On the signed total domatic number of a graph

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Abstract

In this paper, we define the signed total domatic number of a graph in an analogous way to that of the fractional domatic number defined by Rall (A fractional version of domatic number. *Congr. Numer.* **74** (1990), 100–106). A function $f: V(G) \to \{-1, 1\}$ defined on the vertices of a graph G is a signed total dominating function if the sum of its function values over any open neighborhood is at least one. A set $\{f_1, \dots, f_d\}$ of signed total dominating functions on G such that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each vertex $v \in V(G)$ is called a signed total dominating family of functions on G. The signed total domatic number of G is the maximum number of functions in a signed total dominating family of G. In this paper we investigate the signed total domatic number for special classes of graphs

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1 Introduction

The domatic number of a graph and its variants is now well studied in graph theory and the literature on this parameter has been surveyed and

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detailed in the two books by Haynes, Hedetniemi, and Slater [4, 5]. The domatic number, introduced by Cockayne and Hedetniemi [1], is related to its domination number in the same way as the chromatic number is to the independence number: the *domatic number* of a graph G is the maximum number of elements in a partition of $V(G)$ into dominating sets, while the chromatic number is the minimum number of elements in a partition of $V(G)$ into independent sets. An excellent survey on the domatic number and its variants is given by Zelinka in Chapter 13 of [5].

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Since each dominating set of G can be thought of in terms of its characteristic function, Rall [7] viewed the domatic number as the maximum number of characteristic functions $\{f_1, \ldots, f_d\}$ (defined on V) of subsets of V such that each function f_i is a dominating function of G (that is, the sum of its function values over any closed neighborhood is at least one) and $\sum_{i=1}^{d} f_i(v) = 1$ for each $v \in V$. Rall [7] then extended this definition in a natural way to fractional dominating functions by defining the fractional domatic number $d_f(G)$ of G as the maximum number of fractional dominating functions $\{f_1, \ldots, f_d\}$ of G satisfying the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V$. In this paper, we define the signed total domatic number of G in an analogous way to that of the fractional domatic number defined by Rall [7].

For notation and graph theory terminology we in general follow [4]. Specifically, let $G = (V, E)$ be a graph and let v be a vertex in V. The open neighborhood of v is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. For disjoint subsets U and W of vertices, we let $[U, W]$ denote the set of edges between U and W. The subgraph of G induced by a set $S \subseteq V$ is denoted by $G[S]$. The minimum degree among the vertices of G is denoted by $\delta(G)$. If G does not contain a graph F as an induced subgraph, then we say that G is F-free. We call $K_4 - e$ a diamond. In particular, we say a graph is *diamond-free* if it is $(K_4 - e)$-free.

A *total dominating set* (TDS) of a graph G is a set S of vertices of G such that every vertex is adjacent to a vertex in S (other than itself). Equivalently, $S \subseteq V$ is a TDS of G if $|N(v) \cap S| \geq 1$ for every $v \in S$. Every graph without isolated vertices has a TDS, since $S = V$ is such a set. A TDS S of G such that $|N(v) \cap S| = 1$ for every $v \in S$ is called a *perfect total dominating set* (PTDS) of G.

Let $f: V \rightarrow \{-1, 1\}$ be a function which assigns to each vertex of a graph G an element of the set $\{-1, 1\}$. We define the *weight* of f by $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. For a vertex v in V, we denote $f(N(v))$ by $f[v]$ for notational convenience.

Zelinka [8] defined the function f to be a *signed total dominating function* (STDF) of G if $f[v] \geq 1$ for every $v \in V$. The *signed total domination number*, denoted $\gamma_t^s(G)$, of G is the minimum weight of a STDF on G. The study of signed total domination in graphs, started by Zelinka [8], is continued in [2, 3, 6] and elsewhere.

We now define the signed total domatic number of G in an analogous way to that of the fractional domatic number defined by Rall [7]. We call a set $\{f_1, \ldots, f_d\}$ of STDFs on G such that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each vertex $v \in V$ a *signed total dominating family* (STD-family) of functions on G. The *signed total domatic number* of G, denoted $d_t^s(G)$, is the maximum number of functions in a STD-family of G. Since every graph G with no isolated vertex has a STDF (simply assign a weight of 1 to each vertex of the graph), the signed total domatic number is well-defined for all graphs G with no isolated vertex (the set consisting of any one STDF forms a STD-family of G) and $d_t^s(G) \geq 1$.

In this paper we investigate the signed total domatic number for special classes of graphs, including complete graphs, complete bipartite graphs, and cubic graphs.

2 Basic Properties

In this section, we present some basic properties of the signed total domatic number. Rall [7] showed that the product of the fractional domatic number and the fractional domination number is bounded above the order of the graph. Using an identical proof, we shall prove an analogous result for the signed total domatic number. The domatic number of a graph G is bounded above by $\delta(G) + 1$. We show that this bound can be improved slightly for the signed total domatic number. We show further that the signed total domatic number is an odd integer.

Proposition 1 *Let G be a graph of order n with $\delta(G) \geq 1$. Then,*
(i) $d_t^s(G) \cdot \gamma_t^s(G) \leq n$,
(ii) $d_t^s(G) \leq \delta(G)$, *and*
(iii) $d_t^s(G)$ *is an odd integer.*

Proof. Let $G = (V, E)$ and let $d = d_t^s(G)$. Let $\{f_1, \ldots, f_d\}$ be a STD-family of G. Then,

$$d \cdot \gamma_t^s(G) = \sum_{i=1}^{d} \gamma_t^s(G) \leq \sum_{i=1}^{d} \sum_{v \in V} f_i(v) = \sum_{v \in V} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V} 1 = n.$$

This establishes (i). Next, let v be a vertex of minimum degree $\delta(G)$. Then,

$$d = \sum_{i=1}^{d} 1 \leq \sum_{i=1}^{d} \sum_{u \in N(v)} f_i(u) = \sum_{u \in N(v)} \sum_{i=1}^{d} f_i(u) \leq \sum_{u \in N(v)} 1 = \delta(G). \quad (1)$$

This establishes (ii). To verify (iii), suppose to the contrary that d is even. Since the sum of an even number of odd numbers is even, it follows that $\sum_{i=1}^{d} f_i(u) \leq 0$ for every $u \in V$. Hence for $v \in V$,

$$1 \leq d \leq \sum_{u \in N(v)} \sum_{i=1}^{d} f_i(u) \leq \sum_{u \in N(v)} 0 = 0,$$

which is impossible. Hence, d is odd as claimed. □

As an immediate consequence of Proposition 1, we have the following results.

Corollary 2 *If G is a graph with $1 \leq \delta(G) \leq 2$, then $d_t^s(G) = 1$. In particular, if G is a cycle or a tree, then $d_t^s(G) = 1$.*

3 Complete Graphs

Our aim in this section is to determine the signed total domatic number of a complete graph K_n. For $n \geq 3$ odd, $\gamma_t^s(K_n) = 3$, while for $n \geq 2$ even, $\gamma_t^s(K_n) = 2$. We begin with the following lemma.

Lemma 3 *For $n \geq 4$, $d_t^s(K_n) \geq d_t^s(K_{n-2})$.*

Proof. By Proposition 1(iii), the signed total domatic number is an odd positive integer. If $d_t^s(K_{n-2}) = 1$, then the result is immediate. Hence we may assume that $d_t^s(K_{n-2}) = d$ where $d \geq 3$. Let $G = K_n$ and let u and v be distinct vertices of G. Let $G' = G - u - v$. Then, $G' = K_{n-2}$. Let $\{f_1', f_2', \ldots, f_d'\}$ be a STD-family of G'. If n is odd, then $\gamma_t^s(K_n) = 3$, and so $w(f_i') \geq 3$, while if n is even, then $\gamma_t^s(K_n) = 2$, and so $w(f_i') \geq 2$ for all $i = 1, \ldots, d$. For $i = 1, \ldots, d$, let $f_i: V(G) \to \{-1, 1\}$ be the function defined as follows. Let $f_i(w) = f_i'(w)$ if $w \in V(G) - \{u, v\}$. For $i = 1, \ldots, \lceil d/2 \rceil$, let $f_i(u) = 1$ and $f_i(v) = -1$, while for $i = \lceil d/2 \rceil + 1, \ldots, d$, let $f_i(u) = -1$ and $f_i(v) = 1$. Then, $\{f_1, f_2, \ldots, f_d\}$ is a STD-family of G, and so $d_t^s(G) \geq d = d_t^s(G')$. Since $G = K_n$ and $G' = K_{n-2}$, the desired result follows. □

We shall prove:

Theorem 4 *For $n \geq 2$, $d_t^s(K_n) = \lfloor (n+1)/3 \rfloor - \lceil n/3 \rceil + \lfloor n/3 \rfloor$ if n is odd, and $d_t^s(K_n) = n/2 - \lceil (n+2)/4 \rceil + \lfloor (n+2)/4 \rfloor$ if n is even.*

Proof. Let $G = K_n$ have vertex set V and let $d = d_t^s(G)$. Let $\{f_1, \ldots, f_d\}$ be a STD-family of G. Let $g(n) = \lfloor (n+1)/3 \rfloor - \lceil n/3 \rceil + \lfloor n/3 \rfloor$ and let $h(n) = n/2 - \lceil (n+2)/4 \rceil + \lfloor (n+2)/4 \rfloor$.

Claim 1 *If n is odd, then $d = g(n)$.*

Proof. For n odd, $g(n) = n/3$ if $n \equiv 0 \pmod 3$, $g(n) = (n-2)/3$ if $n \equiv 2 \pmod 3$, and $g(n) = (n-4)/3$ if $n \equiv 1 \pmod 3$. If $n = 3$, then $d = 1 = g(3)$ by Corollary 2, and the desired result holds. Hence we may assume that $n \geq 5$. Since n is odd, $\gamma_t^s(K_n) = 3$, and so $w(f_i) \geq 3$ for $i = 1, \ldots, d$. Thus,

$$3d \leq \sum_{i=1}^{d} w(f_i) \leq \sum_{i=1}^{d} \sum_{v \in V} f_i(v) = \sum_{v \in V} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V} 1 = n,$$

and so $d \leq n/3$. If $n \equiv 0 \pmod 3$ (and still n is odd), then $n/3$ is odd, and so, by Proposition 1(iii), $d \leq n/3 = g(n)$. If $n \equiv 1 \pmod 3$, then $(n-1)/3$ is even, and so, by Proposition 1(iii), $d \leq (n-4)/3 = g(n)$. If $n \equiv 2 \pmod 3$, then $(n-2)/3$ is odd, and so, by Proposition 1(iii), $d \leq (n-2)/3 = g(n)$. Hence, $d \leq g(n)$ for n odd.

To show that $d \geq g(n)$, suppose first that $n \equiv 0 \pmod 3$ (and still $n \geq 5$ is odd). Then, $n = 3(2k+1)$ for some $k \geq 1$. Let $V_1, V_2, \ldots, V_{2k+1}$ be a partition of V into $2k+1$ sets each of cardinality 3. For $i = 1, 2, \ldots, 2k+1$, let $f_i: V \to \{-1, 1\}$ be the function defined by

$$f_i(v) = \begin{cases} -1 & \text{if } v \in \bigcup_{j=i}^{i+k-1} V_j \\ +1 & \text{otherwise,} \end{cases}$$

where addition is taken modulo $2k+1$. Then each f_i is a STDF of G and $\sum_{i=1}^{2k+1} f_i(v) = 1$ for every $v \in V$. Hence, $\{f_1, f_2, \ldots, f_{2k+1}\}$ is a STD-family of G, and so $d \geq 2k+1 = n/3 = g(n)$. Consequently, $d = g(n)$ if $n \equiv 0 \pmod 3$ (and $n \geq 5$ is odd).

Suppose, secondly, that $n \equiv 2 \pmod 3$ (and still $n \geq 5$ is odd). Then, $g(n) = g(n-2) = (n-2)/3$. By Lemma 3, $d = d_t^s(K_n) \geq d_t^s(K_{n-2})$. Since $n-2$ is odd and $n-2 \equiv 0 \pmod 3$, $d_t^s(K_{n-2}) = g(n-2) = g(n)$. Thus, $d \geq g(n)$. Consequently, $d = g(n)$ if $n \equiv 2 \pmod 3$.

Suppose, finally, that $n \equiv 1 \pmod 3$ (and still $n \geq 5$ is odd). Then, $g(n) = g(n-4) = (n-4)/3$. By Lemma 3, $d = d_t^s(K_n) \geq d_t^s(K_{n-4})$. Since $n-4$ is odd and $n-4 \equiv 0 \pmod 3$, $d_t^s(K_{n-4}) = g(n-4) = g(n)$. Thus, $d \geq g(n)$. Consequently, $d = g(n)$ if $n \equiv 1 \pmod 3$. Thus if n is odd, then $d = g(n)$, as claimed. □

Claim 2 *If n is even, then $d = h(n)$.*

Proof. For n even, $h(n) = n/2$ if $n \equiv 2 \pmod 4$ and $h(n) = n/2 - 1$ if $n \equiv 0 \pmod 4$. If $n = 2$, then $d = 1 = h(2)$ by Corollary 2, and the desired result holds. Hence we may assume that $n \geq 4$. Since n is even, $\gamma_t^s(K_n) = 2$, and so $w(f_i) \geq 2$ for $i = 1, \ldots, d$. Thus using a similar argument to that used in the proof of Claim 1 (to show that $d \leq n/3$) we now have $d \leq n/2$. Since d is odd, $d \leq n/2 = h(n)$ if $n \equiv 2 \pmod 4$ and $d \leq n/2 - 1 = h(n)$ if $n \equiv 0 \pmod 4$. Hence, $d \leq h(n)$ for n even.

To show that $d \geq h(n)$, suppose first that $n \equiv 2 \pmod 4$. Then, $n = 2(2k+1)$ for some $k \geq 1$. Let $V_1, V_2, \ldots, V_{2k+1}$ be a partition of V into $2k+1$ sets each of cardinality 2. For $i = 1, 2, \ldots, 2k+1$, let $f_i \colon V \to \{-1, 1\}$ be the function defined as in the proof of Claim 1. Then each f_i is a STDF of G and $\sum_{i=1}^{2k+1} f_i(v) = 1$ for every $v \in V$. Hence, $\{f_1, f_2, \ldots, f_{2k+1}\}$ is a STD-family of G, and so $d \geq 2k+1 = n/2 = h(n)$. Consequently, $d = h(n)$ if $n \equiv 2 \pmod 4$.

Suppose, secondly, that $n \equiv 0 \pmod 4$. Then, $h(n) = h(n-2) = (n-2)/2$. By Lemma 3, $d = d_t^s(K_n) \geq d_t^s(K_{n-2})$. Since $n-2 \equiv 2 \pmod 4$, $d_t^s(K_{n-2}) = h(n-2) = h(n)$. Thus, $d \geq h(n)$. Consequently, $d = h(n)$ if $n \equiv 0 \pmod 4$. Thus if n is even, then $d = h(n)$, as claimed. □

The desired result now follows from Claims 1 and 2. □

4 Complete Bipartite Graphs

Our aim in this section is to determine the signed total domatic number of a complete bipartite graph $K_{m,n}$. We begin with the following lemma, a proof of which is along similar lines to that of the proof of Lemma 3 and is therefore omitted.

Lemma 5 *For $m \geq 3$ and $n \geq 3$, $d_t^s(K_{m,n}) \geq d_t^s(K_{m,n-2})$ and $d_t^s(K_{m,n}) \geq d_t^s(K_{m-2,n})$.*

We shall prove:

Theorem 6 *For $m \geq n \geq 1$, $d_t^s(K_{m,n}) = n/2 - \lceil (n+2)/4 \rceil + \lfloor (n+2)/4 \rfloor$ if n is even, $d_t^s(K_{m,n}) = n$ if n and m are odd, and $d_t^s(K_{m,n}) = \min\{n, m/2 - \lceil (m+2)/4 \rceil + \lfloor (m+2)/4 \rfloor\}$ if n is odd and m is even.*

Proof. Let $G = K_{m,n}$ and let $d = d_t^s(G)$. Let $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{m-1}\}$ be the partite sets of G, and so $|X| = n$ and $|Y| = m$. Let $\{f_1, \ldots, f_d\}$ be a STD-family of G. Let $h(n) = n/2 - \lceil (n+2)/4 \rceil + \lfloor (n+2)/4 \rfloor$. If $n \leq 2$, then $\delta(G) \leq 2$, and so, by Corollary 2, $d = 1$ and the desired result holds. Hence we may assume that $m \geq n \geq 3$.

Claim 3 *If n is even, then $d = h(n)$.*

Proof. For n even, $h(n) = n/2$ if $n \equiv 2 \pmod 4$ and $h(n) = n/2 - 1$ if $n \equiv 0 \pmod 4$. For any STDF f of G, and for any $y \in Y$, $f[y] = f(X) \geq 2$ since $|X| = n$ is even. Let $y \in Y$. Then,

$$2d \leq \sum_{i=1}^{d} f_i(X) = \sum_{i=1}^{d} \sum_{v \in N(y)} f_i(v) = \sum_{v \in N(y)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in N(y)} 1 = |X| = n,$$

and so $d \leq n/2$. Since d is odd, $d \leq n/2 = h(n)$ if $n \equiv 2 \pmod 4$ and $d \leq n/2 - 1 = h(n)$ if $n \equiv 0 \pmod 4$. Hence, $d \leq h(n)$ for n even. To show that $d \geq h(n)$, we consider two possibilities.

Suppose first that $n \equiv 2 \pmod 4$. Let $k = n/2$. Then, $k \geq 3$ is an odd integer. For $i = 1, 2, \ldots, k$, let X_i be defined by

$$X_i = \bigcup_{j=(i-1)(k-1)}^{i(k-1)-1} \{x_i\},$$

where addition is taken modulo n. Then, $X_1, X_2, \ldots, X_k$ are distinct subsets of X (and so, $X_i \neq X_j$ for $1 \leq i < j \leq k$) each of cardinality $k-1$. Suppose $m \not\equiv 0 \pmod 4$. Let $\ell = \lceil m/2 \rceil$. Then, $m = 2\ell$ if $m \equiv 2 \pmod 4$ and $m = 2\ell - 1$ if m is odd. Further, $3 \leq k \leq \ell$. For $i = 1, 2, \ldots, k$, let Y_i be defined by

$$Y_i = \bigcup_{j=(i-1)(\ell-1)}^{i(\ell-1)-1} \{y_i\},$$

where addition is taken modulo m. Then, $Y_1, Y_2, \ldots, Y_k$ are distinct subsets of Y (and so, $Y_i \neq Y_j$ for $1 \leq i < j \leq k$) each of cardinality $\ell - 1$. For $i = 1, 2, \ldots, k$, let $f_i: V \to \{-1, 1\}$ be the function defined by

$$f_i(v) = \begin{cases} -1 & \text{if } v \in X_i \cup Y_i \\ +1 & \text{otherwise.} \end{cases}$$

For each $i = 1, 2, \ldots, k$, $f_i(X) = 2$ while $f_i(Y) = 2$ if $m \equiv 2 \pmod 4$ and $f_i(Y) = 1$ if m is odd. Thus, f_i is a STDF of G. Moreover, $\sum_{i=1}^{k} f_i(v) = 1$ for every $v \in X$ and, since k is odd, $\sum_{i=1}^{k} f_i(v) \leq 1$ for every $v \in Y$. Hence, $\{f_1, f_2, \ldots, f_k\}$ is a STD-family of G, and so $d \geq k = h(n)$. Consequently, $d = h(n)$ if $n \equiv 2 \pmod 4$ and $m \not\equiv 0 \pmod 4$.

Suppose $m \equiv 0 \pmod 4$ (and still $n \equiv 2 \pmod 4$). By Lemma 5, $d = d_t^s(K_{m,n}) \geq d_t^s(K_{m-2,n})$. Since $n \leq m-2$, and since $m - 2 \equiv 2 \pmod 4$ and $n \equiv 2 \pmod 4$, $d_t^s(K_{m-2,n}) = h(n)$. Thus, $d \geq h(n)$. Consequently, $d = h(n)$ if $m \equiv 0 \pmod 4$. Thus if $n \equiv 2 \pmod 4$, then $d = h(n)$, as claimed.

Suppose, secondly, that $n \equiv 0 \pmod 4$. Then, $h(n) = (n-2)/2 = h(n-2)$. By Lemma 5, $d = d_t^s(K_{m,n}) \geq d_t^s(K_{m,n-2})$. Since $n - 2 \leq m$ and $n - 2 \equiv 2 \pmod 4$, $d_t^s(K_{m,n-2}) = h(n-2) = h(n)$. Thus if $n \equiv 0 \pmod 4$, then $d = h(n)$, as claimed. □

Claim 4 *If n and m are odd, then $d = n$.*

Proof. For any STDF f of G, and for any $y \in Y$, $f[y] = f(X) \geq 1$ since $|X| = n$ is odd. Using a similar argument to that used in the proof of Claim 3 (to show that $d \leq n/2$) we now have $d \leq n$. To show that $d \geq n$, let $k = (n+1)/2$ and let $\ell = (m+1)/2$. For $i = 1, 2, \ldots, n$, let X_i and Y_i be defined as in the proof of Claim 3. Then for each $i = 1, 2, \ldots, n$, $f_i(X) = f_i(Y) = 1$, and so f_i is a STDF of G. Moreover, $\sum_{i=1}^{n} f_i(v) = 1$ for every $v \in X$ and, since n is odd, $\sum_{i=1}^{n} f_i(v) \leq 1$ for every $v \in Y$. Hence, $\{f_1, f_2, \ldots, f_n\}$ is a STD-family of G, and so $d \geq n$. Consequently, $d = n$. □

Claim 5 *If n is odd and m is even, then $d = \min\{n, h(m)\}$.*

Proof. Since n is odd, $d \leq n$ as shown in the proof of Claim 4. Since $|Y| = m$ is even, $f(Y) \geq 2$ for any STDF f of G. Thus using a similar argument to that used in the proof of Claim 3, $d \leq m/2$. Since d is odd, $d \leq m/2 = h(m)$ if $m \equiv 2 \,(\text{mod}\, 4)$ and $d \leq m/2 - 1 = h(m)$ if $m \equiv 0 \,(\text{mod}\, 4)$. Hence, $d \leq h(m)$. Thus, $d \leq \min\{n, h(m)\}$. To show that $d \geq \min\{n, h(m)\}$, we consider two possibilities. Let $p = \min\{n, h(m)\}$. Then, p is odd.

Suppose first that $m \equiv 2 \,(\text{mod}\, 4)$. Let $k = (n+1)/2$ and let $\ell = m/2$. For $i = 1, 2, \ldots, p$, let X_i and Y_i be defined as in the proof of Claim 3. Then for each $i = 1, 2, \ldots, p$, $f_i(X) = 1$ and $f_i(Y) = 2$, and so f_i is a STDF of G. Moreover, since p is odd, $\sum_{i=1}^{n} f_i(v) \leq 1$ for every $v \in V$. Hence, $\{f_1, f_2, \ldots, f_p\}$ is a STD-family of G, and so $d \geq p$. Consequently, $d = p = \min\{n, h(m)\}$.

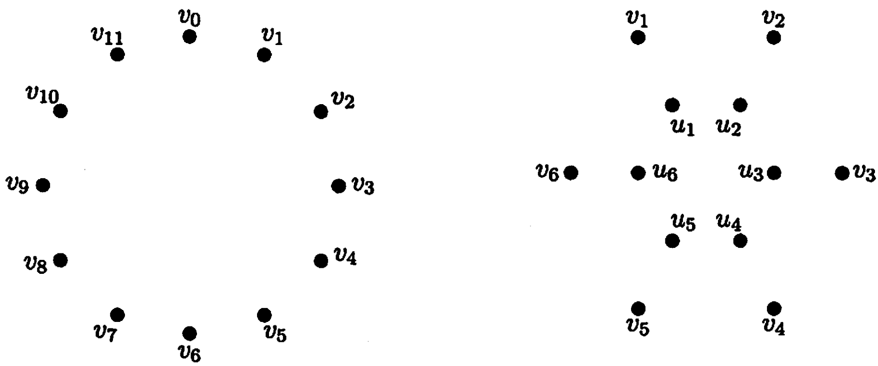
Suppose secondly that $m \equiv 0 \,(\text{mod}\, 4)$. Then, $h(m) = (m-2)/2 = h(m-2)$. By Lemma 5, $d = d_t^s(K_{m,n}) \geq d_t^s(K_{m-2,n})$. Since n is odd and $m - 2 \equiv 2 \,(\text{mod}\, 4)$, $d_t^s(K_{m-2,n}) = \min\{n, h(m-2)\} = \min\{n, h(m)\} = p$. Hence, $d \geq p$. Thus if $m \equiv 0 \,(\text{mod}\, 4)$, then $d = p = \min\{n, h(m)\}$, as claimed. □

The proof of Theorem 6 follows from Claims 3, 4, and 5. □

5 Cubic Graphs

By Proposition 1, if G is a cubic graph, then $d_t^s(G) = 1$ or $d_t^s(G) = 3$. A natural problem, then, is to classify cubic graphs according to whether their signed total domatic number equals 1 or 3.

Example 1. For $k \geq 1$ an integer, consider the circulant graph $G = C_{6k}\langle 1, 3k \rangle$ (i.e., the graph with vertex set $\{v_0, v_1, \ldots, v_{6k-1}\}$ and edge set $\{v_i v_{i+j \,(\text{mod}\, 6k)} \mid i \in \{0, 1, \ldots, 6k-1\}$ and $j \in \{1, 3k\}\}$). The circulant $C_{12}\langle 1, 6 \rangle$ is shown in Figure 1(a). For $t \in \{0, 1, 2\}$, let $V_t = \{v_i \mid 0 \leq i \leq 6k-1, i \equiv t \,(\text{mod}\, 3)\}$ and let $f_t \colon V(G) \to \{-1, 1\}$ be the function defined by $f(v_i) = -1$ if $v_i \in V_t$ and $f(v_i) = 1$ otherwise.



(a) The circulant $C_{12}\langle 1,6\rangle$ (b) The prism $K_2\square C_6$

Figure 1.

Example 2. For $k \geq 1$ an integer, consider the prism $G = K_2\square C_{3k}$ (i.e., the graph obtained from two disjoint cycles $v_1, v_2, \ldots, v_{3k}, v_1$ and $u_1, u_2, \ldots, u_{3k}, u_1$ by adding the edges u_iv_i for $i = 1, 2, \ldots, 3k$). The prism $K_2\square C_6$ is shown in Figure 1(b). For $t \in \{0, 1, 2\}$, let $V_t = \{u_i, v_i \mid 1 \leq i \leq 3k, i \equiv t \pmod 3\}$ and let $f_t : V(G) \to \{-1, 1\}$ be the function defined by $f(v_i) = -1$ if $v_i \in V_t$ and $f(v_i) = 1$ otherwise.

In both Examples 1 and 2, V_0, V_1 and V_2 is a partition of $V(G)$ into three perfect total dominating set, and $\{f_0, f_1, f_2\}$ is a STD-family of G. Consequently, $d_t^s(G) = 3$. These two examples serve to illustrate infinite families of cubic graphs G satisfying $d_t^s(G) = 3$. In both examples, $V(G)$ can be partitioned into three perfect total dominating sets. We show that this is true for all cubic graphs with signed total domatic number equal to 3.

Theorem 7 *Let G be a cubic graph. Then, $d_t^s(G) = 3$ if and only if $V(G)$ can be partitioned into three perfect total dominating sets.*

Proof. We shall follow the notation introduced in the proof of Proposition 1. Suppose first that $d = 3$. Then we must have equality throughout Equation (1) in the proof of Proposition 1. Hence for each $v \in V$ and for each $i = 1, 2, 3$,

$$\sum_{i=1}^{3} f_i(v) = 1, \quad \text{and} \tag{2}$$

$$\sum_{u \in N(v)} f_i(u) = 1. \tag{3}$$

For $i = 1, 2, 3$, let $V_i = \{v \in V \mid f_i(v) = -1\}$. By Equation (2), V_1, V_2, V_3 is a partition of V. By Equation (3), $|N(v) \cap V_i| = 1$ for every $v \in V$ and for each $i = 1, 2, 3$. Thus, V can be partitioned into three PTDSs.

On the other hand, suppose that V can be partitioned into three PTDSs S_1, S_2 and S_3. For $i = 1, 2, 3$, let $f_i: V \to \{-1, 1\}$ be the function defined by $f_i(v) = -1$ if $v \in S_i$ and $f_i(v) = 1$ if $v \in V - S_i$. Then for each $v \in V$ and for each $i = 1, 2, 3$, Equations (2) and (3) are satisfied. Thus, $\{f_1, f_2, f_3\}$ is a STD-family of G, and so $d_t^s(G) \geq 3$. Consequently, $d_t^s(G) = 3$. □

As a consequence of Theorem 7, we have the following result.

Corollary 8 *Let G be a cubic graph of order n. If $d_t^s(G) = 3$, then* (i) $n \equiv 0 \pmod 6$, (ii) *G is diamond-free, and* (iii) *G is 1-factorable.*

Proof. We shall follow the notation introduced in the proof of Proposition 1. Since $d_t^s(G) = 3$, V can be partitioned into three PTDSs V_1, V_2 and V_3 by Theorem 7. Let $S = V_1$.

(i) The set V can be partitioned into $|S|$ 3-element subsets, namely the sets $\{N(v) \mid v \in S\}$. Thus, $n = 3|S|$. Since n is even, $|S|$ must be even, whence $n \equiv 0 \pmod 6$. Let $n = 6s$, where $s \geq 1$.

(ii) Suppose that G contain a diamond. Let u be a vertex of degree 3 in a diamond in G. We may assume that $u \in S$. Let $N(u) \cap S = \{v\}$. But then u and v have a common neighbor w, and so $|N(w) \cap S| \geq 2$, contradicting the fact that S is a PTDS. Hence, G is diamond-free.

(iii) Each set V_i, $1 \leq i \leq 3$, is a PTDS of G, and so $G[V_i] = sK_2$. Consider distinct integers i, j and k where $1 \leq i, j, k \leq 3$. Since no two vertices of V_j have a common neighbor, each vertex in V_i is adjacent to at most one vertex in V_j. However since V_j is a TDS of G, each vertex of V_i is adjacent to at least one vertex in V_j. Consequently, each vertex of V_i is adjacent to exactly one vertex in V_j. Similarly, each vertex of V_j is adjacent to exactly one vertex in V_i. It follows that the set $[V_i, V_j]$ of edges between V_i and V_j induce a perfect matching in $G[V_i \cup V_j]$ (isomorphic to $2sK_2$). This set of $2s$ edges of G, together with the set of s edges in $G[V_k]$, is a perfect matching in G. Similarly, the set of edges $[V_i, V_k]$ between V_i and V_k together with the set of edges in $G[V_j]$ is a perfect matching in G, as is the set of edge $[V_j, V_k]$ between V_j and V_k together with the set of edges in $G[V_i]$. These three perfect matchings in G partition the edge set of G. Thus, G is 1-factorable. □

The necessary conditions in Corollary 8 for a cubic graph to have signed total domatic number equal to 3 are not sufficient, even for bipartite cubic graphs (which are 1-factorable).

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