With this equation and because $c$ divides $c(\frac{n}{2})$, we see that $\gcd(d(\frac{m}{2}), c) = 1$. Similarly, $\gcd(d, c(\frac{n}{2})) = 1$. By assumption $\gcd(\frac{m}{2}, \frac{n}{2}) = 1$. Using these equations, we see that $\gcd(d(\frac{m}{2}), c(\frac{n}{2})) = 1$ or $\gcd(dm, cn) = 2$. This finishes one direction of the theorem.

Now suppose conditions (1), (2), and (3). The first two conditions imply that $\mathbb{Z}_A \times \mathbb{Z}_B$ has a $(\mathbb{Z}_1 \times \mathbb{Z}_1)$-hyperhamiltonian circuit $H'$ with total knot class $(m,n)$. We construct a spanning subgraph $H$ of by requiring each vertex $(x,y)$ in to travel in the same direction as the vertex $(x \mod A, y \mod B)$ in $\mathbb{Z}_A \times \mathbb{Z}_B$. With this construction, the vertices of $\mathbb{Z}_c \times \mathbb{Z}_d$ have degree $(2,2)$ in $H$, and all other vertices have degree $(1,1)$. All that is left to show is that $H$ is connected. Since every directed loop in $H'$ contains the vertex $(0,0)$, every directed loop in $H$ contains a vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$. By Corrolary 3.5, we need only show that $H$ contains a directed path from any vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$ to any other vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$.

When $\gcd(m,n) = 1$, let $(m_1,n_1)$ and $(m_2,n_2)$ be the knot classes of the two directed loops in $H'$. There are directed paths from any vertex $(\alpha A, \beta B)$ of $\mathbb{Z}_c \times \mathbb{Z}_d$ to the vertices $(\alpha A + m_1 A, \beta B + n_1 B)$ and $(\alpha A + m_2 A, \beta B + n_2 B)$, so we want to show that $(m_1,n_1)$ and $(m_2,n_2)$ generate the group $\mathbb{Z}_c \times \mathbb{Z}_d$. This is equivalent to showing that for any element $(x,y)$ of $\mathbb{Z}_c \times \mathbb{Z}_d$, we have two integers $e$ and $f$ satisfying the equation

$$(x,y) = e(m_1,n_1) + f(m_2,n_2).$$

We can write this as the matrix equation

$$\begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The determinant of the matrix on the left is 1 or $-1$, because of the remarks after Theorem 2.5. The inverse of this matrix is again an integer matrix, which gives us equations for $e$ and $f$. This finishes the case when $\gcd(m,n) = 1$.

When $\gcd(m,n) = 2$, we have condition (3) which states that $\gcd(dm, cn) = 2$. It follows that $\gcd(d(\frac{m}{2}), c(\frac{n}{2})) = 1$. The two directed loops of $H'$ both have knot class $(\frac{m}{2}, \frac{n}{2})$, so we want to show that $(\frac{m}{2}, \frac{n}{2})$ generates $\mathbb{Z}_c \times \mathbb{Z}_d$. Since $c$ and $d$ are relatively prime, $\mathbb{Z}_c \times \mathbb{Z}_d$ is cyclic; we again use the isomorphism $\phi$ from above.

Now $\phi(\frac{m}{2}, \frac{n}{2}) = d(\frac{m}{2}) + c(\frac{n}{2})$. The element $(\frac{m}{2}, \frac{n}{2})$ is a generator of $\mathbb{Z}_c \times \mathbb{Z}_d$ if and only if $\gcd(cd, d(\frac{m}{2}) + c(\frac{n}{2})) = 1$. This last equation follows from the facts that $\gcd(c,d) = 1$, $\gcd(c, \frac{m}{2}) = 1$, and $\gcd(d, \frac{n}{2}) = 1$. □

## 4. Questions

Most questions about hamiltonian circuits on digraphs have analogies about hyperhamiltonian circuits. We end with a few specific examples. The first question extends our problem to larger dimensions.

**Question 4.1.** *When does the graph $\mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \cdots \times \mathbb{Z}_{a_n}$ have a $(\mathbb{Z}_{c_1} \times \mathbb{Z}_{c_2} \times \cdots \times \mathbb{Z}_{c_n})$-hyperhamiltonian circuit?*

Not every subgroup of $\mathbb{Z}_a \times \mathbb{Z}_b$ is of the form $\mathbb{Z}_c \times \mathbb{Z}_d$. This leads to our next question.

**Question 4.2.** *Let $A$ be any subgroup of $\mathbb{Z}_a \times \mathbb{Z}_b$. When does $\mathbb{Z}_a \times \mathbb{Z}_b$ have an $A$-hyperhamiltonian circuit?*

Instead of just considering the vertices of one subgroup, it is also possible to consider the vertices belonging to more than one coset of a subgroup.

**Question 4.3.** *Choose a positive number $r$. If $V$ is a disjoint union of $r$ cosets of $\mathbb{Z}_c \times \mathbb{Z}_d$ in $\mathbb{Z}_a \times \mathbb{Z}_b$, when is $\mathbb{Z}_a \times \mathbb{Z}_b$ $V$-hyperhamiltonian?*

Gallian and Witte [4] determined when the digraph $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_1 \times \mathbb{Z}_1)$ is hyperhamiltonian.

**Question 4.4.** *If $V$ is a coset of $\mathbb{Z}_c \times \mathbb{Z}_d$ in $\mathbb{Z}_a \times \mathbb{Z}_b$, when does the digraph $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$ have a $V$-hyperhamiltonian circuit?*

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Bowdoin College, 432 Smith Union, Brunswick, ME 04011

*E-mail address*: mmiller@bowdoin.edu