Note on Indices of Convergence of Digraphs

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Abstract

Let $k(D)$ be the index of convergence of a digraph D of order $n \geq 8$. It is proved that if D is not strong with only minimally strong components and the greatest common divisor of the cycle lengths of D is at least two, then

$$k(D) \leq \begin{cases} \frac{1}{2}(n^2 - 8n + 24) & \text{if } n \text{ is even,} \\ \frac{1}{2}(n^2 - 10n + 35) & \text{if } n \text{ is odd.} \end{cases}$$

The cases of equality are also characterised.

1 Introduction

Let D be a digraph and m be a nonnegative integer. The mth power D^m of D has the same vertex set as D and an arc from x to y if and only if D has a walk of length m from x to y. The period $p(D)$ and the index of convergence $k(D)$ are defined as follows: $p(D)$ is the least positive integer p such that for all sufficient large t, $D^t = D^{t+p}$ and $k(D)$ is the least nonnegative integer such that $D^t = D^{t+p(D)}$ whenever $t \geq k(D)$.

If D has only trivial strong components, then $p(D) = 1$. Otherwise, Rosenblatt [4] proved that if D is strongly connected (or strong), then $p(D)$ is the greatest common divisor (gcd for short) of the distinct cycle lengths of D, and if D is not strong, then $p(D)$ is the least common multiple (lcm for short) of the periods of all the nontrivial strong components of D. Since

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our knowledge about the period $p(D)$ is quite complete, we are interested in the study of the index $k(D)$.

Let $f_0(D)$ be the gcd of the cycle length of a digraph D. If D is strong, then $f_0(D) = p(D)$. Let $M_n = \{D|D$ is a digraph of order n which is not strong and $f_0(D) \geq 2\}$ for $n \geq 7$. Jiang [3] proved that

$$k(D) \leq \begin{cases} \frac{1}{2}(n^2 - 8n + 24) & \text{if } n \text{ is even,} \\ \frac{1}{2}(n^2 - 6n + 15) & \text{if } n \text{ is odd,} \end{cases}$$

and characterized the cases of inequality.

A strong digraph D is minimally strong if no digraph obtained from D by removal of an arc is strong. Let $MM_n = \{D|D \in M_n,\ D$ has only minimally strong components$\}$ for $n \geq 8$. In this paper, we will prove

$$k(D) \leq \begin{cases} \frac{1}{2}(n^2 - 8n + 24) & \text{if } n \text{ is even,} \\ \frac{1}{2}(n^2 - 10n + 35) & \text{if } n \text{ is odd,} \end{cases}$$

and characterize the cases of equality. Since result follows from [3] when n is even, we are interested in the case when n is odd.

2 Results

Let D be a digraph with $p(D) = p$ and let $x, y \in V(D)$. The local index from x to y, denoted by $k_D(x,y)$, is defined to be the least integer k such that for each $t \geq k$ there exists a walk of length t from x to y if and only if there exists a walk of length $t + p$ from x to y. It is easy to see that $k(D) = \max\{k_D(x,y): x,y \in V(D)\}$.

Denote by $P(x,y)$ the set of paths from x to y in D, and $l(W)$ the length of a walk W ($l(Q) = 0$ if $x = y$). Let $Q \in P(x,y)$, and let $C(Q)$ be the set of nontrivial strong components of D which have vertex in common with Q. If $C(Q) \neq \emptyset$, define $f(Q) = \gcd\{p(C_i): C_i \in C(Q)\}$. Let $L(C(Q))$ be the set of cycle lengths in those nontrivial strong components of $C(Q)$, let $s(Q)$ be the smallest number in $L(C(Q))$, and let n_0 be the largest order the nontrivial strong components of D with n vertices. Further define

$$W(Q) = \begin{cases} n + s(Q)\left(\frac{n_0}{f(Q)} - 2\right) & \text{if } C(Q) \neq \emptyset, \\ l(Q) + 1 & \text{if } C(Q) = \emptyset. \end{cases}$$

Lemma 1. ([2, 6]) *Let D be a digraph with $x, y \in V(D)$. If $P(x,y) \neq \emptyset$, then*

$$k_D(x,y) \leq \max\{W(Q): Q \in P(x,y)\}.$$

Lemma 2. [5] *Let D be a digraph of order n with at least a nontrivial strong component. Then*

$$k(D) \leq n + s_0\left(\frac{n_0}{f_o(D)} - 2\right),$$

where s_0 is the maximum of a shortest cycle length of the nontrivial strong components of D.

Lemma 3. [3] *Let D be a digraph of order n that is not strong and has a unique nontrivial strong component, say D_0, of order m. If $f_0(D) \geq 2$, then $k(D) \leq k(D_0)+n-m$, and equality holds if and only if there is a path Q from x to y in D and an ordered pair of vertices x_1, y_1 of Q in D_0 such that $k_D(x_1, y_1) = k(D_0)$ and $l_1 + l_2 = n - m$, where l_1 (l_2) is the length of the subpath from x to x_1 (from y_1 to y) of Q.*

Suppose D is a strong digraph of order n and period p. Let $CL(D)$ be the set of cycle lengths of D. Then [7, 8]

$$k_D(x,y) \leq \max\{d_{CL_0(D)}(x,y) + \tilde{\phi}(r_1, \ldots r_\lambda) - p + 1, 0\}$$

where $CL_0(D) = \{r_1, \ldots, r_\lambda\} \subseteq CL(D)$ with $\gcd(r_1, \ldots, r_\lambda) = p$, $d_{CL_0(D)}$ (x,y) is the relative distance from x to y, that is, the length of a shortest walk from x to y which meets at least one cycle of length r_i for $i = 1, \ldots, \lambda$, $\tilde{\phi}(r_1, \ldots, r_\lambda)$ is the generalized Frobenius number. For two positive integer q_1, q_2 with $\gcd(q_1, q_2) = p$, we have $\tilde{\phi}(q_1, q_2) = p(q_1/p - 1)(q_2/p - 1)$.

Lemma 4. [1] *Let D be a strong digraph of order n with period p and let s be the smallest cycle length of D. Then*

$$k(D) \leq n + s\left(\left\lfloor \frac{n}{p} \right\rfloor - 2\right).$$

Let F_n be the digraph that contains a cycle $(1, 2, \ldots, n-1, 1)$ and three additional arcs $(n-6, n-1)$, $(n-1, n)$ and $(n, 1)$, and let G_n be the digraph that contains a cycle $(1, 2, \ldots, n-1, 1)$ and two additional arcs $(n-4, n)$ and $(n, 1)$.

The upper bound in the following proposition follows from the results of [8], and the case when n is odd follows from [3, Lemma 4], however for completeness, a more self-contained proof is provided here.

Proposition 1. *Let D be a minimally strong digraph of order $n \geq 7$ with $p = p(D) \geq 2$. Then*

$$k(D) \leq \begin{cases} \frac{1}{2}(n^2 - 8n + 24) & \text{if } n \text{ is even,} \\ \frac{1}{2}(n^2 - 6n + 15) & \text{if } n \text{ is odd,} \end{cases} \tag{1}$$

equality holds if and only if D is isomorphic to F_n when n is even and G_n when n is odd.

Proof. Let s be the smallest cycle length of D. If $s = n$, clearly $k(D) = 0$. Suppose that $s < n$. D is minimally strong, any cycle in D has no chords, in particular, D has no cycle of length n if $s < n$.

First suppose that n is odd. If $s = n-1$ $(n-2)$, then $p(D) = n-1$ $(p(D) = n-2)$ and hence by Lemma 4, $k(D) \leq 2 < (n^2 - 6n + 15)/2$.

Suppose that $s \leq n-3$. Then by Lemma 4,

$$k(D) \leq n + (n-3)\left(\frac{n-1}{2} - 2\right) = \frac{1}{2}(n^2 - 6n + 15).$$

If equality in (1) holds, then $s = n-3$, $p = 2$ and hence D has a cycle of length $n-1$ and D is isomorphic to G_n.

Now suppose that n is even. If $s = n-1$ or $n-2$, by the same argument as above, $k(D) \leq 2 < (n^2 - 8n + 24)/2$.

Case 1: $s = n-3$. Then $p(D) = n-3$ and hence $k(D) \leq n + (n-3)(1-2) = 3 < (n^2 - 8n + 24)/2$.

Case 2: $s = n-4$. Then we have $CL(D) = \{n-4\}$ or $\{n-4, n-2\}$. If $CL(D) = \{n-4\}$, then by Lemma 4, $k(D) \leq 4 < (n^2 - 8n + 24)/2$. Suppose $CL(D) = \{n-4, n-2\}$. Then it can be easily checked that D is isomorphic to F_n (and hence $k(D) = (n^2 - 8n + 24)/2$), or one of the digraphs obtained by duplicating a vertex of any vertex of G_{n-1} (and hence $k(D) = (n^2 - 8n + 24)/2 - 1$), or one of the three digraphs obtained by adding to G_{n-1} arc pairs $(n-4, n)$ and $(n, 2)$, or $(n-3, n)$ and $(n, 3)$, or $(n-2, n)$ and $(n, 4)$ (and hence $k(D) = (n^2 - 8n + 24)/2 - 2$ or $n^2 - 8n + 24)/2 - 1$ or $(n^2 - 8n + 24)/2 - 1$). We have proved that in this case $k(D) \leq (n^2 - 8n + 24)/2$, equality holds only if $D \cong F_n$.

Case 3: $s = n-5$. Then we have two possibilities:

(1) $CL(D) = \{n-5\}$. Then $k(D) \leq 5 < (n^2 - 8n + 24)/2$.

(2) $CL(D) = \{n-5, n-2\}$, $n \equiv 2 \bmod 3$). Then $p(D) = 3$ and hence

$$\begin{aligned} k(D) &\leq n + (n-5)\left(\frac{n-2}{3} - 2\right) \\ &= \frac{1}{3}(n^2 - 10n + 40) < \frac{1}{2}(n^2 - 8n + 24). \end{aligned}$$

Case 4: $s \leq n-6$. Then we have

$$k(D) \leq n + (n-6)\left(\frac{n}{2} - 2\right) = \frac{1}{2}(n^2 - 8n + 24),$$

equality holds only if $s = n-6$, $p(D) = 2$ and hence $n = 8$, $n-4 \in CL(D)$ or $n-2 \in CL(D)$. Suppose $s = n-6$ and $p(D) = 2$. If $n = 8$, neither $n-4$ nor $n-2$ is in $CL(D)$, then clearly D can be obtained by replacing each edge of an undirected path of order 8 by two opposite arcs and hence we have $k(D) = 6 < (n^2 - 8n + 24)/2$. If $n-4 \in CL(D)$, then (let $R = \{n-6, n-4\}$)

$$
\begin{aligned}
k(D) &\le \max_{x,y\in V(D)} d_R(x,y) + \tilde{\phi}(n-6,n-4) - p(D) + 1 \\
&\le n+1+\tilde{\phi}(n-6,n-4) - 1 \\
&= \frac{1}{2}(n^2-12n+48) < \frac{1}{2}(n^2-8n+24).
\end{aligned}
$$

We are left with $n-4 \notin CL(D)$. Then $n-2 \in CL(D)$ and $\gcd(n-6,n-2) = 2$. Similarly, we have

$$
\begin{aligned}
k(D) &\le n+1+\tilde{\phi}(n-6,n-2) - 1 \\
&= \frac{1}{2}(n^2-10n+32) < \frac{1}{2}(n^2-8n+24).
\end{aligned}
$$

It follows that in this case we have $k(D) < (n^2-8n+24)/2$.

By combining Cases 1-4, we get (1).

From the above argument we know that the equality in (1) holds if and only if D is isomorphic to F_n when n is even and G_n when n is odd. □

Remark. Let D be a minimally strong digraph of order n with $2 \le n \le 6$ and with period ≥ 2. By listing all the non-isomorphic minimally strong digraphs of order $n \le 6$ with period ≥ 2, or by [8] directly, we get $k(D) \le n-2$ if $n = 2,3,4$, $k(D) \le 5$ if $n = 5$ and $k(D) \le 7$ if $n = 6$, and the bounds are best possible.

Let $H_n^{(1)}(H_n^{(2)})$ be digraph obtained by adding to G_{n-1} a vertex n and an arc $(n,n-4)((n-2,n))$. Let $L_n^{(1)}(L_n^{(2)})$ be the digraph by adding to F_{n-1} a vertex n and an arc $(n,n-6)((n-3,n))$. Let $L_n^{(3)}(L_n^{(4)},L_n^{(5)})$ be the digraph obtained by adding to G_{n-2} two vertices $n-1,n$ and two additional arc pair $(n,n-1)$ and $(n-1,n-5)((n-3,n-1)$ and $(n-1,n)$, $(n-1,n-5)$ and $(n-3,n))$.

The main result of this note is the following.

Proposition 2. *Let $D \in MM_n$ with $n \ge 8$. Then*

$$
k(D) \le \begin{cases} \frac{1}{2}(n^2-8n+24) & \text{if } n \text{ is even,} \\ \frac{1}{2}(n^2-10n+35) & \text{if } n \text{ is odd,} \end{cases} \tag{2}
$$

equality holds if and only if D is isomorphic to $H_n^{(1)}$ or $H_n^{(2)}$ when n is even, and one of $L_n^{(i)}$ for $i = 1,\dots,5$ when n is odd.

Proof. The case when n is even follows from [3, Theorems 1,3]. Suppose that n is odd and D_0 is a strong component of order n_0.

Case 1: $n_0 = n-1$. By Lemma 3 and Proposition 1,

$$
\begin{aligned}
k(D) &\le k(D_0) + 1 \\
&\le \frac{1}{2}((n-1)^2 - 8(n-1) + 24) + 1 \\
&= \frac{1}{2}(n^2-10n+35).
\end{aligned}
$$

Case 2: $n_0 = n - 2$.

Case 2.1: D_0 is the unique nontrivial strong component. By Lemma 3 and Proposition 1,

$$\begin{aligned} k(D) &\leq k(D_0) + 2 \\ &\leq \frac{1}{2}((n-2)^2 - 6(n-2) + 15) + 2 \\ &= \frac{1}{2}(n^2 - 10n + 35). \end{aligned}$$

Case 2.2: D has at least two nontrivial strong component, say D_0, D_1. Then the two vertices not in D_0 form a cycle of length 2. Take $x, y \in V(D)$ with $k_D(x, y) = k(D)$. If $x, y \in V(D_1)$ or there is no path from x to y, then $k(D) = k_D(x, y) = 0$. If $x, y \in V(D_0)$, then

$$\begin{aligned} k(D) &= k_D(x, y) \leq k(D_0) \\ &\leq \frac{1}{2}(n^2 - 10n + 31) \\ &= \frac{1}{2}(n^2 - 10n + 35). \end{aligned}$$

Otherwise, suppose that there is a path Q from x to y. Then $C(Q) \neq \emptyset$, $s(Q) = f(Q) = 2$ and

$$\begin{aligned} W(Q) &= n + s(Q)\left(\frac{n_0}{f(Q)} - 2\right) \\ &\leq n + 2\left(\frac{n-2}{2} - 2\right) \\ &= 2n - 6 < \frac{1}{2}(n^2 - 10n + 35). \end{aligned}$$

By Lemma 1,

$$k(D) = k_D(x, y) < \frac{1}{2}(n^2 - 10n + 35).$$

Case 3: $n_0 = n - 3$.

Case 3.1: D_0 is the unique nontrivial strong component. By Lemma 3 and Proposition 1, for $n \geq 11$, we have

$$\begin{aligned} k(D) &\leq k(D_0) + 3 \\ &\leq \frac{1}{2}((n-3)^2 - 8(n-3) + 24) + 3 \\ &= \frac{1}{2}(n^2 - 14n + 63) \\ &< \frac{1}{2}(n^2 - 10n + 35). \end{aligned}$$

For $n = 9$, $k(D_0) \leq 8$ by Lemma 4, we have $k(D) \leq k(D_0) + 3 < (n^2 - 10n + 35)/2$.

Case 3.2: D has at least two nontrivial strong components, say D_0, D_1. Then $CL(D_1) = \{2\}$ or $\{3\}$. Take $x, y \in V(D)$ with $k_D(x,y) = k(D)$. If $x, y \in V(D_1)$ or there is no path from x to y, then $k(D_0) = k_D(x,y) \leq 1$. If $x, y \in V(D_0)$, then

$$\begin{aligned} k(D) &= k_D(x,y) \leq k(D_0) \\ &\leq \frac{1}{2}(n^2 - 10n + 31) \\ &= \frac{1}{2}(n^2 - 10n + 35). \end{aligned}$$

Otherwise, suppose that there is a path Q from x to y. If $C(Q) = \emptyset$, then $W(Q) \leq n$. If $C(Q) \neq \emptyset$, then $s(Q) = f(Q) = 2$ or 3, and hence either

$$\begin{aligned} W(Q) &= n + s(Q)\left(\frac{n_0}{f(Q)} - 2\right) \\ &\leq n + 2\left(\frac{n-3}{2} - 2\right) \\ &= 2n - 7 < \frac{1}{2}(n^2 - 10n + 35) \end{aligned}$$

or

$$\begin{aligned} W(Q) &= n + s(Q)\left(\frac{n_0}{f(Q)} - 2\right) \\ &\leq n + 3\left(\frac{n-3}{3} - 2\right) \\ &= 2n - 9 < \frac{1}{2}(n^2 - 10n + 35). \end{aligned}$$

By Lemma 1,

$$k(D) = k_D(x,y) < \frac{1}{2}(n^2 - 10n + 35).$$

Case 4: $n_0 \leq n - 4$. By Lemma 2,

$$\begin{aligned} k(D_0) &\leq n + (n-4)\left(\frac{n-4}{2} - 2\right) \\ &= \frac{1}{2}(n^2 - 10n + 32) < \frac{1}{2}(n^2 - 10n + 35). \end{aligned}$$

By combining cases 1–4, we get (2).

If D is isomorphic to $L_n^{(i)}$ for $i = 1, \ldots, 5$, then it is easy to check that the equality in (2) holds.

Suppose that the equality in (2) holds. The we have either (1) $n_0 = n-1$, $k(D_0) = ((n-1)^2 - 8(n-1) + 24)/2 = (n^2 - 10n + 33)/2$ and hence by Proposition 1 $D_0 \cong F_{n-1}$ or (2) $n_0 = n-2$, $k(D_0) = ((n-2)^2 - 6(n-2) + 15)/2 = (n^2 - 10n + 31)/2$, and hence by Proposition 1 $D_0 \cong G_{n-2}$. It is easy to see that there is an unique ordered pair of vertices $n-6, n-3$ in F_{n-1} such that

$$k(F_{n-1}) = k_{F_{n-1}}(n-6, n-3) = \frac{1}{2}(n^2 - 10n + 33)$$

and there is an unique ordered pair of vertices $n-5, n-3$ in G_{n-2} such that

$$k(G_{n-2}) = k_{G_{n-2}}(n-5, n-3) = \frac{1}{2}(n^2 - 10n + 31).$$

Note that D_0 is the only nontrivial strong component. By Lemma 3, $k(D) = (n^2 - 10n + 35)/2$ implies that D is isomorphic to some $L_n^{(i)}$ for $i = 1, \ldots, 5$. □

References

[1] E.A. Bender and T.W. Tucker, The exponents of strongly connected graphs, *Canadian J. Math.* **21** (1969), 769–782.

[2] Jiang Zhiming and Shao Jiayu, On the set of indices of convergence for reducible matrices, *Linear Algebra Appl.* **148** (1991), 265–278.

[3] Jiang Zhiming, On the indices of convergence of reducible Boolean matrices with the generalized period $f_0 \geq 2$, *Linear Algebra Appl.* **248** (1996), 355–374.

[4] D. Rosenblatt, On the graphs and asymptotic forms of finite Boolean relation matrices and stochastic matrices, *Naval. Res. Logist. Quart.* **4** (1957), 151–167.

[5] Shao Jiayu, The indices of convergence of reducible Boolean matrices, *Acta Math. Sinica* **33** (1990), 13–28.

[6] Zhou Bo, The set of indices of convergence for a Boolean matrix class, *Util. Math.* **61** (2002), 23–32.

[7] Shao Jiayu and Li Qiao, The index set for the class of irreducible Boolean matrices with given period, *Linear and Multilinear Algebra* **22** (1988), 285–303.

[8] Shao Jiayu, The indices of convergence of irreducible and nearly reducible Boolean matrices, *Acta Math. Appl. Sinica* **15** (1992), 333–344.