The Graphs $C_7^{(t)}$ are Graceful for $t \equiv 0, 1 \pmod 4$ *

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Abstract

Let C_n denote the cycle with n vertices , and $C_n^{(t)}$ denote the graphs consisting of t copies of C_n with a vertex in common . Koh et al. conjectured that the graphs $C_n^{(t)}$ are graceful if and only if $nt \equiv 0, 3 \pmod 4$. The conjecture has been shown true for $n = 3, 5, 6, 4k$. In this paper, the conjecture is shown to be true for $n = 7$.
Keywords: *graceful graph, vertex labeling, edge labeling*

1 Introduction

The graceful labeling traced its origin to one introduced by Rosa [1] in 1967 as a way of decomposing a complete graph into isomorphic subgraphs. Let $G = (V, E)$ be a simple graph with $|V|$ vertices and $|E|$ edges. Let

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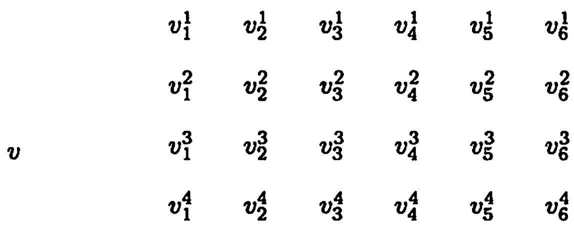

Figure 2.1: The graph $C_7^{(4)}$.

$f : V \to \{0, 1, \ldots, |E|\}$ be an injective mapping. Define an induced function $g : E \to \{1, 2, \ldots, |E|\}$ by setting $g(p, q) = |f(p) - f(q)|$, for every $(p, q) \in E$. If g maps E onto $\{1, 2, \ldots, |E|\}$, then f is said to be a graceful labeling of G. A graph is graceful if it has a graceful labeling.

Let C_n denote the cycle with n vertices , and $C_n^{(t)}$ denote the graphs consisting of t copies of C_n with a vertex in common . Koh et al. [5] conjectured that the graphs $C_n^{(t)}$ are graceful if and only if $nt \equiv 0, 3 \pmod 4$, and proved that the graphs $C_{4k}^{(t)}$ and $C_6^{(2t)}$ are graceful for $t \geqslant 1$. Qian [7] proved that the graphs $C_{2k}^{(2)}$ are graceful. Bermond et al. [2, 3] proved that the graphs $C_3^{(t)}$ (i.e, the friendship graph or Dutch t-windmill) are graceful if and only if $t \equiv 0$ *or* $1 \pmod 4$. The first author [6] of this paper proved that the graphs $C_5^{(t)}$ are graceful for $t \equiv 0, 3 \pmod 4$. So the conjecture has been shown true for $n = 3, 5, 6, 4k$. In this paper, the conjecture is shown to be true for $n = 7$.

For the literature on graceful graphs we refer to [4] and the relevant references given in it.

2 The graphs $C_7^{(t)}$

Now, we consider the graphs $C_7^{(t)}$.

Let $v_0^i, v_1^i, v_2^i, v_3^i, v_4^i, v_5^i, v_6^i$ be the vertices of the i-th cycle , $v_0^i = v$ for all i. The graph of $C_7^{(4)}$ is shown in Figure 2.1.

Theorem 2.1. The graphs $C_7^{(t)}$ are graceful for $t \equiv 0, 1$ ($mod\, 4$).

Proof. Let $\theta = t \bmod 2$, we define a vertex labeling f as follows:

$$
\begin{aligned}
f(v) &= 0, \\
f(v_1^i) &= 7t+1-i, && 1 \le i \le t, \\
f(v_2^i) &= \begin{cases} 2t+i+\theta, & 1 \le i \le (t-\theta)/2, \\ i+\theta, & (t-\theta)/2+1 \le i \le t, \end{cases} \\
f(v_3^i) &= \begin{cases} (9t-\theta)/2+1-i+2\theta, & 1 \le i \le (t-\theta)/2, \\ (7t-\theta)/2+2-i+\theta, & (t-\theta)/2+1 \le i \le t, \end{cases} \\
f(v_4^i) &= t+i+\theta, && 1 \le i \le t, \\
f(v_5^i) &= (11t-\theta)/2+2-i+\theta, && 1 \le i \le t, \\
f(v_6^i) &= \begin{cases} i, & 1 \le i \le (t+\theta)/2, \\ 5t+1+i-\theta, & (t+\theta)/2+1 \le i \le t \text{ and } \\ & i \; mod \; 2 = 1, \\ 3t-1+i+\theta, & (t+\theta)/2+1 \le i \le t \text{ and } \\ & i \; mod \; 2 = 0. \end{cases}
\end{aligned}
$$

Denote by

$$S_j = \{f(v_j^i) \; | 1 \le i \le t\}, \qquad 0 \le j \le 6,$$

$$D_j = \{g(v_j^i, v_{(j+1) \text{ mod } 7}^i) | \; 1 \le i \le t\}, \qquad 0 \le j \le 6,$$

$$g(v_j^i, v_{(j+1) \text{ mod } 7}^i) = |f(v_{(j+1) \text{ mod } 7}^i) - f(v_j^i)|, \quad 1 \le i \le t, \quad 0 \le j \le 6.$$

Case 1. Suppose that $t \equiv 0 \pmod 4$, say $t = 4k$, then we have:

$$
\begin{aligned}
& S_0 = \{0\}, \\
& S_1 = \{6t+1, 6t+2, \ldots, 7t\}, \\
& S_2 = S_{21} \cup S_{22} = \{2t+1, 2t+2, \ldots, 5t/2\} \cup \{t/2+1, t/2+2, \ldots, t\}, \\
& S_3 = S_{31} \cup S_{32} = \{9t/2, 9t/2-1, \ldots, 4t+1\} \cup \{3t+1, 3t, \ldots, 5t/2+2\}, \\
& S_4 = \{t+1, t+2, \ldots, 2t\}, \\
& S_5 = \{11t/2+1, 11t/2, \ldots, 9t/2+2\}, \\
& S_6 = S_{61} \cup S_{62} \cup S_{63} \\
& \quad = \{1, 2, \ldots, t/2\} \cup \{11t/2+2, 11t/2+4, \ldots, 6t\} \\
& \qquad \cup \{7t/2+1, 7t/2+3, \ldots, 4t-1\}.
\end{aligned}
$$

Hence, $S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$ is the set of labels of all vertices , and

$$\begin{aligned}
& S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \\
= \ & S_0 \cup S_1 \cup S_{21} \cup S_{22} \cup S_{31} \cup S_{32} \cup S_4 \cup S_5 \cup S_{61} \cup S_{62} \cup S_{63} \\
= \ & (S_0 \cup S_{61} \cup S_{22} \cup S_4 \cup S_{21}) \cup S_{32} \cup S_{63} \cup S_{31} \cup (S_5 \cup S_{62} \cup S_1) \\
= \ & \{\ 0, 1, \ldots, t/2, t/2+1, t/2+2, \ldots, t, t+1, t+2, \ldots, 2t, 2t+1, 2t+2, \\
& \ldots, 5t/2, 5t/2+2, 5t/2+3, \ldots, 3t+1, \ \ 7t/2+1, 7t/2+3, \ldots, 4t-1, \\
& 4t+1, 4t+2, \ldots, 9t/2, \quad 9t/2+2, 9t/2+3, \ldots, 11t/2+1, 11t/2+2, \\
& 11t/2+4, \quad \ldots, \quad 6t-2, \quad 6t, \quad 6t+1, 6t+2, \ldots, 7t\}.
\end{aligned}$$

It is clear that the labels of the vertices are different, and $Max\{f(v_j^i) | 1 \le i \le t, \ 0 \le j \le 6\} = 7t = |E|$. We thus conclude that f is an injective mapping from the vertex set of G into $\{\ 0, 1, \ldots, |E|\}$.

Now, we verify that g maps E onto $\{1, 2, \ldots, |E|\}$.

$$\begin{aligned}
D_0 &= \{6t+1, 6t+2, \ldots, 7t\}, \\
D_1 &= D_{11} \cup D_{12} = \{5t-1, 5t-3, \ldots, 4t+1\} \cup \{6t-1, 6t-3, \ldots, 5t+1\}, \\
D_2 &= D_{21} \cup D_{22} \\
&= \{5t/2-1, 5t/2-3, \ldots, 3t/2+1\} \cup \{5t/2, 5t/2-2, \ldots, 3t/2+2\}, \\
D_3 &= D_{31} \cup D_{32} \\
&= \{7t/2-1, 7t/2-3, \ldots, 5t/2+1\} \cup \{3t/2, 3t/2-2, \ldots, t/2+2\}, \\
D_4 &= \{9t/2, 9t/2-2, \ldots, 5t/2+2\}, \\
D_5 &= D_{51} \cup D_{52} \\
&= \{11t/2, 11t/2-2, \ldots, 9t/2+2\} \cup \{t/2+1, t/2+3, \ldots, 3t/2-1\}, \\
D_6 &= D_{61} \cup D_{62} \cup D_{63} = \{1, 2, \ldots, t/2\} \cup \{11t/2+2, 11t/2+4, \ldots, 6t\} \\
&\qquad \cup \{7t/2+1, 7t/2+3, \ldots, 4t-1\}.
\end{aligned}$$

Hence, $D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6$ is the set of labels of all edges, and

$$\begin{aligned}
& D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6 \\
= \ & D_0 \cup D_{11} \cup D_{12} \cup D_{21} \cup D_{22} \cup D_{31} \cup D_{32} \cup D_4 \cup D_{51} \cup D_{52} \\
& \cup D_{61} \cup D_{62} \cup D_{63} \\
= \ & D_{61} \cup (D_{52} \cup D_{32}) \cup (D_{21} \cup D_{22}) \cup (D_{31} \cup D_{63} \cup D_{11} \cup D_{12}) \\
& \cup (D_4 \cup D_{51} \cup D_{62}) \cup D_0 \\
= \ & \{1, 2, \ldots, t/2\} \cup \{t/2+1, t/2+2, \ldots, 3t/2-1, 3t/2\} \\
& \cup \{3t/2+1, 3t/2+2, \ldots, 5t/2-1, 5t/2\} \\
& \cup \{5t/2+1, 5t/2+3, \ldots, 7t/2-1, 7t/2+1, 7t/2+3, \ldots, 4t-1, \\
& \quad 4t+1, 4t+3, \ldots, 5t-1, 5t+1, 5t+3, \ldots, 6t-1\} \\
& \cup \{5t/2+2, 5t/2+4, \ldots, 9t/2, 9t/2+2, 9t/2+4, \ldots, 11t/2, 11t/2+2, \\
& 11t/2+4, \ldots, 6t\} \cup \{6t+1, 6t+2, \ldots, 7t\} \\
= \ & \{1, 2, \ldots, 7t\}.
\end{aligned}$$

It is clear that the labels of the edges are different. So, g maps E onto $\{1, 2, \ldots, |E|\}$. By the definition of graceful graph, we can conclude that the graphs $C_7^{(4k)}$ are graceful.

Case 2. Suppose that $t \equiv 1 \pmod 4$, say $t = 4k+1$, then we have:

$$\begin{aligned}
S_0 &= \{0\},\\
S_1 &= \{6t+1, 6t+2, \ldots, 7t\},\\
S_2 &= S_{21} \cup S_{22}\\
&= \{2t+2, 2t+3, \ldots, (5t+1)/2\} \cup \{(t+3)/2, (t+5)/2, \ldots, t+1\},\\
S_3 &= S_{31} \cup S_{32}\\
&= \{(9t+3)/2, (9t+1)/2, \ldots, 4t+3\} \cup \{3t+2, 3t+1, \ldots, (5t+5)/2\},\\
S_4 &= \{t+2, t+3, \ldots, 2t+1\},\\
S_5 &= \{(11t+3)/2, (11t+1)/2, \ldots, (9t+5)/2\},\\
S_6 &= S_{61} \cup S_{62} \cup S_{63}\\
&= \{1, 2, \ldots, (t+1)/2\} \cup \{(11t+5)/2, (11t+9)/2, \ldots, 6t\}\\
&\quad \cup \{(7t+3)/2, (7t+7)/2, \ldots, 4t-1\}.
\end{aligned}$$

Hence, $S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$ is the set of labels of all vertices , and

$$\begin{aligned}
& S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6\\
= \;& S_0 \cup S_1 \cup S_{21} \cup S_{22} \cup S_{31} \cup S_{32} \cup S_4 \cup S_5 \cup S_{61} \cup S_{62} \cup S_{63}\\
= \;& (S_0 \cup S_{61} \cup S_{22} \cup S_4 \cup S_{21}) \cup S_{32} \cup S_{63} \cup (S_{31} \cup S_5 \cup S_{62} \cup S_1)\\
= \;& \{\, 0, 1, \ldots, (t+1)/2, (t+3)/2, (t+5)/2, \ldots, t+1, t+2, t+3, \ldots, 2t+1,\\
& 2t+2, 2t+3, \ldots, (5t+1)/2, \quad (5t+5)/2, (5t+7)/2, \ldots, 3t+2,\\
& (7t+3)/2, (7t+7)/2, \ldots, 4t-1, \quad 4t+3, 4t+4, \ldots, (9t+3)/2,\\
& (9t+5)/2, (9t+7)/2, \ldots, (11t+3)/2, (11t+5)/2, \quad (11t+9)/2,\\
& \ldots \quad 6t-2, \quad 6t, \quad 6t+1, 6t+2, \ldots, 7t\}.
\end{aligned}$$

It is clear that the labels of the vertices are different, and $Max\{f(v_j^i) | 1 \leq i \leq t, 0 \leq j \leq 6\} = 7t = |E|$. We thus conclude that f is an injective mapping from the vertex set of G into $\{\, 0, 1, \ldots, |E|\}$.

Now, we verify that g maps E onto $\{1, 2, \ldots, |E|\}$.

$$\begin{aligned}
D_0 &= \{6t+1, 6t+2, \ldots, 7t\},\\
D_1 &= D_{11} \cup D_{12} = \{5t-2, 5t-4, \ldots, 4t+1\} \cup \{6t-1, 6t-3, \ldots, 5t\},\\
D_2 &= D_{21} \cup D_{22} = \{(5t-1)/2, (5t-5)/2, \ldots, (3t+5)/2\}\\
&\qquad \cup \{(5t+1)/2, (5t-3)/2, \ldots, (3t+3)/2\},\\
D_3 &= D_{31} \cup D_{32} = \{(7t-1)/2, (7t-5)/2, \ldots, (5t+5)/2\}\\
&\qquad \cup \{(3t+1)/2, (3t-3)/2, \ldots, (t+3)/2\},\\
D_4 &= \{(9t-1)/2, (9t-5)/2, \ldots, (5t+3)/2\},\\
D_5 &= D_{51} \cup D_{52} = \{(11t+1)/2, (11t-3)/2, \ldots, (9t+3)/2\}\\
&\qquad \cup \{(t+5)/2, (t+9)/2, \ldots, (3t-1)/2\},\\
D_6 &= D_{61} \cup D_{62} \cup D_{63}\\
&= \{1, 2, \ldots, (t+1)/2\} \cup \{(11t+5)/2, (11t+9)/2, \ldots, 6t\}\\
&\quad \cup \{(7t+3)/2, (7t+7)/2, \ldots, 4t-1\}.
\end{aligned}$$

Hence, $D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6$ is the set of labels of all edges, and

$$\begin{aligned}
& D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6 \\
= & D_0 \cup D_{11} \cup D_{12} \cup D_{21} \cup D_{22} \cup D_{31} \cup D_{32} \cup D_4 \cup D_{51} \cup D_{52} \\
& \cup D_{61} \cup D_{62} \cup D_{63} \\
= & D_{61} \cup (D_{32} \cup D_{52}) \cup (D_{22} \cup D_{21}) \cup (D_4 \cup D_{51} \cup D_{62}) \\
& \cup (D_{31} \cup D_{63} \cup D_{11} \cup D_{12}) \cup D_0 \\
= & \{1, 2, \ldots, (t+1)/2\} \cup \{(t+3)/2, (t+5)/2, \ldots, (3t-1)/2, (3t+1)/2\} \\
& \cup \{(3t+3)/2, (3t+5)/2, \ldots, (5t-1)/2, (5t+1)/2\} \\
& \cup \{(5t+3)/2, (5t+7)/2, \ldots, (9t-1)/2, (9t+3)/2, (9t+7)/2, \ldots, \\
& \quad (11t+1)/2, (11t+5)/2, (11t+9)/2, \ldots, 6t\} \\
& \cup \{(5t+5)/2, (5t+9)/2, \ldots, (7t-1)/2, (7t+3)/2, (7t+7)/2, \ldots, \\
& \quad 4t-1, 4t+1, 4t+3, \ldots, 5t-2, 5t, 5t+2, \ldots, 6t-1\} \\
& \cup \{6t+1, 6t+2, \ldots, 7t\} \\
= & \{1, 2, \ldots, 7t\}.
\end{aligned}$$

It is clear that the labels of the edges are different. So, g maps E onto $\{1, 2, \ldots, |E|\}$. By the definition of graceful graph, we can conclude that the graphs $C_7^{(4k+1)}$ are graceful.

According to the proof of Case 1 and Case 2, the graphs $C_7^{(t)}$ are graceful for $t \equiv 0, 1 \pmod 4$. □

In Figure 2.2 we show our graceful labelings for $C_7^{(8)}$ and $C_7^{(9)}$.

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	56	17	36	9	45	1
	55	18	35	10	44	2
	54	19	34	11	43	3
	53	20	33	12	42	4
0	52	5	25	13	41	46
	51	6	24	14	40	29
	50	7	23	15	39	48
	49	8	22	16	38	31

$C_7^{(8)}$

	63	20	42	11	51	1
	62	21	41	12	50	2
	61	22	40	13	49	3
	60	23	39	14	48	4
0	59	6	29	15	47	5
	58	7	28	16	46	33
	57	8	27	17	45	52
	56	9	26	18	44	35
	55	10	25	19	43	54

$C_7^{(9)}$

Figure 2.2: The graceful labelings of $C_7^{(8)}$ and $C_7^{(9)}$.